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A Theorem on Unit-Regular Rings

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Abstract. Let R be a unit-regular ring and let σ be an endomorphism of R such that $\sigma(e) = e$ for all $e^2 = e \in R$ and let $n \geq 0$. It is proved that every element of $R[x;\sigma]/(x^{n+1})$ is equivalent to an element of the form $e_0 + e_1x + \cdots + e_nx^n$, where the e_i are orthogonal idempotents of R. As an application, it is proved that $R[x;\sigma]/(x^{n+1})$ is left morphic for each $n \geq 0$.

Throughout this note, R is an associative ring with unity. A ring R is called *unit-regular* if, for any $a \in R$, $a = aua$ for some unit u of R. For $a, b \in R$, we say that a is *equivalent to* b if $b = uav$ for some units u and v in R. It is an interesting question in ring theory (in particular in the theory of matrix rings) to ask when an arbitrary element of a ring is equivalent to an element with a certain property. In this note, we consider this question for the ring $R[x;\sigma]/(x^{n+1})$, where R is a unit-regular ring with an endomorphism σ. Our main results are Theorem 2 and Corollary 3.

Let R be a ring. For $a, b \in R$, let $[a, b] = ab - ba$ be the commutator of a and b. For two additive subgroups A and B of R, let $[A, B]$ denote the additive subgroup of R generated by all elements $[a, b]$ for $a \in A$ and $b \in B$. An additive subgroup L of R is called a *Lie ideal* if $[L, R] \subseteq L$.

Proposition 1 *Let R be a semiprime ring and let σ be an endomorphism of R such that $\sigma(e) = e$ for all $e = e^2 \in R$. Then $e(\sigma^k(r) - r)(1 - e) = 0$ for all $r \in R$, all $e^2 = e \in R$, and all positive integers k.*

Proof Since σ^k is also an endomorphism of R and $\sigma^k(e) = e$ for all $e = e^2 \in R$, it suffices to show the case $k = 1$. Let E be the additive subgroup of R generated by all idempotents in R. Note that for $e^2 = e \in R$ and $r \in R$,

$$[r, e] = (e + (1 - e)re) - (e + er(1 - e))$$

is a difference of two idempotents. It follows that E is a Lie ideal of R. Thus, for $r \in R$ and $e = e^2 \in R$, we have $[e, r] \in [E, R] \subseteq E$ and hence

$$[e, r] = \sigma([e, r]) = [\sigma(e), \sigma(r)] = [e, \sigma(r)].$$

So $[e, \sigma(r) - r] = 0$ for all $r \in R$. Right-multiplying the last equality by $1 - e$ yields $e(\sigma(r) - r)(1 - e) = 0$, as asserted. ∎

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For an endomorphism σ of R, let $R[x;\sigma]$ be the ring of left polynomials over R. Thus, elements of $R[x;\sigma]$ are polynomials in x with coefficients in R written on the left, subject to the relation $xr = \sigma(r)x$ for all $r \in R$. Let $S = R[x;\sigma]/(x^{n+1})$ where $n \geq 0$. Then

$$S = \{r_0 + r_1x + \cdots + r_nx^n : r_i \in R, i = 0, 1, \ldots, n\}$$

with $x^{n+1} = 0$ and $xr = \sigma(r)x$ for all $r \in R$. Our aim is to prove the following theorem and Corollary 3.

Theorem 2 *Let σ be an endomorphism of R such that $\sigma(e) = e$ for all $e^2 = e \in R$ and let $S = R[x;\sigma]/(x^{n+1})$ where $n \geq 0$. Then the following are equivalent:*

(i) *R is a unit-regular ring.*
(ii) *Each $\alpha \in S$ is equivalent to $e_0 + e_1x + \cdots + e_nx^n$, where the e_i are orthogonal idempotents of R.*

Proof (ii) $\Rightarrow$ (i). Note that if $r_0 + r_1x + \cdots + r_nx^n \in S$ is a unit, then so is r_0 in R. Let $a \in R$. By hypothesis, there exists $e^2 = e \in R$ such that $uav = e$, where u, v are units in R. Then $a = a(vu)a$ is unit-regular.

(i) $\Rightarrow$ (ii). It suffices to show the following claim: For each integer k with $1 \leq k \leq n$, there exist idempotents $e_0, \ldots, e_{k-1} \in R$ and $r_k, \ldots, r_n \in R$ such that up to equivalence

$$(*) \qquad \alpha = e_0 + e_1x + \cdots + e_{k-1}x^{k-1} + \sum_{j=k}^{n} r_jx^j,$$

where $e_i \in (1 - e_{i-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{i-1})$ for $i = 1, \ldots, k - 1$ and where $r_j \in (1 - e_{k-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{k-1})$ for $j = k, \ldots, n$.

Our theorem is then proved by choosing $k = n$. Indeed, in this case we see that

$$\alpha = e_0 + e_1x + \cdots + e_{n-1}x^{n-1} + r_nx^n,$$

where $e_i \in (1 - e_{i-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{i-1})$ for $i = 1, \ldots, n - 1$ and where $r_n \in hRh$ with $h := (1 - e_0) \cdots (1 - e_{n-1})$. Because hRh is unit-regular by [3, Corollary 4.7], there is a unit u in hRh with inverse v and an idempotent e_n in hRh such that $r_n = ue_n$. Clearly, $(e_0 + \cdots + e_{n-1}) + v$ is a unit in R and

$$(e_0 + \cdots + e_{n-1} + v)\alpha = e_0 + e_1x + \cdots + e_{n-1}x^{n-1} + e_nx^n,$$

as asserted.

We now turn to proving our claim. By induction we first deal with the case $k = 1$. Let $\alpha = r_0 + r_1x + \cdots + r_nx^n \in S$. Since R is unit-regular, every element of R is the product of a unit and an idempotent. Thus, up to equivalence, left-multiplying α by a suitable unit of R, we can assume that $r_0 = e_0$ is an idempotent. Because

$$(1 - (1 - e_0)r_1x)\alpha(1 - r_1x) = e_0 + (1 - e_0)r_1(1 - e_0)x + \cdots,$$

where both $1-(1-e_0)r_1x$ and $1-r_1x$ are units of S, we can further assume that $r_1 \in (1-e_0)R(1-e_0)$. Now

$$(1-(1-e_0)r_2x^2)\alpha(1-r_2x^2) = e_0 + r_1x + (1-e_0)r_2(1-e_0)x^2 + \cdots,$$

where both $1-(1-e_0)r_2x^2$ and $1-r_2x^2$ are units of S, so we can assume that $r_2 \in (1-e_0)R(1-e_0)$. A simple induction shows that we can assume that

$$\alpha = e_0 + r_1x + r_2x^2 + \cdots + r_nx^n, \; r_i \in (1-e_0)R(1-e_0), \quad \text{for } i = 1, \dots, n.$$

Thus the case where $k = 1$ is proved. Fix an integer k with $1 < k < n$ and assume that $(*)$ holds. Clearly, $e_0, \dots, e_{k-1}$ are orthogonal idempotents. We set

$$f_{k-1} := (1-e_0)\cdots(1-e_{k-1}) \quad \text{and} \quad g_{k-1} = e_0 + \cdots + e_{k-1}.$$

Then f_{k-1} and g_{k-1} are orthogonal idempotents and $f_{k-1} + g_{k-1} = 1$. Because $f_{k-1}Rf_{k-1}$ is a unit-regular ring by [3, Corollary 4.7], write $r_k = ue_k$ where e_k is an idempotent of $f_{k-1}Rf_{k-1}$ and u is a unit of $f_{k-1}Rf_{k-1}$ with inverse v. Then $g_{k-1} + v$ is a unit of R with inverse $g_{k-1} + u$. Since

$$(g_{k-1} + v)\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} vr_jx^j,$$

up to equivalence we can assume that

$$\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} r_jx^j,$$

where $e_k^2 = e_k \in f_{k-1}Rf_{k-1}$ and $r_j \in f_{k-1}Rf_{k-1}$ for $j = k+1, \dots, n$. Now

$$\begin{aligned}\alpha' &:= (1-r_{k+1}x)\alpha \\ &= e_0 + e_1x + \cdots + e_kx^k + r_{k+1}(1-e_k)x^{k+1} + \sum_{j=k+2}^{n} r_j'x^j,\end{aligned}$$

where $r_{k+1}, r_{k+2}', \dots, r_n' \in f_{k-1}Rf_{k-1}$. Set $r_{k+1}' := r_{k+1}(1-e_k)$. We then compute

$$\big(1-(1-e_k)r_{k+1}'x\big)\,\alpha'(1-r_{k+1}'x) = \sum_{i=0}^{k} e_ix^i + \sum_{j=k+1}^{n} r_j''x^j,$$

where

$$\begin{aligned}r_{k+1}'' &= r_{k+1}' - e_k\sigma^k(r_{k+1}') - (1-e_k)r_{k+1}'e_k \\ &= e_k(r_{k+1}' - \sigma^k(r_{k+1}')) + (1-e_k)r_{k+1}'(1-e_k) \\ &= e_k(r_{k+1} - \sigma^k(r_{k+1}))(1-e_k) + (1-e_k)r_{k+1}'(1-e_k) \\ &= (1-e_k)r_{k+1}'(1-e_k) \in (1-e_k)f_{k-1}Rf_{k-1}(1-e_k),\end{aligned}$$

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since $e_k(r_{k+1} - \sigma^k(r_{k+1}))(1 - e_k) = 0$ by Proposition 1, and where all $r_i'' \in f_{k-1}Rf_{k-1}$ for $i \geq k+2$.

We set $f_i := (1 - e_0)\cdots(1 - e_i)$ for $i = 0, 1, \ldots, k$. Up to equivalence we may assume that

$$\alpha = \sum_{i=0}^{k} e_i x^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r_j x^j,$$

where $e_i = e_i^2 \in f_{i-1}Rf_{i-1}$ for $i = 1, \ldots, k$, and where $r_{k+1} \in f_kRf_k$, $r_j \in f_{k-1}Rf_{k-1}$ for $j = k+2, \ldots, n$. We then compute

$$\begin{aligned}\alpha' &:= (1 - r_{k+2}x^2)\alpha \\ &= \sum_{i=0}^{k} e_i x^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r_j' x^j,\end{aligned}$$

where $r_j' \in f_{k-1}Rf_{k-1}$ for $j > k+2$ and where $r_{k+2}' = r_{k+2}(1 - e_k)$. We then compute

$$(1 - (1 - e_k)r_{k+2}'x^2)\alpha'(1 - r_{k+2}'x^2) = \sum_{i=0}^{k} e_i x^i + r_{k+1}x^{k+1} + \sum_{j=k+2}^{n} r_j'' x^j,$$

where

$$\begin{aligned}r_{k+2}'' &= r_{k+2}' - e_k\sigma^k(r_{k+2}') - (1 - e_k)r_{k+2}'e_k \\ &= e_k\big(r_{k+2}' - \sigma^k(r_{k+2}')\big) + (1 - e_k)r_{k+2}'(1 - e_k) \\ &= e_k\big(r_{k+2} - \sigma^k(r_{k+2})\big)(1 - e_k) + (1 - e_k)r_{k+2}'(1 - e_k) \\ &= (1 - e_k)r_{k+2}'(1 - e_k) \in (1 - e_k)f_{k-1}Rf_{k-1}(1 - e_k) = f_kRf_k,\end{aligned}$$

since $e_k(r_{k+2} - \sigma^k(r_{k+2}))(1 - e_k) = 0$ by Proposition 1, and where all $r_i'' \in f_{k-1}Rf_{k-1}$ for $i \geq k+3$. Repeating analogous arguments, up to equivalence we may assume that

$$\alpha = e_0 + e_1x + \cdots + e_kx^k + \sum_{j=k+1}^{n} r_j x^j,$$

where $r_j \in f_kRf_k$ for $j = k+1, \ldots, n$. So we complete the inductive step and hence the proof is finished. ■

Following [5], an element $a \in R$ is called *left morphic* if $R/Ra \cong \mathbf{l}(a)$, where $\mathbf{l}(a) = \{r \in R \mid ra = 0\}$ is the left annihilator of a in R, and the ring R is called *left morphic* if every element of R is left morphic. A well known result of Ehrlich says that a ring R is unit-regular if and only if R is both left morphic and (von Neumann) regular (see [2]). The morphic property of the ring $R[x;\sigma]/(x^{n+1})$ was first considered in [5] where it was noticed that if D is a division ring and σ is an endomorphism of D with $\sigma(1) = 1$, then $D[x;\sigma]/(x^2)$ is left morphic. Later in [1], it was proved that if

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R is a strongly regular ring (*i.e.*, a regular ring whose idempotents are central) and σ is an endomorphism of R such that $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x;\sigma]/(x^2)$ is left morphic. Note that every strongly regular ring is unit-regular. Recently, in [4, Theorem 2], it was proved that if R is a unit-regular ring and σ is an endomorphism of R such that $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x;\sigma]/(x^2)$ is left morphic and $R[x]/(x^{n+1})$ is left morphic for each $n \geq 0$. It is worth noting that the proof of [4, Theorem 2] only works for $R[x]/(x^{n+1})$, that is, the case where $\sigma = 1_R$. The assumption that $\sigma(e) = e$ for all $e^2 = e \in R$ in the next corollary is not superfluous (see [4, Example 3]).

Corollary 3 *Let R be a unit-regular ring with an endomorphism σ such that $\sigma(e) = e$ for all $e^2 = e \in R$. Then $R[x;\sigma]/(x^{n+1})$ is left morphic for each $n \geq 0$.*

Proof Let $\alpha \in S := R[x;\sigma]/(x^{n+1})$. We show that α is left morphic in S. By Theorem 3, α is equivalent to $\gamma := e_0 + e_1x + \cdots + e_nx^n$, where

$$e_0^2 = e_0 \in R \quad \text{and} \quad e_i^2 = e_i \in (1 - e_{i-1}) \cdots (1 - e_0)R(1 - e_0) \cdots (1 - e_{i-1})$$

for $i = 1, \ldots, n$. Let $\beta = b_0 + b_1x + \cdots + b_nx^n$, where $b_i = (1 - e_0)(1 - e_1) \cdots (1 - e_{n-i})$ for $i = 0, \ldots, n$. Thus, we have

$$\begin{aligned} S\gamma &= Re_0 + R(e_0 + e_1)x + \cdots + R(e_0 + \cdots + e_n)x^n = \mathbf{l}(\beta), \\ \mathbf{l}(\gamma) &= Rb_0 + Rb_1x + \cdots + Rb_nx^n = S\beta. \end{aligned}$$

So γ is left morphic in S by [5, Lemma 1]. Hence α is left morphic in S by [5, Lemma 3]. ■

In our concluding examples, we present a unit regular ring R that is not strongly regular such that there exists an endomorpism $\sigma \neq 1_R$ with $\sigma(e) = e$ for all $e^2 = e \in R$, and also a unit regular ring R that is not strongly regular such that 1_R is the only endomorphism fixing idempotents and that there exists an endomorphism σ not equal to 1_R.

Example 4 Let $R = S \times T$ where S is a strongly regular ring that is not commutative and T is a unit regular ring that is not strongly regular. Then R is unit regular, but it is not strongly regular. Take a unit v of S that is not central and let $u = (v, 1_T)$. Then u is a unit of R. Let $\sigma \colon R \to R$ be the endomorphism given by $\sigma(r) = u^{-1}ru$. Then $\sigma \neq 1_R$, and $\sigma(e) = e$ for all $e^2 = e \in R$.

Example 5 Let $R = \mathbb{M}_2(\mathbb{Z}_2)$ be the 2×2 matrix ring over the ring of integers modulo 2. Then R is a unit regular ring that is not strongly regular. Because each element of R is either an idempotent or the sum of two idempotents or the sum of three idempotents, we see that 1_R is the only endomorphism fixing idempotents. However, $\sigma \colon R \to R$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$ is an endomorphism of R with $\sigma \neq 1_R$.

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