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ON METRIZATION OF UNIONS OF FUNCTION SPACES ON DIFFERENT INTERVALS

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Abstract

This paper investigates a class of metrics that can be introduced on the set consisting of the union of continuous functions defined on different intervals with values in a fixed metric space, where the union ranges over a family of intervals. Its definition is motivated by the Skorohod metric(s) on càdlàg functions. We show what is essential in transferring the ideas employed in the latter metric to our setting and obtain a general construction for metrics in our case. Next, we define the metric space where elements are sequences of functions from the above mentioned set. We provide conditions that ensure separability and completeness of the constructed metric spaces.

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1. Introduction

The space $C([a, b])$ of continuous functions defined on an interval $[a, b]$ is used in many branches of analysis. Apart from the uniform topology, there are various other useful topologies, metrics and norms that have been considered on $C([a, b])$ for various purposes. The definition of a topology, in particular a metric, on a set consisting of continuous functions defined on *different* intervals has so far attracted less attention, if any. Such sets—and the need for a suitable metric—appear naturally when studying stochastic perturbations of deterministic dynamical systems in metric spaces. They arise when one considers a deterministic system with continuous trajectories, that is subject to interventions at discrete points in time randomly, in which the deterministic system jumps instantaneously to a new position, also randomly; see, for instance, Lasota and Mackey [5]. We have been motivated by the modeling in population

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biology of a fish population with occasional fishing in such a fashion. Other examples are in financial mathematics [2, 4] and impulsive systems [7].

The sample trajectories of the resulting stochastic process in the metric state space S of the unperturbed dynamical system may be viewed as a *concatenation* of the continuous trajectories of the latter in between the times of interventions at which the jump take place. Thus, these continuous paths are taken from a union over $\tau > 0$ of the collection $C_s([0, \tau])$ of all continuous S-valued maps defined on the interval $[0, \tau]$. We denote this union by C_s^τ.

For application of numerous results from probability theory it is advantageous to have a metric on C_s^τ which makes it a complete and separable metric space. The main objective of this paper is to exhibit a general construction for metrics on C_s^τ that yield completeness and separability based on collections of homeomorphisms between intervals $[0, \tau]$ and $[0, \tau']$, and a *penalty function* that measures the amount of deformation that these homeomorphisms cause to the intervals. We believe that the results are of general mathematical interest.

The construction is motivated by the metric on càdlàg functions $D_s([0, 1])$ or $D_s([0, \infty))$ as exhibited in [1, 3, 6]. In fact, it turns out that the penalty function used for càdlàg functions may not be the only possible penalty function that yields a complete (separable) metric space, but it is one that has highly convenient additional properties. Our approach to the stochastic process contrasts with the common approach in probability theory of viewing the trajectories directly in the space of càdlàg functions $D_s([0, \infty))$; see Billingsley [1], Ethier and Kurtz [3] and Skorohod [6].

The structure of the paper is as follows. In Section 2 we present properties on the sets of homeomorphisms between the intervals (D1), (D2) and on the penalty function (P1)–(P4) that allow us to define a metric on C_s^τ. We show that the penalty function for [1, 3, 6] satisfies these properties (and more). In Section 3 we provide additional conditions under which the metric yields a separable or a complete metric space. It turns out that one needs a slightly stronger condition (P3′) instead of (P3) for completeness (Theorem 15). For separability, one needs separability of the spaces $C_s([0, \tau])$. Section 4 is devoted to sequences with values in C_s^τ, which relates to coding of the sample trajectories of the stochastic process that results from the perturbation and a brief discussion of the relation to càdlàg functions. The latter is the subject of further study. We conclude this introduction with an example where a space of functions on different intervals is natural.

Motivating example. Consider a time homogeneous dynamical system on a metric space S given by a continuous map $\Phi : [0, \infty) \times S \to S$. This means that a state $x \in S$ at time $t_0 \geq 0$ evolves to the state $\Phi(t - t_0, x)$ at time t. For instance, S could be $\mathbb{R}^n$ or an infinite-dimensional Banach space and $\Phi(t, x)$ could be the solution operator of a well-posed system of evolutionary ordinary or partial differential equations.

Suppose that the system is subject to random interventions at times $0 < t_1 < t_2 \cdots$, in the sense that the continuous evolution given by Φ stops at time t_k, and the system jumps instantaneously and randomly to a new position after which it starts its evolution

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from there again according to Φ. The position of the system in S at time t, represented by $Y(t)$, can be described in the case of jump sizes being independent random variables, $J_1, J_2, \ldots$, as:

$$Y(t) = Y_k(t - t_k) \quad \text{if } t \in [t_k, t_{k+1}),$$

where

$$Y_0(t) = \Phi(t, x_0) \quad \text{if } t \in [0, t_1],$$
$$Y_k(t) = \Phi(t, Y_{k-1}(t_k - t_{k-1}) + J_k) \quad \text{if } t \in [0, t_{k+1} - t_k],\ k \geq 1,$$

for some initial value $x_0 \in S$. For convenience we have put $t_0 = 0$. The full trajectories Y can thus be encoded as a sequence (Y_k) that takes values in $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$. In the case where the interventions are equidistant, that is $t_{k+1} - t_k = \tau$ for all k, each Y_k is a random element of $C_s([0, \tau])$, and $(Y_k)_{k=0}^{\infty}$ is a Markov chain in the space $C_s([0, \tau])$. In the case that the intervention times are not equidistant, we propose to view each Y_k as an element of the space $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$. Thus, some complications of continuous time Markov processes can be avoided. Such nonequidistant intervention times appear in particular when stochastic times are considered.

Recall that for application of the theory of Markov chains it is a great advantage if the state space is a separable complete metric space. In the subsequent sections we will consider various metrics on $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$ and study separability and completeness.

2. Metrics on the space C_s^τ

Let (S, d_s) be a metric space and let $\tau = (\tau_\alpha)_{\alpha \in A}$ be a family of distinct elements τ_α of $[0, \infty)$; that is, $\tau_\alpha \neq \tau_\beta$ when $\alpha \neq \beta$ and $(\tau_\alpha) \subset [0, \infty)$. Put $C_s^\tau := \bigcup_\alpha C_s([0, \tau_\alpha])$, where

$$C_s([0, \tau_\alpha]) := \{f : [0, \tau_\alpha] \to S :\ f \text{ continuous}\}.$$

We will define a class of metrics on C_s^τ. For each pair (α, β), let there be a nonempty collection, $\Delta_{\alpha,\beta}$, of maps λ:

$$\Delta_{\alpha,\beta} \subseteq \{\lambda : [0, \tau_\alpha] \to [0, \tau_\beta], \lambda \text{ strictly increasing continuous}; \lambda(0) = 0, \lambda(\tau_\alpha) = \tau_\beta\}.$$

Note that each λ is invertible. Assume that these collections satisfy the following properties:

(D1) $\Delta_{\beta,\alpha} = \Delta_{\alpha,\beta}^{-1} := \{\lambda^{-1} :\ \lambda \in \Delta_{\alpha,\beta}\}$;
(D2) $\Delta_{\beta,\gamma} \circ \Delta_{\alpha,\beta} := \{\lambda_1 \circ \lambda_2 :\ \lambda_1 \in \Delta_{\beta,\gamma},\ \lambda_2 \in \Delta_{\alpha,\beta}\} \subset \Delta_{\alpha,\gamma}$.

Note that (D1) and (D2) imply $I \in \Delta_{\alpha,\alpha}$, where I is the identity map. Moreover, any $\lambda \in \Delta_{\alpha,\beta}$ must be a homeomorphism. Put $\Delta := \bigcup_{\alpha,\beta} \Delta_{\alpha,\beta}$.

Assume that there is a *penalty function* $P : \Delta \to [0, \infty)$ that satisfies the following properties:

(P1) $P(I) = 0$.
(P2) $P(\lambda^{-1}) = P(\lambda)$.

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(P3) $P(\lambda_1 \circ \lambda_2) \le P(\lambda_1) + P(\lambda_2)$.
(P4) If $\lambda_n \in \Delta_{\alpha,\beta}$ and $P(\lambda_n) \to 0$ as $n \to \infty$, then $\alpha = \beta$ and $\lambda_n \to I$ uniformly on $[0, \tau_\alpha]$.

The function $P(\lambda)$ 'penalizes' the stretching of an interval by the map λ, as illustrated by the following good candidates for such Δ and P (as will be proved below).

Example 1. $\Delta_{\alpha,\beta}$ consists of all strictly increasing, continuous mappings of $[0, \tau_\alpha]$ onto $[0, \tau_\beta]$,

$$P(\lambda) = \|\lambda - I\| = \sup_{t\in[0,\tau_\alpha]} |\lambda(t) - t|, \qquad \lambda \in \Delta_{\alpha,\beta}. \tag{1}$$

Example 2. $\Delta_{\alpha,\beta}$ consists of all increasing lipeomorphisms[1],

$$P(\lambda) = \sup_{0\le t<s\le\tau_\alpha} \left|\log \frac{\lambda(s) - \lambda(t)}{s - t}\right|, \qquad \lambda \in \Delta_{\alpha,\beta}. \tag{2}$$

We proceed to define a metric on C_s^τ making use of the data (Δ, P). Given a family $\Delta_{\alpha,\beta}$ and P as above satisfying (D1)–(D2) and (P1)–(P4), we define a function $\hat{d}$ on $C_s^\tau \times C_s^\tau$ as follows. For $f, g \in C_s^\tau$ with $f \in C_s([0, \tau_\alpha])$, $g \in C_s([0, \tau_\beta])$, let

$$d_\lambda(f, g) := \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g \circ \lambda(t)),$$
$$\hat{d}(f, g) := \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f, g) + P(\lambda)\}.$$

Remark 3. If $f, g \in C_s([0, \tau_\alpha])$ then

$$\hat{d}(f, g) = \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f, g) + P(\lambda)\} \le \{d_{\lambda=I}(f, g) + P(I)\} = \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)),$$

according to the assumptions $I \in \Delta_{\alpha,\alpha}$ and (P1).

Proposition 4. *If* $\Delta_{\alpha,\beta}$ *and* $P : \Delta \to [0, \infty)$ *satisfy* (D1)–(D2) *and* (P1)–(P4) *respectively, then* $\hat{d}$ *is a metric on* C_s^τ.

Proof. For $f, g \in C_s^\tau$ with $f \in C_s([0, \tau_\alpha])$, $g \in C_s([0, \tau_\beta])$, let $\hat{d}(f, g) = 0$. Then there exist $\lambda_n : [0, \tau_\alpha] \to [0, \tau_\beta] \in \Delta_{\alpha,\beta}$ such that $\lim_{n\to\infty}\{d_{\lambda_n}(f, g) + P(\lambda_n)\} = 0$. Consequently, $P(\lambda_n) \to 0$, $d_{\lambda_n}(f, g) \to 0$. Now, $P(\lambda_n) \to 0$ gives $\tau_\alpha = \tau_\beta$ and $\lambda_n \to I$ uniformly on $[0, \tau_\alpha]$ according to (P4). Therefore, $\lim_{n\to\infty} d_{\lambda_n}(f, g) = 0$ gives

$$\lim_{n\to\infty} \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g \circ \lambda_n(t)) = 0$$

and this yields

$$\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g \circ I(t)) = 0.$$

[1] A function is called a *lipeomorphism* if it is Lipschitz, bijective and has a Lipschitz inverse. Also, an increasing lipeomorphism should be strictly increasing.

Thus, $f = g$. For the converse, if $f = g$ and $f, g : [0, \tau_\alpha] \to S$, then, as mentioned above, $\hat{d}(f, g) \le \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t))$ and having d_s a metric $\hat{d}(f, g) = 0$. For symmetry,

$$d_\lambda(f, g) = \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g \circ \lambda(t)) = \sup_{t\in[0,\tau_\beta]} d_s(f(\lambda^{-1}(t)), g(t)) = d_{\lambda^{-1}}(g, f).$$

By assumption (D1) and (P2),

$$\begin{aligned}\hat{d}(f, g) &:= \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f, g) + P(\lambda)\}\\ &= \inf_{\lambda^{-1}\in\Delta^{-1}_{\alpha,\beta}=\Delta_{\beta,\alpha}} \{d_{\lambda^{-1}}(g, f) + P(\lambda^{-1})\} = \hat{d}(g, f).\end{aligned}$$

For the triangle inequality, let f, g be as stated above and let $h \in C_s([0, \tau_\gamma])$. Then

$$\begin{aligned}d_{\lambda_1}(f, g) + d_{\lambda_2}(g, h) &= \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(\lambda_1(t))) + \sup_{t\in[0,\tau_\beta]} d_s(g(t), h(\lambda_2(t)))\\ &\ge \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(\lambda_1(t))) + \sup_{s\in[0,\tau_\alpha]} d_s(g(\lambda_1(t)), h(\lambda_2\lambda_1(t)))\\ &\ge \sup_{t\in[0,\tau_\alpha]} d_s(f(t), h(\lambda_2\lambda_1(t))) = d_{\lambda_2\lambda_1}(f, h).\end{aligned}$$

Note that $\lambda_2\lambda_1 \in \Delta_{\beta,\gamma}\Delta_{\alpha,\beta}$ and, by assumption (D2), $\lambda_2\lambda_1 \in \Delta_{\alpha,\gamma}$. Therefore,

$$\begin{aligned}\hat{d}(f, g) + \hat{d}(g, h) &= \inf_{\lambda_1\in\Delta_{\alpha,\beta}} \{d_{\lambda_1}(f, g) + P(\lambda_1)\} + \inf_{\lambda_2\in\Delta_{\beta,\gamma}} \{d_{\lambda_2}(g, h) + P(\lambda_2)\}\\ &= \inf_{\{\lambda_1\in\Delta_{\alpha,\beta},\lambda_2\in\Delta_{\beta,\gamma}\}} \{d_{\lambda_1}(f, g) + d_{\lambda_2}(g, h) + P(\lambda_1) + P(\lambda_2)\}\\ &\ge \inf_{\{\lambda_1\in\Delta_{\alpha,\beta},\lambda_2\in\Delta_{\beta,\gamma}\}} \{d_{\lambda_2\lambda_1}(f, h) + P(\lambda_2\lambda_1)\}\\ &\ge \inf_{\lambda_2\lambda_1\in\Delta_{\alpha,\gamma}} \{d_{\lambda_2\lambda_1}(f, h) + P(\lambda_2\lambda_1)\}\\ &\ge \inf_{\lambda\in\Delta_{\alpha,\gamma}} \{d_\lambda(f, h) + P(\lambda)\} = \hat{d}(f, h).\end{aligned}$$

□

We introduce the following condition:

(P4′) If $\lambda_n \in \Delta_{\alpha,\beta_n}$ and $P(\lambda_n) \to 0$ as $n \to \infty$, then $\tau_{\beta_n} \to \tau_\alpha$ as $n \to \infty$ and $\lambda_n \to I$ uniformly on $[0, \tau_\alpha]$.

Note that (P4′) implies (P4). With (P4′) one can clarify the concept of convergence under the metric $\hat{d}$. Indeed, if it is given that $(f_n) \in C_s([0, \tau_{\alpha_n}])$ and $f \in C_s([0, \tau_\alpha])$ such that $f_n \to f$ under the metric $\hat{d}$ then we conclude that there exist $\lambda_n \in \Delta_{\alpha_n,\alpha}$ such that

$$d_{\lambda_n}(f_n, f) = \sup_{t\in[0,\tau_\alpha]} d_s(f_n(\lambda_n^{-1}(t)), f(t)) \to 0,$$

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that is, $f_n \circ \lambda_n^{-1} \to f$ uniformly on the interval on which f is defined, $[0, \tau_\alpha]$, under the metric d_s. Moreover, $P(\lambda_n) \to 0$ which, by (P2) and (P4′), implies that

$$\tau_{\alpha_n} \to \tau_\alpha.$$

Conversely, if we have $(f_n) \in C_s([0, \tau_{\alpha_n}])$ and we were to prove that $\hat{d}(f_n, f) \to 0$ then we should find $f \in C_s([0, \tau_\alpha])$ and functions $\lambda_n \in \Delta_{\alpha_n,\alpha}$ (we should only consider $\lambda_n : [0, \tau_{\alpha_n}] \to [0, \tau_\alpha]$, or $\lambda_n^{-1} : [0, \tau_\alpha] \to [0, \tau_{\alpha_n}]$, that is, either the domain or the range of λ_n is independent of n) such that $P(\lambda_n) \to 0$ and

$$\sup_{t\in[0,\tau_{\alpha_n}]} d_s(f_n(t), f \circ \lambda_n(t)) = \sup_{t\in[0,\tau_\alpha]} d_s(f_n \circ \lambda_n^{-1}(t), f(t)) \to 0.$$

We will establish more properties of the penalty function (2):

Proposition 5. *$P(\lambda)$ defined by (2) satisfies properties* (P1)–(P3) *mentioned above. Moreover, if $\lambda_n \in \Delta_{\alpha_n,\beta_n}$ and $P(\lambda_n) \to 0$ as $n \to \infty$, then $\tau_{\beta_n}/\tau_{\alpha_n} \to 1$ as $n \to \infty$ and*

$$\frac{1}{\tau_{\alpha_n}} \sup_{s\in[0,\tau_{\alpha_n}]} |\lambda_n(s) - s| \to 0.$$

As a special case, if (τ_{α_n}) is bounded, then $\tau_{\beta_n} - \tau_{\alpha_n} \to 0$ as $n \to \infty$. Consequently, $P(\lambda)$ satisfies (P4′) *and, therefore,* (P4).

Proof. For (P1) and (P2) the proof is obvious. For (P3), let $\lambda_2 : [0, \tau_\alpha] \to [0, \tau_\beta]$ and $\lambda_1 : [0, \tau_\beta] \to [0, \tau_\gamma]$. Then (abbreviating $\lambda_1 \circ \lambda_2$ to $\lambda_1\lambda_2$)

$$\begin{aligned}\left|\log \frac{\lambda_1\lambda_2(s) - \lambda_1\lambda_2(t)}{s - t}\right| &= \left|\log \frac{\lambda_1\lambda_2(s) - \lambda_1\lambda_2(t)}{\lambda_2(s) - \lambda_2(t)} \cdot \frac{\lambda_2(s) - \lambda_2(t)}{s - t}\right| \\ &\leq \left|\log \frac{\lambda_1\lambda_2(s) - \lambda_1\lambda_2(t)}{\lambda_2(s) - \lambda_2(t)}\right| + \left|\log \frac{\lambda_2(s) - \lambda_2(t)}{s - t}\right| \\ &= \left|\log \frac{\lambda_1(w) - \lambda_1(v)}{w - v}\right| + \left|\log \frac{\lambda_2(s) - \lambda_2(t)}{s - t}\right|.\end{aligned}$$

Taking the supremum on the interval $[0, \tau_\alpha]$ for both sides of the inequality, we get (P3). Furthermore, let $\lambda_n \in \Delta_{\alpha_n,\beta_n}$ be such that $P(\lambda_n) \to 0$ as $n \to \infty$. Since

$$\sup_{0\leq t<s\leq\tau_{\alpha_n}} \left|\log \frac{\lambda_n(s) - \lambda_n(t)}{s - t}\right| \to 0,$$

with $t = 0$ and $s = \tau_{\alpha_n}$ we obtain $|\log \tau_{\beta_n}/\tau_{\alpha_n}| \to 0$. Hence, $\tau_{\beta_n}/\tau_{\alpha_n} \to 1$ as $n \to \infty$. Given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that

$$\max\{(\mathrm{e}^{\epsilon_1} - 1), (1 - \mathrm{e}^{-\epsilon_1})\} < \epsilon.$$

There exists N such that $|\log (\lambda_n(s) - \lambda_n(t))/(s - t)| < \epsilon_1$ for all $\tau_{\alpha_n} \geq s > t \geq 0$ and for all $n \geq N$. Thus, at $t = 0$, $\mathrm{e}^{-\epsilon_1} s < \lambda_n(s) < \mathrm{e}^{\epsilon_1} s$ for all $s > 0$ and for all $n \geq N$. Hence, $-(1 - \mathrm{e}^{-\epsilon_1})s < \lambda_n(s) - s < (\mathrm{e}^{\epsilon_1} - 1)s$ for all $s > 0$ and for all $n \geq N$. Hence,

$$\frac{1}{\tau_{\alpha_n}} \sup_{s\in[0,\tau_{\alpha_n}]} |\lambda_n(s) - s| \leq \max\{(\mathrm{e}^{\epsilon_1} - 1), (1 - \mathrm{e}^{-\epsilon_1})\} < \epsilon$$

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for all $n \geq N$. Thus, if (τ_{α_n}) is bounded, then

$$\tau_{\beta_n} - \tau_{\alpha_n} = \left(\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} - 1\right)\tau_{\alpha_n} \to 0$$

as $n \to \infty$. The proof that $P(\lambda)$ satisfies (P4′) follows immediately. □

Remark 6. The definitions (1) and (2) for a penalty function may be used for nondecreasing functions λ, not necessarily continuous, when allowing $+\infty$ as a value of the penalty function. In case (2), let λ be a nondecreasing function that maps $[0, \tau_\alpha]$ to $[0, \tau_\beta]$ satisfying $\lambda(0) = 0$ and $\lambda(\tau_\alpha) = \tau_\beta$. If $P(\lambda) < \infty$, then λ is strictly increasing and Lipschitz continuous. The inverse satisfies $P(\lambda^{-1}) = P(\lambda) < \infty$. Thus, λ^{-1} is also Lipschitz. Moreover, $P(\lambda) < \infty$ if and only if λ is an increasing lipeomorphism. Conversely, there exist strictly increasing continuous functions λ for which $P(\lambda) = \infty$.

Proof. If $P(\lambda) < \infty$, then the slopes $(\lambda(s) - \lambda(t))/(s - t)$ should satisfy $0 < (\lambda(s) - \lambda(t))/(s - t) < \infty$ for $s > t$. Therefore, λ is strictly increasing. Now to show that λ is a lipeomorphism when $P(\lambda)$ is finite, let $P(\lambda) = c < \infty$, then $|\log (\lambda(s) - \lambda(t))/(s - t)| \leq c$ for all $\tau_\alpha \geq s > t \geq 0$, so, $\mathrm{e}^{-\mathrm{c}}(s - t) \leq \lambda(s) - \lambda(t) \leq \mathrm{e}^{\mathrm{c}}(s - t)$. Thus, $|\lambda(s) - \lambda(t)| \leq \max\{\mathrm{e}^{-\mathrm{c}}, \mathrm{e}^{\mathrm{c}}\}|s - t| = L|s - t|$, for all $\tau_\alpha \geq s > t \geq 0$, where $L = \max\{\mathrm{e}^{-\mathrm{c}}, \mathrm{e}^{\mathrm{c}}\}$. So, λ is Lipschitz. By (P2) and using the same argument as above, λ^{-1} is Lipschitz. Therefore, λ is a lipeomorphism and hence is a member of $\Delta_{\alpha,\beta}$. Conversely, if λ is a lipeomorphism, then $|\lambda(s) - \lambda(t)| \leq L|s - t|$, and $|\lambda^{-1}(s) - \lambda^{-1}(t)| \leq \ell|s - t|$ with $\ell, L \geq 1$. Consequently, $0 < 1/\ell \leq |(\lambda(s) - \lambda(t))/(s - t)| \leq L$ for $s \neq t$. Hence, $|\log (\lambda(s) - \lambda(t))/(s - t)| \leq c$ for all $\tau_\alpha \geq s > t \geq 0$, where $c = \max(-\log \ell, \log L)$. Thus, $P(\lambda) \leq c < \infty$. □

Remark 7. The penalty function introduced in (2) satisfies, for $\lambda \in \Delta_{\alpha,\beta}$,

$$P(\lambda) = \sup_{0 \leq t < s \leq \tau_\alpha} \left|\log \frac{\lambda(s) - \lambda(t)}{s - t}\right| = \operatorname{ess} \sup_{t \in [0, \tau_\alpha]} |\log \lambda'(t)|$$

(see [3, p. 117]). This function is used in [3] in the definition of the Skorohod metric on càdlàg functions.

The following example shows a function P such that $\hat{d}$ is not a metric—the definiteness property fails—but it is a semimetric.

Example 8. Let $P(\lambda) = \log(|\lambda|_\ell |\lambda^{-1}|_\ell)$, where $|\lambda|_\ell$ denotes the Lipschitz constant of a Lipschitz λ, with $\Delta_{\alpha,\beta} := \{\lambda{:}[0, \tau_\alpha] \to [0, \tau_\beta] : \lambda$ an increasing lipeomorphism$\}$. Taking $f(t) = t,\ t \in [0, 1],\ g(t) = 2t,\ t \in [0, \frac{1}{2}]$ gives that $\hat{d}(f, g) = 0$ but $f \neq g$.

In a special case one obtain does a metric space with this P.

Example 9. For a fixed τ the space $(C_s([0, \tau]), \hat{d})$ with $P(\lambda) = \log(|\lambda|_\ell \mid \lambda^{-1}|_\ell)$ and $\Delta_{\alpha,\beta} := \{\lambda{:}[0, \tau] \to [0, \tau] : \lambda$ an increasing lipeomorphism$\}$ is a metric space. Here we need to show that P satisfies (P4). For this purpose we need the following lemma.

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LEMMA 10. *Let* $\lambda_n : [0, \tau] \to [0, \tau]$ *be strictly increasing lipeomorphisms such that* $|\lambda_n|_\ell |\lambda_n^{-1}|_\ell \to 1$. *Then:*

(1) $|\lambda_n|_\ell \to 1$ and $|\lambda_n^{-1}|_\ell \to 1$*;*

(2) $\lambda_n(t) \to t$, for all t (pointwise convergence)*;*

(3) $\lambda_n \to I$ uniformly ($\|\lambda_n - I\| \to 0$).

PROOF. We have $|\lambda_n(\tau) - \lambda_n(0)| \le |\lambda_n|_\ell |\tau - 0|$, but having $\lambda_n(0) = 0, \lambda_n(\tau) = \tau$ gives $|\lambda_n(\tau) - \lambda_n(0)| = |\tau - 0|$. So, $|\lambda_n|_\ell \ge 1$. λ_n is increasing, therefore λ_n^{-1} is increasing and hence $|\lambda_n^{-1}|_\ell \ge 1$ as well. Together with $|\lambda_n|_\ell |\lambda_n^{-1}|_\ell \to 1$, we get $|\lambda_n|_\ell \to 1$ and $|\lambda_n^{-1}|_\ell \to 1$. For the second point of the lemma we need to show that $\lim_{n\to\infty} \lambda_n(t) = t$ for all t. We will do this by contradiction. So, assume that there exists t_0 such that $\lambda_n(t_0)$ does not converge to t_0. Then $t_0 \ne 0$ and $t_0 \ne \tau$. Moreover, there exist $\epsilon > 0$ and a subsequence (n_k) such that either $\lambda_{n_k}(t_0) \le t_0 - \epsilon$ or $\lambda_{n_k}(t_0) \le t_0 + \epsilon$ for all k. In the first case, $|\lambda_n(t_0) - \lambda_n(\tau)| \le |\lambda_n|_\ell |t_0 - \tau|$ or $|\lambda_n|_\ell \ge |\lambda_n(t_0) - \lambda_n(\tau)|/|t_0 - \tau|$:

$$|\lambda_{n_k}|_\ell \ge \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(\tau)|}{|t_0 - \tau|} = \frac{\tau - \lambda_{n_k}(t_0)}{\tau - t_0} \ge \frac{\tau - t_0 + \epsilon}{\tau - t_0} = 1 + \frac{\epsilon}{\tau - t_0} > 1.$$

In the second case,

$$|\lambda_{n_k}|_\ell \ge \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(0)|}{|t_0 - 0|} = \frac{\lambda_{n_k}(t_0)}{t_0} \ge \frac{t_0 + \epsilon}{t_0} = 1 + \frac{\epsilon}{t_0} > 1.$$

This contradicts $|\lambda_n|_\ell \to 1$ as $n \to \infty$. For the last part of the lemma, let $\epsilon > 0$. From the pointwise convergence, there exist N_x such that $|\lambda_n(x) - x| < \epsilon/3$ for all $n \ge N_x$. Let $M = \sup_n |\lambda_n|_\ell$. Then $M \ge 1$. Now if $|x - y| < \epsilon/3M$ then

$$\begin{aligned}
|\lambda_n(y) - y| &\le |\lambda_n(y) - \lambda_n(x)| + |\lambda_n(x) - x| + |x - y| \\
&\le |\lambda_n|_\ell |y - x| + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \\
&< \frac{\epsilon|\lambda_n|_\ell}{3M} + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \quad \left(\text{note that } \frac{|\lambda_n|_\ell}{M} \le 1, \text{ also } \frac{1}{M} \le 1\right) \\
&\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Since $[0, \tau]$ is compact, there exist finitely many $x_i \in [0, \tau]$ such that $[0, \tau]$ is covered by open intervals $(x_i - (\epsilon/3M), x_i + (\epsilon/3M))$. Put $N = \max_i(N_{x_i})$; then

$$\sup_{y\in[0,\tau]} |\lambda_n(y) - y| < \epsilon$$

for all $n \ge N$. That is, $\|\lambda_n - I\| \to 0$. □

3. Separability and completeness of the space C_s^τ

The main result on separability is given by the following proposition.

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Proposition 11. *Suppose that there exists a countable subset*

$$Q \subseteq \{\tau_\alpha : \tau_\alpha \in \mathbb{R}_+, \alpha \in \Lambda\}$$

such that for every τ_α there exist $\tau_n \in Q$, and there exist $\lambda_n \in \Delta_{\alpha,n}$, such that $P(\lambda_n) \to 0$. If $C_s([0,\tau_\alpha])$ is separable for all α, then the space $C_s^\tau := \bigcup_\alpha C_s([0,\tau_\alpha])$ equipped with the metric $\hat{d}$ is separable.

Proof. For each α, let $M[0,\tau_\alpha]$ be a countable dense subset of $C_s([0,\tau_\alpha])$. We claim that the set $M = \bigcup_{\tau_\alpha \in Q} M[0,\tau_\alpha]$, is a countable dense subset in C_s^τ. Obviously, taking the disjoint union over $\tau_\alpha \in Q$ makes M countable. Now, to show that the defined set M is dense we only need to show that if $f \in C_s[0,\tau_\alpha]$, $\tau_\alpha \notin Q$, then f can be approximated by functions from M (the other case, $f \in C_s[0,\tau_\alpha]$, $\tau_\alpha \in Q$, is obvious). So, let $f \in C_s([0,\tau_\alpha])$, $\tau_\alpha \notin Q$, and take $\tau_n \in Q$ and $\lambda_n : [0,\tau_\alpha] \to [0,\tau_n]$ such that $P(\lambda_n) \to 0$. Define $f_n \in C_s([0,\tau_n])$ as $f_n = f \circ \lambda_n^{-1}$. Then

$$f_n(\lambda_n(t)) = f(\lambda_n^{-1}(\lambda_n(t))) = f(t).$$

Therefore, $d_{\lambda_n^{-1}}(f_n, f) = 0$. Thus,

$$\hat{d}(f_n, f) \le d_{\lambda_n^{-1}}(f_n, f) + P(\lambda_n) = P(\lambda_n) \to 0$$

as $n \to \infty$. Let $g_n \in M[0,\tau_n]$ such that $\hat{d}(g_n, f_n) < 1/n$. Then

$$\hat{d}(g_n, f) \le \hat{d}(g_n, f_n) + \hat{d}(f_n, f) \le \frac{1}{n} + P(\lambda_n) \to 0. \qquad \square$$

Remark 12. The penalty functions in Examples 1 and 2 satisfy the required condition in the previous proposition when $\{\tau_\alpha : \alpha \in \Lambda\} = (0,\infty)$.

Proof. Choose $Q = \mathbb{Q} \cap (0,\infty)$. For $\tau_\alpha \in (0,\infty)$ choose τ_n in Q such that $\tau_n \to \tau_\alpha$. Define $\lambda_n(t) := (\tau_n/\tau_\alpha)t$. Clearly, $\lambda_n \in \Delta_{\alpha,n}$. Moreover, in the first case,

$$P(\lambda_n) = \|\lambda_n - I\| = \sup_{t\in[0,\tau_\alpha]} |\lambda_n(t) - t| = \sup_{t\in[0,\tau_\alpha]} \left|\frac{\tau_n}{\tau_\alpha} - 1\right| t \le \left|\frac{\tau_n}{\tau_\alpha} - 1\right| \tau_\alpha \to 0.$$

In the second case,

$$P(\lambda_n) = \sup_{\tau_\alpha \ge s > t \ge 0} \left|\log \frac{\lambda_n(s) - \lambda_n(t)}{s - t}\right| = \sup_{\tau_\alpha \ge s > t \ge 0} \left|\log \frac{\tau_n}{\tau_\alpha}\right| \to 0. \qquad \square$$

Whether the spaces $C_s([0,\tau_\alpha])$ are separable depends on S. Separability of S is not sufficient; pathwise connectedness conditions seem to be required. The following particular result suffices in most practical settings.

Proposition 13. *Let X be a normed space. Let S be a convex subset of X which is separable in the relative topology. Then the space $C_s([0,\tau_\alpha])$ equipped with the uniform metric is separable. In particular, $(C_s([0,\tau_\alpha]), \hat{d})$ is separable for any α.*

Proof. Let E be a countable dense subset of S and let $\xi = \{f : [0, \tau_\alpha] \to S\}$ be such that there exist $0 = t_1 < t_2 < \cdots < t_n = \tau_\alpha$, $t_i \in \mathbb{Q}$, $f(t_1), f(t_2), \ldots, f(t_n) \in E$ satisfying

$$f(\lambda t_i + (1 - \lambda)t_{i+1}) = \lambda f(t_i) + (1 - \lambda)f(t_{i+1})$$

for all $1 \le i < n$ and $\lambda \in [0, 1]$. Countability of ξ is obvious. We need to show that ξ is dense in $C_s([0, \tau_\alpha])$. For this purpose let $f \in C_s([0, \tau_\alpha])$. Given $\epsilon > 0$, choose $\delta > 0$ such that $d_s(f(s), f(t)) < \epsilon$ whenever $|s - t| < \delta$. Choose $0 = t_1 < t_2 < \cdots < t_n = \tau_\alpha$ in $[0, \tau_\alpha] \cap \mathbb{Q}$ such that $t_{i+1} - t_i < \delta$. Choose $e_1, e_2, \ldots, e_n \in E$ such that $d_s(f(t_i), e_i) < \epsilon$. Construct $g \in \xi$ such that $g(t_i) = e_i$. One can check that $\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)) \le 5\epsilon$; then Remark 3 yields $\hat{d}(f, g) < 5\epsilon$, and this ends the proof. □

Establishing the property of completeness requires more work. In particular, we need a condition stronger than (P3). For the completeness of the general metric space $(C_s^\tau, \hat{d})$ we will need the following condition (P3′) which will replace (P3).

(P3′) If $\mu_m \in \Delta_{m,m+1}$ for $m = 1, 2, \ldots$ are such that $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, then the sequence $(\mu^{(m)})_{m=1}^{\infty}$ defined by $\mu^{(m)} := \mu_m \circ \mu_{m-1} \circ \cdots \circ \mu_1$ converges uniformly to some $\lambda \in \Delta$ and $P(\lambda) \le \sum_{m=1}^{\infty} P(\mu_m)$.

Note that (P3′) implies (P3).

Let $(\mu_n^{(m)})_{m=1}^{\infty}$ denote the sequence in Δ defined by $\mu_n^{(m)} := \mu_{n+m} \circ \cdots \circ \mu_{n+1} \circ \mu_n$.

Remark 14. Suppose that (Δ, P) satisfy (P3′) and (μ_n) is a sequence satisfying the conditions of (P3′). Define

$$\lambda_n := \lim_{n\to\infty} \mu_n^{(m)} = \lim_{m\to\infty} \mu_{n+m} \ldots \mu_{n+1}\mu_n.$$

Then $P(\lambda_n) \to 0$ as $n \to \infty$. Moreover, there exists τ_β; $0 < \tau_\beta < \infty$ such that $\lambda_n \in \Delta_{n,\beta}$ and $\tau_n \to \tau_\beta$. Hence,

$$\lambda_n : [0, \tau_n] \to \Big[0, \lim_{m\to\infty} \tau_{n+m+1}\Big] = [0, \tau_\beta]$$

and $\tau_{n+m+1} \le M$ for some $M \in \mathbb{R}$.

Proof. By (P3′), $P(\lambda_n) \le \sum_{m=n}^{\infty} P(\mu_m)$. Thus, $P(\lambda_n) \to 0$ as $n \to \infty$. Also the other part follows from (μ_n) satisfying (P3′), which in turn yields that λ_n is the uniform limit of $(\mu_n^{(m)})$ and $\lambda_n \in \Delta$. That is, there exist τ_{β_n}; $0 < \tau_{\beta_n} < \infty$ such that $\lambda_n \in \Delta_{n,\beta_n}$. Then, since $\tau_n = \mu_{n-1} \ldots \mu_1(\tau_1)$,

$$\lambda_n(\tau_n) = \lim_{m\to\infty} \mu_{n+m} \ldots \mu_{n+1}\mu_n(\tau_n) = \lim_{m\to\infty} \mu_{n+m} \ldots \mu_1(\tau_1) = \lambda_1(\tau_1),$$

for each n, so $\tau_{\beta_n} = \tau_\beta := \lambda_1(\tau_1)$ for all n. Further, $\tau_n \to \tau_\beta$. Also

$$\lim_{m\to\infty} \tau_{n+m+1} = \lambda_1(\tau_1) = \tau_\beta.$$

Thus, for fixed n, $(\tau_{n+m+1})_m$ converges, so it is bounded. □

Theorem 15. *Let* (Δ, P) *satisfy conditions* (D1), (D2), (P1), (P2), (P3′) *and* (P4). *If* (S, d_s) *is a complete metric space, then the space* $(C_s^\tau, \hat{d})$ *is complete.*

Proof. The proof is inspired by Billingsley [1]. Let (f_k) be a Cauchy sequence in $C_s^\tau = \bigcup_\alpha C_s([0, \tau_\alpha])$ under the metric $\hat{d}$ associated to (Δ, P). It contains a subsequence $(g_n) = (f_{k_n})$ such that $g_n \in C_s([0, \tau_n])$ and $\hat{d}(g_n, g_{n+1}) < 1/2^n$. We shall prove that (g_n) is convergent. Now, by $\hat{d}(g_n, g_{n+1}) < 1/2^n$, Δ contains $\mu_n : [0, \tau_n] \to [0, \tau_{n+1}]$ such that $d_{\mu_n}(g_n, g_{n+1}) + P(\mu_n) < 1/2^n$. This implies that

$$d_{\mu_n}(g_n, g_{n+1}) = \sup_{t\in[0,\tau_n]} d_s(g_n(t), g_{n+1}(\mu_n t)) < \frac{1}{2^n} \tag{3}$$

and

$$P(\mu_n) < \frac{1}{2^n}.$$

We need to find a function g in $C_s([0, \tau_\beta])$, $\tau_\beta \neq \infty$, and functions λ_n in $\Delta_{n,\beta}$ for which $P(\lambda_n) \to 0$ and $g_n(\lambda_n^{-1}(t)) \to g(t)$ uniformly in t with respect to the metric d_s. Since $P(\mu_n) < 1/2^n$, $\sum_{n=1}^\infty P(\mu_n) < \infty$. Thus, by (P3′), $\lambda_n = \lim_{m\to\infty} \mu_n^{(m)}$ exists for each n, $\lambda_n \in \Delta$ and $P(\lambda_n) \leq \sum_{m=n}^\infty P(\mu_m) \to 0$ as $n \to \infty$. Since

$$\lambda_n = \lim_{m\to\infty} \mu_{n+m} \ldots \mu_{n+1}\mu_n,$$

it follows that

$$\lambda_n = \lambda_{n+1} \circ \mu_n. \tag{4}$$

Therefore, λ_n and λ_{n+1} must have the same range. Since $\lambda_n, \lambda_{n+1} \in \Delta$, $\lambda_n \in \Delta_{n,\beta}$ for some fixed τ_β. This means that the range of $\lambda_n : [0, \tau_n] \to [0, \tau_\beta]$ is independent of n, and this will make it possible to evaluate the supremum below. Now (4) is equivalent to $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. Therefore, by (3),

$$\sup_{t\in[0,\tau_\beta]} d_s(g_n(\lambda_n^{-1} t), g_{n+1}(\lambda_{n+1}^{-1} t)) = \sup_{\omega\in[0,\tau_n]} d_s(g_n(\omega), g_{n+1}(\mu_n \omega)) < \frac{1}{2^n}.$$

It follows that the functions $g_n(\lambda_n^{-1} t)$, which are elements of $C_s([0, \tau_\beta])$, are uniformly Cauchy and hence converge uniformly to a limit function $g(t)$. Finally, since

$$\sup_{t\in[0,\tau_n]} d_s(g_n(t), g(\lambda_n t)) \to 0$$

and $P(\lambda_n) \to 0$, we have $\hat{d}(g_n, g) \to 0$. □

Note that the proof of Theorem 15 will not work for the penalty function (1). This is because $\sup_t |\lambda_n(t) - t| \leq 1/2^{n-1}$ does not imply that λ_n is strictly increasing and therefore we cannot assume that the limit function $\lambda_n = \lim_{m\to\infty} \mu_{n+m} \ldots \mu_{n+1}\mu_n$ will still be invertible. The penalty function given by (2), however, is particularly nice. It satisfies (P3′). Hence it yields a complete metric space when S is complete, for example. To prove this we need a lemma.

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LEMMA 16. *Let $P(\lambda)$ be the penalty function (2). If $\lambda \in \Delta$, then*

$$\sup_{t\in[0,\tau_\alpha]} |\lambda(t) - t| \le \tau_\alpha(\mathrm{e}^{P(\lambda)} - 1).$$

PROOF. Since $|x - 1| \le \mathrm{e}^{|\log x|} - 1$ for $x > 0$,

$$\begin{aligned}|\lambda(t) - t| &= t\left|\frac{\lambda(t) - \lambda(0)}{t - 0} - 1\right| \le t(\mathrm{e}^{|\log(\lambda(t)-\lambda(0))/(t-0)|} - 1)\\ &\le t(\mathrm{e}^{\sup_{0\le s<r\le\tau_\alpha}|\log(\lambda(r)-\lambda(s))/(r-s)|} - 1)\\ &= t(\mathrm{e}^{P(\lambda)} - 1).\end{aligned}$$

Taking the supremum over the interval $[0, \tau_\alpha]$, we get the required result. □

PROPOSITION 17. *Let $P(\lambda)$ be the penalty function (2). Let*

$$\mu_m : [0, \tau_m] \to [0, \tau_{m+1}] \in \Delta_{m,m+1}$$

be such that $\sum_{m=1}^{\infty} P(\mu_m) < \infty$. Then the following results hold.

(1) *There exist m, M, $0 < m < M$, such that $m \le \tau_n \le M$ for all n.*

(2) *For each n, the sequence $(\mu_n^{(m)})_{m=1}^{\infty}$ in Δ defined by*

$$\mu_n^{(m)} := \mu_{n+m} \circ \cdots \circ \mu_{n+1} \circ \mu_n$$

is Cauchy in the uniform metric.

(3) *The uniform limit $\lambda_n := \lim_{m\to\infty} \mu_n^{(m)}$ is in Δ, $P(\lambda_n) \le \sum_{m=0}^{\infty} P(\mu_{n+m})$ and consequently $P(\lambda_n) \to 0$ as $n \to \infty$.*

In particular, (Δ, P) *satisfies* (P3′).

PROOF. In order to prove that there exist m, M, $0 < m < M$, such that $0 < m \le \tau_n \le M$, we consider

$$P(\mu_n) = \sup_{0\le t<s\le\tau_n}\left|\log\frac{\mu_n(s) - \mu_n(t)}{s - t}\right|$$

and, with $t = 0$, $s = \tau_n$,

$$\left|\log\frac{\mu_n(\tau_n) - \mu_n(0)}{\tau_n - 0}\right| = |\log(\tau_{n+1}) - \log(\tau_n)| \le P(\mu_n).$$

Since $P(\mu_n)$ is summable, $(\log(\tau_n))$ is Cauchy. Hence, $(\log(\tau_n))$ converges and it is bounded. That is, $|\log\tau_n| \le L$, for some $L \in \mathbb{R}$. Thus, $\mathrm{e}^{-L} \le \tau_n \le \mathrm{e}^{L}$. Or $0 < m \le \tau_n \le M$, where $m = \mathrm{e}^{-L}$ and $M = \mathrm{e}^{L}$. Also, $\log(\tau_n) \to \zeta$ implies that $\tau_n = \mathrm{e}^{\log\tau_n} \to \mathrm{e}^{\zeta} := \tau_\beta$. That is, $\lim_{n\to\infty} \tau_n = \tau_\beta$ where $0 < \tau_\beta < \infty$. Now to prove $\mu_n^{(m)}$ is Cauchy in the uniform metric we observe that, by Lemma 16,

$$\begin{aligned}\sup_{t\in[0,\tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &= \sup_{t\in[0,\tau_n]} |\mu_{n+m+1}\ldots\mu_{n+1}\mu_n(t) - \mu_{n+m}\ldots\mu_{n+1}\mu_n(t)|\\ &= \sup_{s\in[0,\tau_{n+m+1}]} |\mu_{n+m+1}(s) - s|\\ &\le \tau_{n+m+1}(\mathrm{e}^{P(\mu_{n+m+1})} - 1).\end{aligned}$$

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Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\lim_{m\to\infty} P(\mu_m) = 0$. Let N be such that $P(\mu_m) \le \frac{1}{2}$ for all $m \ge N$. Since $\mathrm{e}^x - 1 \le 2x$ for $0 \le x \le \frac{1}{2}$, it follows that, for all $m \ge N$,

$$\sup_{t\in[0,\tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| \le \tau_{n+m+1} 2P(\mu_{n+m+1}) \le 2MP(\mu_{n+m+1}).$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} 2MP(\mu_{n+m+1}) < \infty$. Thus, $(\mu_n^{(m)})_{m=1}^{\infty}$ is uniformly Cauchy and there exists $\lambda_n = \lim_{m\to\infty} \mu_n^{(m)}$. Moreover, for the third part, for fixed n, the function λ_n is a uniform limit of continuous functions and thus continuous. Also it is nondecreasing and satisfies $\lambda_n(0) = 0$,

$$\lambda_n(\tau_n) = \lim_{m\to\infty} \mu_{n+m} \circ \cdots \circ \mu_{n+1} \circ \mu_n(\tau_n) = \lim_{m\to\infty} \tau_{n+m+1} = \tau_\beta.$$

Since

$$\begin{aligned}
&\left|\log \frac{\mu_{n+m} \ldots \mu_{n+1}\mu_n(t) - \mu_{n+m} \ldots \mu_{n+1}\mu_n(s)}{t-s}\right| \\
&\quad \le P(\mu_{n+m} \ldots \mu_{n+1}\mu_n) \\
&\quad \le P(\mu_{n+m}) + \cdots + P(\mu_{n+1}) + P(\mu_n) = \sum_{i=n}^{n+m} P(\mu_i),
\end{aligned}$$

taking the limit as $m \to \infty$ gives

$$\left|\log \frac{\lambda_n(t) - \lambda_n(s)}{t-s}\right| \le \lim_{m\to\infty} \sum_{i=n}^{n+m} P(\mu_i) = \sum_{i=n}^{\infty} P(\mu_i).$$

Thus $P(\lambda_n) \le \sum_{i=n}^{\infty} P(\mu_i) < \infty$. Hence, by Remark 6, $\lambda_n \in \Delta$. Also $P(\lambda_n) \to 0$ as $n \to \infty$. □

COROLLARY 18. *If (S, d_s) is a complete metric space and $\hat{d}$ is considered with the penalty function (2), then the space $(C_s^\tau, \hat{d})$ is complete.*

Part of the nice properties established in Proposition 17 may already be derived from the following general property:

(P5) There exists a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = L$$

(exists) and such that, for $\lambda \in \Delta_{\alpha,\beta}$,

$$\sup_{t\in[0,\tau_\alpha]} |\lambda(t) - t| \le \tau_\alpha f(P(\lambda)).$$

According to Lemma 16, the function $P(\lambda)$ given by (2) satisfies (P5).

Note that (P5) implies (P4). Moreover, we have the following partial result towards establishing (P3′).

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LEMMA 19. *Assume that* (Δ, P) *satisfies condition* (P5). *Let* $\mu_m : [0, \tau_m] \to [0, \tau_{m+1}] \in \Delta_{m,m+1}$ *be such that* $\sum_{m=1}^{\infty} P(\mu_m) < \infty$. *Then the following results hold.*

(1) *There exist* m, M, $0 < m < M$, *such that* $m \le \tau_n \le M$ *for all* n.

(2) *For each* n, *the sequence* $(\mu_n^{(m)})_{m=1}^{\infty}$ *in* Δ *defined by*

$$\mu_n^{(m)} := \mu_{n+m} \circ \cdots \circ \mu_{n+1} \circ \mu_n$$

is Cauchy in the uniform metric.

(3) *If* $\lambda_n := \lim_{m\to\infty} \mu_n^{(m)}$ *is in* Δ *for some* $n = n_0 \in \mathbb{N}$, *then* $\lambda_n \in \Delta$ *for all* $n \ge n_0$.

PROOF. Since $\lim_{x\to 0^+} f(x)/x = L$, $f(x) \le (L + \epsilon)x$ for sufficiently small x and $f(x) \to 0$ as $x \to 0$. Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $P(\mu_m) \to 0$ as $m \to \infty$. Thus, $f(P(\mu_m)) \to 0$ as $m \to \infty$. Moreover, since $f(P(\mu_m)) \le (L + \epsilon)P(\mu_m)$ and $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$. By (P5),

$$\begin{aligned} |\tau_{m+1} - \tau_m| &= |\mu_m(\tau_m) - \tau_m| \\ &\le \sup_{0\le t\le\tau_m} |\mu_m(t) - t| \le \tau_m f(P(\mu_m)). \end{aligned}$$

Thus $|\tau_{m+1}/\tau_m - 1| \le f(P(\mu_m)) \to 0$ as $m \to \infty$. Or $\tau_{m+1}/\tau_m \to 1$ as $m \to \infty$. Since $\log x \le x - 1$ for all $x > 0$,

$$\begin{aligned} |\log(\tau_{m+1}) - \log(\tau_m)| &\le \max\Big(\frac{\tau_{m+1}}{\tau_m} - 1, \frac{\tau_m}{\tau_{m+1}}\Big(1 - \frac{\tau_{m+1}}{\tau_m}\Big)\Big) \\ &\le \Big(1 + \frac{\tau_m}{\tau_{m+1}}\Big)\Big|\frac{\tau_{m+1}}{\tau_m} - 1\Big|. \end{aligned}$$

Since $\tau_m/\tau_{m+1} \to 1$ as $m \to \infty$, $1 + \tau_m/\tau_{m+1} \le 3$. Hence,

$$|\log(\tau_{m+1}) - \log(\tau_m)| \le 3\Big|\frac{\tau_{m+1}}{\tau_m} - 1\Big| \le 3 f(P(\mu_m)).$$

Since $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$, $(\log \tau_m)_{m=1}^{\infty}$ is Cauchy. Hence, it is convergent and bounded. That is to say, there exist m, M, $0 < m < M$, such that $m \le \tau_n \le M$ for all n. For the second part,

$$\begin{aligned} \sup_{t\in[0,\tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &= \sup_{s\in[0,\tau_{n+m+1}]} |\mu_{n+m+1}(s) - s| \\ &\le \tau_{n+m+1} f(P(\mu_{n+m+1})). \end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\lim_{m\to\infty} P(\mu_m) = 0$. Let N be such that $P(\mu_m) \le \delta$ for all $m \ge N$. Then for all $m \ge N$,

$$\sup_{t\in[0,\tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| \le \tau_{n+m+1}(L + \epsilon)P(\mu_{n+m+1}) \le M(L + \epsilon)P(\mu_{n+m+1}).$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} M(L + \epsilon)P(\mu_{n+m+1}) < \infty$. Thus, $(\mu_n^{(m)})_{m=1}^{\infty}$ is uniformly Cauchy. Moreover, there exists $\lambda_n = \lim_{m\to\infty} \mu_n^{(m)}$. For the third part, let $\lambda_n = \lim_{m\to\infty} \mu_n^{(m)}$ be in Δ for some $n \ge n_0$. Then $\lambda_n^{-1} \in \Delta$. By (4), $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. So by induction $\lambda_n^{-1} \in \Delta$ for all $n \ge n_0$. Hence, $\lambda_n \in \Delta$ for all $n \ge n_0$. □

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Remark 20. In the general case, unfortunately (P5) does not seem to allow any conclusions to be drawn on the domination of $P(\lambda_n)$ by $\sum_{m=n}^{\infty} P(\mu_m)$ as required in (P3′). Additional properties of (Δ, P) are required to establish that part.

4. Spaces of sequences in C_s^τ and càdlàg functions

For an introduction to càdlàg functions, see, for instance, Billingsley [1]. As mentioned in the introduction, sequences of functions in C_s^τ are a means of coding the continuous deterministic parts of a sample trajectory of the stochastic process that results from random jump interventions on a dynamical system in S. In this section we investigate the two'classical' metric properties of sequence spaces, and their relation to càdlàg functions. We denote the space of all sequences of continuous functions from C_s^τ by $(C_s^\tau)^{\mathbb{N}}$ or $(C_s^{\mathbb{N}})$, that is,

$$(C_s^{\mathbb{N}}) := \{(f_n) = (f_1, f_2, \ldots) : \ f_i \in C_s([0, \tau_i])\}.$$

We would now like to define a metric on $(C_s^{\mathbb{N}})$. For this purpose we will make use of the metric $\hat{d}$. Let $\hat{d}_0(f, g) = \hat{d}(f, g)/(1 + \hat{d}(f, g))$. Then $\hat{d}_0$ is a metric on C_s^τ equivalent to $\hat{d}$, which gives C_s^τ finite diameter. Moreover, completeness of C_s^τ under $\hat{d}$ yields that C_s^τ is complete under $\hat{d}_0$. If $(f_n), (g_n) \in (C_s^{\mathbb{N}})$, then we define a metric d as follows:

$$d((f_n), (g_n)) = \sum_{i=1}^{\infty} 2^{-i} \hat{d}_0(f_i, g_i). \tag{5}$$

Proposition 21. *The space $((C_s^{\mathbb{N}}), d)$ is complete, whenever $(C_s^\tau, \hat{d})$ is complete.*

Proof. The space $((C_s^{\mathbb{N}}), d)$ is the Cartesian product of the complete metric spaces $(C_s^\tau, \hat{d})$ and hence it is complete. □

Proposition 22. *If $(C_s^\tau, \hat{d})$ is separable, then the space $((C_s^{\mathbb{N}}), d)$ is separable.*

Proof. The space $((C_s^{\mathbb{N}}), d)$ is the countable Cartesian product of the separable metric spaces $(C_s^\tau, \hat{d}_0)$ which are of finite diameter. Thus, the product $(C_s^{\mathbb{N}})$ with the metric (5) is separable. Moreover, if M is a countable dense subset of C_s^τ and we fix $f_0 \in M$, then a countable dense subset of the the space $((C_s^{\mathbb{N}}), d)$, $\mathfrak{M}$, could be defined as follows:

$$\mathfrak{M} = \{(f_1, f_2, f_3, \ldots, f_m, f_0, f_0, f_0, \ldots) : f_i \in M \forall i, m \in \mathbb{N}\}.$$ □

The components of a sequence (f_k) in $(C_s^{\mathbb{N}})$ may be *concatenated* to yield an S-valued function on $[0, \tau_\infty)$ where $\tau_\infty = \sum_n \tau_n$ whenever $f_n \in C_s([0, \tau_n])$. This function has jump discontinuities at times $t_k = \sum_{n=1}^{k} \tau_n$. At these points one needs to decide on the value of the concatenated function. For instance, one may pick $f_{k+1}(0)$. Thus, one obtains a right-continuous functions on $[0, \tau_\infty)$ with left-limits. That is, we may define a concatenation map as follows:

$$\gamma : (C_s^{\mathbb{N}}) \to \bigcup_{0<\tau\leq\infty} S^{[0,\tau)},$$
$$\gamma(f_1, f_2, \ldots)(t) := f_k(t), \qquad \tau_1 + \cdots + \tau_{k-1} \leq t < \tau_1 + \cdots + \tau_k.$$

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In order to get a map into the càdlàg functions on $[0, \infty)$ one therefore needs to restrict to a subcollection of sequences, $(C_s^{\mathbb{N}})_\infty$, consisting of all sequences $(f_n) \in (C_s^{\mathbb{N}})$ such that $\sum_{i=1}^{\infty} \tau_i = \infty$. THis is a proper subspace of $(C_s^{\mathbb{N}})$. Moreover, we may restrict the concatenation map γ, from $(C_s^{\mathbb{N}})_\infty$ into the space of all càdlàg functions $D_s([0, \infty))$,

$$\gamma : (C_s^{\mathbb{N}})_\infty \to D_s([0, \infty)),$$

with the same definition as above.

It is interesting to observe that this concatenation map is not injective. This can easily be seen from the following example.

EXAMPLE 23. Put $f_n(t) = 7$, for $t \in [0, 2^n]$, $g_n(t) = 7$, for $t \in [0, 1]$. Then clearly $f_n \in C_s([0, 2^n])$, $g_n \in C_s([0, 1])$, $(f_n) \neq (g_n)$ and $\gamma((f_n)) = \gamma((g_n))$, namely the constant function 7 in $D_s([0, \infty))$.

Continuity or measurability properties of the concatenation map are not yet completely clear, and will be the topic of further investigation in subsequent work.

We continue by providing some properties of $(C_s^{\mathbb{N}})_\infty$.

LEMMA 24. *The space $(C_s^{\mathbb{N}})_\infty$ is not a closed subspace of $(C_s^{\mathbb{N}})$. Hence, $((C_s^{\mathbb{N}})_\infty, d)$ is not complete.*

PROOF. We give an example of a sequence $(f_k^{(n)})_{k=1}^{\infty} \in (C_s^{\mathbb{N}})_\infty$ such that $(f_k^{(n)}) \to (f_k)$ and $(f_k) \notin (C_s^{\mathbb{N}})_\infty$. Let $(f_k^{(n)}) \in C_s([0, (1/n^2) + 1/2^k])$ be such that $(f_k^{(n)}) \to (f_k)$ for some $(f_k) \in (C_s^{\mathbb{N}})$. Then $(f_k^{(n)}) \in (C_s^{\mathbb{N}})_\infty$ since $\sum_k \sum_n 1/n^2 + 1/2^k = \infty$. But $(f_k) \notin (C_s^{\mathbb{N}})_\infty$ since $\sum_k \tau_k = \sum_k 1/2^k \neq \infty$. □

PROPOSITION 25. *If $(C_s^{\tau}, \hat{d})$ is separable, then the space $((C_s^{\mathbb{N}})_\infty, d)$ is separable.*

PROOF. The proof is similar to that for the space $((C_s^{\mathbb{N}}), d)$, Proposition 22, with slight differences. As f_0 is fixed, so is its interval of definition, therefore, $\sum_{i=1}^{\infty} \tau_i = \infty$. □

LEMMA 26. *$(C_s^{\mathbb{N}})_\infty$ is dense in $((C_s^{\mathbb{N}}), d)$. In particular, when $(C_s^{\tau}, \hat{d})$ is complete, $((C_s^{\mathbb{N}}), d)$ is a completion of the space $((C_s^{\mathbb{N}})_\infty, d)$.*

PROOF. From the proof of Proposition 22, $\mathfrak{M}$ is dense in $((C_s^{\mathbb{N}}), d)$ and

$$\mathfrak{M} \subseteq (C_s^{\mathbb{N}})_\infty \subseteq (C_s^{\mathbb{N}}),$$

so $(C_s^{\mathbb{N}})_\infty$ is dense in $(C_s^{\mathbb{N}})$. Thus, when $(C_s^{\tau}, \hat{d})$ is complete, it follows from Lemma 21 that $((C_s^{\mathbb{N}}), d)$ is a completion of $(C_s^{\mathbb{N}})_\infty$. □

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