Problematic Objects between Mathematics and Mechanics

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The relationship between the objects of mathematics and physics has been a recurrent source of philosophical debate. Rationalist philosophers can minimize the distance between mathematical and physical domains by appealing to transcendental categories, but then are left with the problem of where to locate those categories ontologically. Empiricists can locate their objects in the material realm, but then have difficulty explaining certain peculiar "transcendental" features of mathematics like the timelessness of its objects and the unfalsifiability of (at least some of) its truths. During the past twenty years, the relationship between mathematics and physics has come to seem particularly problematic, in part because of a strong interest in "naturalized epistemology" among American philosophers. The tendency to construe epistemological relations in causal and materialist terms seems to enforce a sharp distinction between mathematical and physical entities, and makes the former seem at best uncomfortably inaccessible and at worst irrelevant. Paul Benacerraf, for example, poses this situation as a conundrum. (Benacerraf 1965) And Hartry Field suggests that we might do best simply to scuttle any appeal to mathematical entities in a physico-philosophical description of the world. (Field 1980)

I would like to argue that what exists in mathematics are the items that figure in problems, especially the constellations of problems that we see as constituting its diverse domains. Mathematical entities only occur enmeshed in interesting problems; indeed, they come into view because they are both determinate and opaque and therefore problematic: one can formulate questions about them, and the questions don't have obvious answers. This way of talking about mathematical existence might seem psychological or historical, but my formulation points out its objectivity: once seen, a mathematical problem is not an accident of someone's subjectivity, nor a by-product of historical institutions.

What kind of epistemological picture is needed to explain how we can apprehend the items of mathematics as problematic? To see a triangle, for example, as a complex unity establishing a determinate but opaque relationship among its sides and interior angles, requires some transcendental as well as causal-material activity on our part. A causal-material account might be able to explain how we come to have an image of a triangle, or how we abstract the shape of a triangle from instances that we gradually

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learn to see as triangular, but it cannot explain why the resultant apprehension poses a problem.

Mathematics exists as a multiplicity of diverse domains. Problems and the items that figure in them form clusters whose discursive boundaries mark them off from other clusters. This also testifies to the objectivity of mathematical knowledge: certain kinds of problems, and not others, arise with respect to certain kinds of items. You can't say whatever you wish about a mathematical object, and you can't pose arbitrary questions about it.

But mathematical domains are not unrelated; they overlap at their boundaries, though the nature of the overlap always reveals opacities as it is determined. The partial unification of domains is not trivial because the domains really are distinct. Thus, establishing correspondences between domains is itself an interesting mathematical problem. But what items figure in such problems? In various studies published over the last decade, I have investigated mathematical research at the overlap of domains and found that one of its common features is the occurrence of "hybrids." The Cantor space at the intersection of logic and topology, and the Cartesian parabola at the intersection of algebra and geometry are examples of such hybrids. (Grosholz, 1985, 1990/91)

Hybrids are characteristically items that would not have arisen in one or the other domain investigated in isolation. Sometimes problems arise in a domain, but cannot be solved there. (The objectivity of items is attested not only by the problems in which they figure, but also by the intractability of those problems.) Then links with other domains must be forged so that other methods and perspectives can be brought to bear on the original problem, and in that process hybrids appear. They hover ambiguously, items not clearly belonging to one or the other domain, sometimes appearing almost contradictorily qualified as belonging to one *and* the other. And they may become the characteristic items of new domains arising at the intersection of old ones, as Leibniz' infinite-sided polygons which are also continuous curves (algebraic and transcendental) become the central focus of the emerging domain of the infinitesimal calculus. (Grosholz forthcoming)

In the seventeenth century, a new correlation was forged between the domains of mathematics and mechanics. This extended episode in the history of science ought to be closely examined, then, by philosophers of science concerned with the relationship between the objects of mathematics and physics. I have been especially interested to note that the interaction between mechanics and the infinitesimal calculus (arising itself during that period out of algebra, geometry and number theory) does not look appreciably different from the interaction between, say, algebra and geometry. Then perhaps the objects of physics are the items that intervene in the problems of physics; and a causal-materialist account of our knowledge of them falls short here just as it does in the case of mathematics. What makes the motion of a piece of string, a swinging pendulum, a planet, or the configuration of a hanging chain or a spinning globe, problematic?

1. Leibniz' Tractrix

I would like to discuss briefly one such example of the interaction of mechanics and mathematics in the seventeenth century, and point out certain of its important features, appealing explicitly to the model of the partial unification of mathematical domains just sketched. Typically, in this kind of unification problem-solving strategies from allied (but still distinct) domains are brought to bear on a family of problems that arise in one domain but cannot be solved there. This interaction changes the shape of the domains involved though they still maintain their autonomy, and tends to precipitate hybrids at the overlap of the domains.

The tractrix, Leibniz claims, is especially well suited to his new calculus. It was introduced to him in a drawing room in Paris when a mathematician dragged his watch across a table, describing a straight line with the free end of the fob, and asked the assembled guests what curve it traced. He added that because of the effect of friction, at any given point on the curve the direction of the motion of the watch is supposed to be along the fob, which is thus seen as the tangent to the curve. That's to say, he would never have asked his companions to examine the path of his watch on the table unless he had already learned to see its fob as a tangent, and its trajectory as an analyzable curve. (Bos 1988, pp. 9-12)

This parlor game owed its interest to the work of two generations of mathematicians on the problem of tangents. Leibniz retells the tale because his own work focussed on the inverse problem of tangents, and because mechanics had played a central role in his attempts to extend the new synthesis of geometry and algebra beyond the Cartesian program. Mechanics in the early 1670's was in part a practical exercise involving fountains, catapults and winches, and in part an emerging theory based on rediscovered texts by pseudo-Aristotle, Archimedes and Heron of Alexandria, as well as the writings of Descartes and Huygens. The use of "mechanical procedures" like this instance of the watch tracing a path on the table allowed Leibniz to locate, systematize and justify the introduction of transcendental curves and numbers, infinite series and reasonings involving infinitesimal magnitudes. And it was also one small step in the unification of practical and mathematical mechanics.

The mathematician asks for a description of motion under certain constraints, and the object of inquiry is the trajectory of the body in motion. Experience offers the watch on the table in Paris, whose motion is a problem; mathematics offers a curve which is already a hybrid because it exists as a geometrical shape, the solution of a differential equation in Leibniz's new calculus, an algebraic equation and a point-wise "mechanical" construction on the basis of other, constructing curves. A trajectory is a peculiar object to locate in experience. Its unity is not that of a physical object in its substantial oneness (a oneness which is indeterminate empirically but absolute transcendentally). Rather, its unity is that of a nexus of forces on the one hand, and on the other hand the unity of a mathematical curve, ambiguously defined as a hybrid.

Leibniz' tractrix can be embedded in two quite different kinds of diagrams. The first is a schematic picture of the drawing-room table and the watch. The x-axis is the base curve, along which the end of the watch-fob is pulled; if the cord-length is set equal to a, the initial position of the watch will be at a distance a up the y-axis and the end of the watch fob at the origin; call the line segment representing the watch and its fob PQ. Then as the end of the fob, Q, is pulled, P traces out the curve. This tracing specifies a differential equation, $dx = -(\sqrt{a^2 - y^2}) / y\, dy$, which follows from the similarity of the characteristic (differential) triangle and the finite triangle with sides a, σ and y. (Diagram 1) In a sense it gives the tractrix in a nutshell: that curve in which the differences between the abscissae and the differences between the ordinates obey this relationship. But what curve is that? The differential equation sums up the situation, but it must be solved. (Bos 1988, pp. 21-22)

Integrating both sides (the variables are already separated) yields $ax = -a \int a^2 - y^2 / y\, dy$. In a second, geometrical rather than mechanical, diagram Leibniz shows how to construct the curve $z = -a(\sqrt{a^2 - y^2} / y)$. He draws it point by point, making use of certain lines and a circle quadrant to construct it as the auxiliary

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curve ZZA that he calls the *linea tangentium*; and then, assuming that the latter can be integrated, he constructs the tractrix by finding for every point y a point x whose distance from the y-axis is equal to the area under the auxiliary curve at y, divided by the constant a: XY = area YAZ / a. (Diagram 2) (Bos 1988, pp. 22-24)

The role of the tractrix in the two diagrams is very different. Its occurrence in both signals its hybrid nature, for the first is a schematic picture of its mechanical genesis, a watch being dragged across a table, and the second exhibits its relation as a point-wise construction to the circle and the *linea tangentium* in geometrical fashion. The first diagram augments geometry with a mechanical process; and it likewise imposes on mechanics a geometrical interpretation which in fact oversimplifies the dynamical situation. The diagram contains no representation of the variable time; the only vestige of the dynamical aspect of the real situation is the assumption of friction between the watch and the table (the fob itself is assumed frictionless). This can be read off the diagram in the assumption that the motion of the watch is always in the direction of the fob.

The first diagram is also as it were mathematically incomplete; it reveals the exact and determinate genesis of the curve and a few of its important properties, but in itself, like the differential equation correlated with it, does not show how to investigate the tractrix further or how to relate the tractrix to other curves. Thus the second diagram must supplement it. The tractrix can be identified by shape as the same entity in both diagrams, and the sameness of shape is fundamental, for it is what holds the variables associated with the curve together in an intelligible unity. But the bridge between the two diagrams that registers the distinction as well as the relation between the mechanical and geometrical contexts is the expression of the curve in terms of Leibniz' differential equation and its transformation into ordinary algebraic terms, that is, its solution. Thus the tractrix exists as a hybrid to which experience and indeed mechanics alone would never have drawn our attention, and which Cartesian geometry would never have generated without the Leibnizian extension to mechanical constructions.

2. Resnik's Structuralism

The writings of Michael Resnik represent an important counter-proposal to the causal-materialist account of mathematical knowledge. Resnik wishes to assimilate rather than separate the objects of mathematics and physics, and to buttress the ontological and epistemic status of mathematical objects. His structuralist account of the entities of mathematics characterizes them as positions in patterns and likens our acquaintance with them to our acquisition of knowledge about linguistic and musical patterns. This then leads him to explore the extent to which our knowledge of physical entities can also be understood as acquaintance with patterns. Claiming that the entities of quantum physics themselves look like patterns, he finds no reason to think that mathematical patterns could not be instantiated by them. His final pronouncement, however, is that what the mappings between mathematical and physical patterns look like must remain indeterminate, since there is no master pattern which includes them both. An important feature of his structuralist theory is that a fact of the matter about how structures correspond exists only when they are subpatterns of the same pattern. (Resnik 1981, 1982, 1990)

In this section, I want to criticize Resnik's account of mathematical entities as positions in patterns, because I think its logicist bias makes certain kinds of complex unities in mathematics difficult to see. Since they are just the kind of item which I have discussed as hybrids at the overlap of domains, their absence in his account also makes the application of mathematics to mechanics harder to understand. All the same, the central tenet of Resnik's structuralist theory that reference from structure to structure may be indeterminate is quite consonant with my own convictions. For our recurrent attempts to elaborate and revise inter-structural correspondences play an important role in the advancement of mathematical and scientific knowledge.

Resnik claims that mathematics does not present objects with an "internal" composition, given in isolation and with features independent of the structures in which they happen to occur. Rather, he writes, "The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure." Nor, he adds, do they have any "internal structures." This rather polemical claim is immediately modified in what follows: clearly Resnik considers the objects of mathematical study to include structures themselves as well as points or positions.

Yet the suggestion persists that what mathematics is really about is points or positions, with stuctures intervening in a secondary way as relations among points or positions. For Resnik's two leading illustrations of what he means by positions or points are the natural numbers in sequence, and geometrical points in a (discrete and finite) spatial array. After all, mathematics has been regarded as the offspring of the Adam of numbers and the Eve of geometrical points. And the natural numbers considered as iterated units, and geometrical points, seem to have no independent presentability and no internal composition apart from the structures in which they occur.

Resnik defines a structure (or, to use the term he prefers, pattern) as "a complex entity consisting of one or more objects, which I call positions, standing in various relationships (and having various characteristics, distinguished positions and operations.)" (Resnik 1981, p. 530) Examples of structures are models of formal theories, like (N,S) the natural numbers with the successor function, and a finite, discrete pattern of geometrical points. In general, Resnik says, "patterns are specific models of theories (up to isomorphism)." (Resnik 1981, p. 536) So it seems that while patterns have an internal composition, or rather while they are a composition (whether internal or not is unclear), they cannot be given independent of the points or positions whose relations they are. There is nothing more to them than those relations.

Herein lies the difficulty. Neither points or positions, nor patterns, seem to be the kind of thing about which one could pose an interesting mathematical problem. What is there to say about a point, or the unit? If there is nothing to say, why should there be anything to say about relations among entities about which there is nothing to say? This puzzle is not an empty bit of sophistry. Rather, it indicates Resnik's failure to locate a middle ground of mathematical objects that exhibit an interesting, complex and problematic unity, objects that can indeed be given independently and which have internal composition.

In order to show the importance of this middle ground of mathematical objects, I need to exhibit the circularity in Resnik's exposition that rules them out ad hoc and to explain why he doesn't notice the circle. Take the case of geometry. Resnik sees geometry as a model of a formal theory couched in the language of predicate logic with some geometrical vocabulary added in the extralogical axioms. The quantifiers of the formal theory range over geometrical points, which then seem to be the true objects of geometry, and the model supplies relations among those objects. Geometrical points have no internal composition; and the logicization of geometry stipulates that points are the only object of geometry; so geometrical objects have no internal composition.

Resnik doesn't recognize this circularity (at least not in the papers cited), perhaps because of certain prejudices he shares with many other contemporary philosophers who look at mathematics through the lens of logic. The first is the assumption that mathematical domains are structured like logical theories. The second is Quine's dictum that the objects of a domain are what the quantifiers of such logical theories quantify over. The third is the logicist misunderstanding of analytic geometry, which views the domain of geometry as reduced to that of number, the continuous as reduced to the (infinitely iterated) discrete. Logicist reduction plays down the difficulty of establishing correlations between domains, and the opacities that remain in such correlations once they have been set up.

Descartes in the seventeenth century offered a similar misinterpretation of geometry. He wanted to present his mathematics as a relational structure instantiated by items that would be mere place-holders. These place-holders would have no internal structure of their own, and so would not impugn the generality of, or threaten to disrupt, the structures in which they stood. He chose straight line segments. This was in many ways a useful choice, but the place-holders were far from neutral. They changed both the geometry and the algebra of the problem-context in which Descartes worked, and excluded many important objects (areas, volumes, infinitesimals, curves) from his geometry of ratios and proportions, thus limiting in important ways the kinds of problems and solutions he could entertain. Moreover, his insistence that his relational structure was transparent to reason blocked his ability to see that his mathematics was in fact suspended between two nonequivalent kinds of structures (equations, and proportions), and that the way these structures worked was the product of historical debate, and certain decisions on the part of his mathematical antecedents. (Grosholz 1990/91, ch. 1 and 2)

Likewise, Resnik doesn't examine the consequences of the debt which his theory of patterns owes to twentieth century predicate logic and the set theory that arose alongside it. To suppose that any geometrical item is a logical-relational structure holding among points depends on two historical projects. The attempt to assimilate geometry to the realm of number inspired and continues to challenge transfinite set theory. The attempt to assimilate the realm of number to logic characterized the first decades of the development of predicate logic, leaving an indelible imprint. The former gives us the habit of thinking of the continuum as an infinite concatenation of discrete (number-like) points, and the latter of supposing that numbers behave like well-formed formulae.

Though in his discussion of congruences between patterns, Resnik rejects strict reductionism for the good reason that there is no master pattern in terms of which congruences might be established univocally, still he does not escape the influence of these venerable logicist projects of reduction. Predicate logic pretends to be a transparent structure, but actually logicians have chosen its characteristic place-holders, certain well-formed formulae, and that choice has consequences. It affects the shape of other structures to which predicate logic is applied both by amplification and suppression. Domains, objects and problems which do not lend themselves to the combinatorial, boolean shape of predicate logic tend to fade away in the eyes of logicians working under its insistent light. (Grosholz 1982)

I have argued, however, that the true objects of a mathematical domain are those entities about which problems arise, the foci of mathematical investigations. The objects that inspire mathematical research programs are profoundly interesting; their recalcitrance and mystery are as challenging as their revelations. No geometer would waste his or her time investigating dots. Moreover, the relations among mathematical

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domains like logic and geometry, geometry and number theory, or set theory and analysis, are not relations of reduction in any simple sense. The objects, problems and methods of these domains are too heterogeneous to be captured by a single morphism; indeed, part of the mystery that inspires mathematical research is the investigation of what happens at the indeterminate overlap of mathematical domains. Resnik himself makes this point, though expressed in terms of positions, patterns and the congruences that hold between patterns. He claims, "there is no fact of the matter whether an occurrence of a pattern is or is not the same as another except when they are both subpatterns of the same pattern," to explain the phenomena of multiple reductions between domains. (Resnik 1981, p. 546)

Resnik connects the indeterminacy of reference between patterns with the absence of any single master pattern, and of any true individuals in mathematics, with respect to which some absolute reference might be fixed. Both points are well taken. But I would observe that citing the absence of a master pattern also indicates the multiplicity of mathematical domains, though in a very abstract and general way. Also, the absence of true individuals (noninstantiables) (Gracia 1988) in mathematics is consistent with my claim that among mathematical instantiables are unities that have interesting internal complexity and can be given independently. A large part of mathematics is about these unities, which are purely formal and yet isolable and complex. I would add that such formal unities, because they are isolable and complex, can thus sometimes function especially well as schemata for individuals encountered in the natural world.

Euclid certainly did not take points as the basic objects of geometry. Euclidean geometry studies a variety of objects which, he is quite careful to point out, are very different from each other: points, lines (which are bounded by but not composed of points), and plane figures (which are bounded by but not composed of lines). (Heath 1956, pp. 153-232) In Euclid's view, the more complex entities could not be reduced to the simpler; indeed, their heterogeneity is to him so important that he bans the yoking together of points, lines and plane figures in ratios and proportions. His treatment of the objects of geometry reveals that the unity of points is trivial; the unity of lines is somewhat more interesting, since lines can be measured by lines; and the unity of plane figures, like triangles and circles, is so rich and various that it constitutes a research program of which the Pythagorean Theorem is the signpost and flag. (Heath 1956, pp. 349-368)

Euclidean plane figures like the right triangle are objects about which an important set of problems can be posed; these problems are not problems about points. They concern the endlessly interesting internal composition of the triangle, as a whole greater than the sum of its parts, the points or vertices that bound its sides, the lines that join its vertices, the angles that exist between those lines. That whole has the unity of shape that allows it to be given in a diagram, in isolation from all the other possible objects of geometrical study. (Susan Hale, in arguing for a distinction between the relational and intrinsic properties of geometrical entities, underscores my point here. Some mathematical entities such as curves, she concludes, do have properties which are not merely extrinsic and relational. (Hale forthcoming))

Thus, I claim that Resnik cannot explain the research program of Euclidean geometry simply by reference to structures as he defines them. Nor can his account of congruence between patterns explain the synthesis of geometry and algebra in the work of Descartes, of geometry and mechanics in Newton's Principia, or the synthesis of geometry, algebra, number theory and mechanics in the late seventeenth century writings of Leibniz, for those partial unifications were posed and elaborated in terms of isolable, internally complex mathematical objects. Descartes' Cartesian parabola, Leibniz' tractrix and Newton's planetary ellipse were not treated as congruences in which the positions or points of one pattern are mapped onto or occur within the positions or points of another pattern. They were hypotheses about certain possible relations among domains made treatable as problems, that is, made into a research program, by the problematic unity of shape exhibited by higher algebraic and transcendental curves.

The realm of number, the hierarchy of sets, the formulae of logic will never give rise to the peculiar unity of geometrical shape. (Though one might argue that they have interesting, nontrivial unities of their own.) And points augmented by logically specified relations will never yield lines or figures. A triangle is not a position or a pattern in Resnik's sense, and it is not even a subpattern, since a subpattern is only a collection of points or positions united by logically specified relations. The founding of research programs in mathematics, and the partial unification of mathematical domains, indeed the very possibility of mathematical knowledge, requires the existence of objects that, while not individuals (noninstantiables) are yet isolable, internally articulated unities.

3. Structuralism and Space

In Euclid's geometry, space as a whole is not itself an object of study. One might say something about its properties: it has no boundaries, no regions, no parts, no separable components, no holes: it is isometric, isotropic, homogeneous, infinite, continuous, dense, non-separable. (But note how anachronistic it sounds to say most of that.) However, space as a whole does not intervene in problems: Euclid's problems have to do with points, lines and bounded plane figures. I have been urging that it makes sense to think of the objects of Euclidean geometry as heterogeneous, as many-sorted; now I want to urge the difference between the objects which articulate Euclidean space, and the space itself. The former are finite and bounded; they have parts and discernible regions apropos those parts. In the case of plane figures, they have the problematic unity of shape. Though we might want to say that they reveal the properties of space as a whole through the way they articulate that space, in almost every important respect they are unlike it.

Does space taken as a whole have a unity? As far as I know, no one in classical antiquity ever asked that question. We might say in retrospect that it has the unity of being without parts, separable components, boundaries, regions, etc. But that's a trivial unity, a unity without internal composition, a unity due to sheer indeterminateness or structurelessness. While it allows the objects of Euclidean geometry, and their congruences and similarities, to appear, in itself it has none of the features that makes them objects of study.

Perhaps I am being unduly enigmatic; I may make it seem like a mystery how determinate geometrical objects could "emerge" from indeterminate geometrical space. Historically, though, it was space as a whole that "emerged" as an object of study from the study of geometrical objects like points, lines and plane figures. And this process did not take place within geometry, but only after geometry had been combined with the domain of number in the seventeenth, eighteenth and nineteenth centuries, and then combined with set theory at the turn of the twentieth century. For once a continuum is seen in analogy with the discrete realm of number, and the discrete realm of number is reorganized to mimic the continuum, space itself becomes a hybrid, novel object which for certain purposes of problem-solving may be decomposed to point-numbers. And once set theory has been introduced, these infinitary col-

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lections of points can be seen as (geometric-numerical) sets, thus as bearers of internal structure and objects of study.

In twentieth century mathematics, we are used to talking about spaces as objects of study, not only Euclidean spaces, but non-Euclidean spaces and a variety of topological spaces whose properties are very different from Euclidean spaces (they may have separable components, holes, bumps, etc.), as well as highly infinitary function spaces whose points or positions are functions. Yet it is well to remember two things. First, these spaces are not objects of set theory per se, for set theory taken in itself would never have discovered them as objects, nor the problems in which they intervene. Rather, they are hybrids hovering ambiguously between analysis and set theory. To call them merely sets of points is to forget their origins.

Second, the difference between Euclidean space and for example Hilbert space brings to the fore an important tension in the modern framework between the demand for reduction and the demand for hierarchy. (Once again I would point out that while Resnik's structuralism is not reductionist, it includes and masks some reductionist assumptions.) We say that the points or positions in Hilbert space are functions; this make them look anomalous on Resnik's account, since points or positions aren't supposed to have any internal composition. The obvious response is that functions themselves are just sets of points. But then the difference between Euclidean space and Hilbert space is obscured, for they have both become just sets of points. Set theory alone (and, I would add, Resnik's structuralism) cannot reinstate the distinction, that is, the middle ground in which functions have a unity different from the unity of points and from the unity of space.

Resnik wants to talk about Hilbert spaces as structures with peculiar points or positions, Euclidean space as a structure of points, and the ordinary objects of Euclidean geometry as substructures of points. I have just argued that the kinds of unity in question are very different, as are the problems in which they figure, and that therefore something very important is lost in this account. But returning to Resnik's way of talking raises a further question: what kind of unity do structures have? For Resnik certainly thinks that we study structures as well as points or positions in mathematics; indeed, the fact that they intervene in structures is what makes points or positions susceptible of study. And if we study structures, they must exhibit some kind of unity or bring mathematical reality into some kind of unity, for nothing is knowable that isn't unified.

When Resnik talks about structures as models of formal theories in predicate logic, as he often does, the unity invoked is the unity of an axiomatized set of statements formulated in the language of predicate logic. For the pervasive, almost invisible logicism that runs through contemporary philosophy of mathematics imposes the attributes of logic on everything it touches. Now the unity of deductive systems is an extremely interesting kind of unity; even Euclid was interested in it. It has been studied in the twentieth century with the help of Boolean algebra and recursion theory. But it is not the kind of unity a geometric space has; and it is not the kind of unity a triangle or a function has.

Resnik's choice of the words "structure" or "pattern" reminds us that the objects of mathematics are not individuals (noninstantiables). All the objects of mathematics are instantiables and so look like universals; this explains in part their transcendence. But the objects of mathematics also exhibit kinds of formal unity which occupy a middle ground between the trivial unity of a point, unit or the empty set, and the imperfectly understood transfinitudes of set theory. Recognizing this aspect of mathematical reality helps to explain how mathematical domains are constituted as collec-

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tions of related problems about certain kinds of objects; and how they are partially unified around hybrid objects, a synthesis that contributes to the growth of mathematical knowledge in important ways. It also helps to explain how mathematics can be instantiated by physical reality, in virtue of a process that gradually (though never completely) unifies mathematics and physics, despite the many ways in which physical objects differ from mathematical objects.

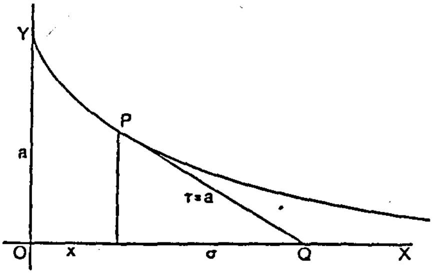

Diagram 1

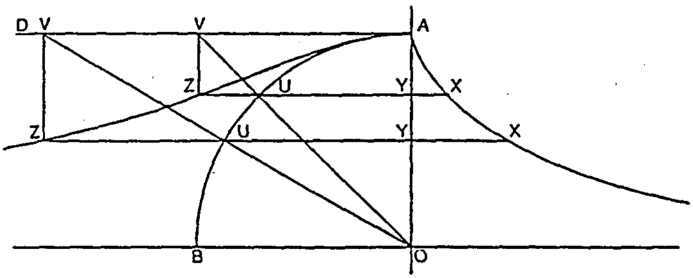

Diagram 2

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