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Nagoya Math. J.
Vol. 142 (1996), 67–93

GENERALIZED HYPERGROUPS AND ORTHOGONAL POLYNOMIALS

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Introduction

We study in this paper a generalization of the notion of a discrete hypergroup with particular emphasis on the relation with systems of orthogonal polynomials. The concept of a locally compact hypergroup was introduced by Dunkl [8], Jewett [12] and Spector [25]. It generalizes convolution algebras of measures associated to groups as well as linearization formulae of classical families of orthogonal polynomials, and many results of harmonic analysis on locally compact abelian groups can be carried over to the case of commutative hypergroups; see Heyer [11], Litvinov [17], Ross [22], and references cited therein. Orthogonal polynomials have been studied in terms of hypergroups by Lasser [15] and Voit [31], see also the works of Connett and Schwartz [6] and Schwartz [23] where a similar spirit is observed.

The special case of a discrete hypergroup, particularly in the commutative case, goes back earlier. In fact the ground-breaking paper of Frobenius [9] implicitly uses the notion of a hypergroup as the central object upon which the entire edifice of harmonic analysis on a finite (non-abelian) group is built (see Curtis [7] for an interesting discussion of this important historical point, and also Wildberger [34] for an extension of this point of view to Lie groups). Variants of the concept have appeared in many places: the early work of Kawada [13] on C-algebras; the systems of generalized translation operators studied by Levitan [16]; the hypercomplex systems studied by Berezansky and Kalyushnyi [4] and others; the work of the physicists on Racah-Wigner algebras (see for example Sharp [24]); the association schemes studied by combinatoricists (see for example the book of Bannai and Ito [3]); and the work of McMullen [18] and McMullen and Price [19]. More recently we mention also the objects introduced by Arad and Blau [1] called table algebras (see also [2]); the hypergroup-like objects studied by Sunder [26]; the convolution algebras studied by Szwarc [28]; and the fusion rule algebras

Received October 11, 1994.

https://doi.org/10.1017/S002776300000564X Published online by Cambridge University Press

arising in conformal field theories [29].

Here we introduce a very general class of objects which we call 'generalized hypergroups' which include as special cases many of the above. We have, in view of the multitude of approaches outlined above, attempted to keep our axioms down to an essential minimum. A generalized hypergroup is simply a $*$-basis of a $*$-algebra with unit satisfying essentially one axiom on the structure constants (see (A3) below). A main set of examples arises from general systems of orthogonal polynomials on the real line.

The paper is organized as follows: In Section 1 we introduce the notion of a generalized hypergroup and investigate some general features. In Section 2 we discuss characters of a generalized hypergroup and observe how a hypergroup is obtained from a generalized hypergroup by means of renormalization. In Sections 3 and 4 we establish a boundedness criterion that will ensure that the generalized hypergroup can be densely imbedded in a C^*-algebra. An interesting example is given by the Jacobi polynomials. It is also proved that every countable discrete hypergroup satisfies the boundedness criterion. We then show in Section 5 how the Gelfand theory can be used to establish representation of a commutative generalized hypergroup. Examples of orthogonal systems of polynomials are shown to satisfy the boundedness condition so we see that the rudiments of a theory of harmonic analysis are present in this case even if the usual positivity condition of the linearization coefficients (as studied for example by Gasper [10] for the Jacobi polynomials) is absent. In Section 6 we introduce the notion of a generalized eigenvector and study, in the case of commutative generalized hypergroups, characters of it. Finally Section 7 is devoted to a study of the Fourier transform on a commutative generalized hypergroup. We establish analogues of the Plancherel formula and the inversion formula.

Acknowledgements. The present joint work started during the first named author's stay at the University of NSW in 1992. He is most grateful to the School of Mathematics for its hospitality. He also thanks Professor K. Aomoto for many interesting remarks on orthogonal polynomials.

1. Generalized hypergroups and examples

Definition. A (*discrete*) *generalized hypergroup* is a pair $(\mathcal{K}, \mathcal{A}_0)$ where $\mathcal{A}_0$ is a $*$-algebra with unit c_0 over $\mathbf{C}$ and $\mathcal{K} = \{c_0, c_1, c_2, \dots\}$ is a countable (infinite or finite) subset of $\mathcal{A}_0$ satisfying the following conditions:

(A1) $\mathcal{K}^* = \mathcal{K}$;

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(A2) $\mathscr{K}$ is a linear basis of $\mathscr{A}_0$ i.e., every $a \in \mathscr{A}_0$ admits a unique expression of the form $a = \sum_i \alpha_i c_i$ with only finitely many non-zero $\alpha_i \in \mathbf{C}$;

(A3) The *structure constants* $b(i, j, k) \in \mathbf{C}$ defined by

$$c_i c_j = \sum_k b(i, j, k) c_k$$

satisfy the conditions:

$$c_i^* = c_j \Leftrightarrow b(i, j, 0) > 0, \quad c_i^* \neq c_j \Leftrightarrow b(i, j, 0) = 0.$$

If no confusion occurs, we simply say that $\mathscr{K}$ is a generalized hypergroup.

Given a generalized hypergroup $\mathscr{K}$ we define a bijection $\sigma : \{0, 1, 2, \dots\} \to \{0, 1, 2, \dots\}$ by $c_i^* = c_{\sigma(i)}$. Obviously $\sigma^2 = \mathrm{id}$ and $\sigma(0) = 0$. Note that $\sigma = \mathrm{id}$ may happen. It is convenient to put

$$w_i = \frac{1}{b(\sigma(i), i, 0)} > 0,$$

which is called the *weight* of c_i. Obviously, $w_0 = 1$. Note also that

$$b(i, j, 0) = \begin{cases} w_j^{-1} > 0 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Definition. Let $\mathscr{K} = \{c_0, c_1, \dots\}$ be a generalized hypergroup with structure constants $b(i, j, k)$. Then it is called

(i) *hermitian* if $c_i^* = c_i$ for all i;
(ii) *commutative* if $c_i c_j = c_j c_i$ for all i, j;
(iii) *real* if $b(i, j, k) \in \mathbf{R}$ for all i, j, k;
(iv) *positive* if $b(i, j, k) \geq 0$ for all i, j, k;
(v) *normalized* if $\sum_k b(i, j, k) = 1$ for all i, j;
(vi) *signed* if it is both real and normalized.

Remark. By definition a positive, normalized generalized hypergroup is a countable discrete hypergroup and vice versa. A hypergroup-like structure introduced by Sunder [26] is a generalized hypergroup with structure constants being non-negative integers.

We now assemble some general results. In what follows let $\mathscr{K}$ be a generalized hypergroup with structure constants $b(i, j, k)$.

LEMMA 1.1. *It holds that*

$$b(i, 0, k) = b(0, i, k) = \delta_{ik}, \tag{1.2}$$

$$b(\sigma(j), \sigma(i), \sigma(k)) = \overline{b(i, j, k)}, \tag{1.3}$$

$$w_j b(i, j, k) = w_k b(\sigma(k), i, \sigma(j)). \tag{1.4}$$

Proof. Identity (1.2) follows immediately from the fact that c_0 is the unit of $\mathscr{A}_0$. Identity (1.3) is easily verified by applying the $*$-operation to the identity $c_i c_j = \sum_k b(i, j, k) c_k$. We show (1.4). Since $(c_i c_j) c_k = c_i (c_j c_k)$, we have

$$\sum_l b(i, j, l) b(l, k, m) = \sum_l b(j, k, l) b(i, l, m)$$

for any choice of i, j, k, m. In particular, taking $m = 0$ we obtain

$$b(i, j, \sigma(k)) b(\sigma(k), k, 0) = b(j, k, \sigma(i)) b(i, \sigma(i), 0).$$

Namely,

$$w_{\sigma(i)} b(i, j, \sigma(k)) = w_k b(j, k, \sigma(i))$$

and therefore

$$w_i b(\sigma(i), j, \sigma(k)) = w_k b(j, k, i).$$

Then (1.4) follows by changing suffixes. q.e.d.

LEMMA 1.2. *Put*

$$K_{ij} = \{k ; b(i, j, k) \neq 0\}, \quad I_{jk} = \{i ; b(i, b(i, j, k) \neq 0\}, \tag{1.5}$$
$$J_{ik} = \{j ; b(i, j, k) \neq 0\}.$$

Then these are all finite sets. If furthermore $\mathscr{K}$ is positive, each of these sets is non-empty.

Proof. We first prove that $J_{ik} = K_{\sigma(i)k}$. In fact, by definition and by (1.4) we see that

$$j \in J_{ik} \Leftrightarrow b(i, j, k) \neq 0 \Leftrightarrow b(\sigma(k), i, \sigma(j)) \neq 0.$$

In view of (1.3) the last condition is equivalent to

$$\Leftrightarrow b(\sigma(i), k, j) \neq 0 \Leftrightarrow j \in K_{\sigma(i)k}.$$

In a similar manner one sees easily that $I_{jk} = \sigma(K_{j\sigma(k)})$. Since $\#(K_{ij}) < \infty$ for all

i, j by (A2) and (A3), it also follows that $\#(J_{ik}) < \infty$ and $\#(I_{jk}) < \infty$.

Now suppose that $\mathscr{K}$ is positive. We note that $c_i c_j \neq 0$ for any i, j. In fact, suppose $c_i c_j = 0$. Then,

$$0 = c_i(c_j c_j^*) = \sum_k b(j, \sigma(j), k) c_i c_k = \sum_{k,l} b(j, \sigma(j), k) b(i, k, l) c_l.$$

Since $b(i, j, k) \geq 0$ for all i, j, k by assumption, we have

$$b(j, \sigma(j), k) b(i, k, l) = 0$$

for all k, l. Putting $k = 0$ and $l = i$, we obtain

$$b(j, \sigma(j), 0) b(i, 0, i) = 0,$$

which is impossible by (1.1) and (1.2). Thus $c_i c_j \neq 0$ for any i, j. In other words, K_{ij} is not empty for any i, j. It follows from the first part of the present proof that both I_{jk} and J_{ik} are also always non-empty. q.e.d.

By definition we have

LEMMA 1.3. *The following conditions are equivalent*:
(i) $\mathscr{K}$ *is hermition*;
(ii) $\sigma = \mathrm{id}$;
(iii) $b(i, i, 0) > 0$ *for all* i *and* $b(i, j, 0) = 0$ *for distinct* i, j.

LEMMA 1.4. *The following conditions are equivalent*:
(i) $\mathscr{K}$ *is commutative*;
(ii) $\mathscr{A}_0$ *is commutative*;
(iii) $b(i, j, k) = b(j, i, k)$ *for all* i, j, k.

LEMMA 1.5. *Let* $\mathscr{K}$ *be a hermitian generalized hypergroup. Then* $\mathscr{K}$ *is commutative if and only if it is real.*

Proof. By definition

$$c_i c_j = \sum_k b(i, j, k) c_k.$$

Since $\mathscr{K}$ is a hermitian generalized hypergroup, we have

$$c_j c_i = c_j^* c_i^* = (c_i c_j)^* = \sum_k \overline{b(i, j, k)} c_k^* = \sum_k \overline{b(i, j, k)} c_k.$$

Hence $\mathscr{K}$ is commutative if and only if $b(i, j, k) = \overline{b(i, j, k)}$ for any choice of

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i, j, k. q.e.d.

We also make the following definition.

DEFINITION. Two generalized hypergroups $(\mathcal{K}, \mathcal{A}_0)$ and $(\mathcal{D}, \mathcal{B}_0)$ are called *isomorphic* if there exists a $*$-isomorphism $\psi : \mathcal{A}_0 \to \mathcal{B}_0$ such that $\psi(\mathcal{K}) = \mathcal{D}$.

We now introduce our main class of examples. Let μ be a finite measure on $\mathbf{R}$ and assume that $1, x, x^2, \ldots$ belong to $L^2(\mathbf{R}, \mu)$ and are linearly independent. Let $\mathcal{K} = \{p_0 \equiv 1, p_1, p_2, \ldots\}$ be the associated system of orthogonal polynomials obtained in the usual way by performing the Gram-Schmidt orthogonalization to the sequence $1, x, x^2, \ldots$ up to constant multiples. Then each p_i is a polynomial of degree i with real coefficients. For each i, j we may write

$$p_i p_j = \sum_k b(i, j, k) p_k, \quad b(i, j, k) \in \mathbf{R}.$$

Moreover, it is known that $b(i, j, k) = 0$ unless $|i - j| \le k \le i + j$. Let $\mathcal{A}_0$ be the commutative $*$-algebra of all polynomials in x with complex coefficients. Since each p_i has real coefficients, $p_i^* = p_i$ for all i, i.e., $\sigma = \mathrm{id}$. It then follows that $(\mathcal{K}, \mathcal{A}_0)$ is a hermitian, commutative (hence real by Lemma 1.5) generalized hypergroup. In fact, axioms (A1) and (A2) are obvious. Axiom (A3) is immediate from the relation:

$$b(i, j, 0)\mu(\mathbf{R}) = \langle p_i, p_j \rangle_{L^2(\mathbf{R},\mu)}, \tag{1.6}$$

which follows by observing that

$$\begin{aligned} b(i, j, 0) \int_{\mathbf{R}} p_0(x)^2 \mu(dx) &= \sum_k b(i, j, k) \int_{\mathbf{R}} p_k(x) p_0(x) \mu(dx) \\ &= \int_{\mathbf{R}} p_i(x) p_j(x) p_0(x) \mu(dx) = \int_{\mathbf{R}} p_i(x) p_j(x) \mu(dx). \end{aligned}$$

In short, a system of orthogonal polynomials on $\mathbf{R}$ canonically becomes a hermitian, commutative generalized hypergroup, which we call a *generalized hypergroup of orthogonal polynomials*. A particularly interesting question in this connection is to determine conditions on μ that will ensure that $\mathcal{K}$ is positive.

One may generalize this construction by orthogonalizing the sequence $1, z, z^2, \ldots$ with respect to a finite measure on $\mathbf{C}$. In that case, however, axiom (A3) is not satisfied automatically.

2. Renormalization and characters

In this section we discuss how a normalized generalized hypergroup occurs.

LEMMA 2.1. *Let* $(\mathcal{K}, \mathcal{A}_0)$ *be a generalized hypergroup with structure constants* $b(i, j, k)$. *Let* $d_i \in \mathbf{C}$ *be a sequence satisfying*

$$d_0 = 1, \quad d_i \neq 0, \quad d_{\sigma(i)} = \overline{d_i}. \tag{2.1}$$

Define

$$\tilde{c}_i = \frac{c_i}{d_i}, \quad \tilde{\mathcal{K}} = \{\tilde{c}_0 = c_0, \tilde{c}_1, \tilde{c}_2, \ldots\}.$$

Then $(\tilde{\mathcal{K}}, \mathcal{A}_0)$ *is a generalized hypergroup with structure constants*:

$$\tilde{b}(i, j, k) = \frac{d_k}{d_i d_j} b(i, j, k). \tag{2.2}$$

Proof. We must check (A1)-(A3) for $(\tilde{\mathcal{K}}, \mathcal{A}_0)$. In fact, (A1) and (A2) are obvious. As for (A3), it is straightforward that $\tilde{b}(i, j, k)$ are the structure constants of $\tilde{\mathcal{K}}$. Since $(\tilde{c}_i)^* = \tilde{c}_j \Leftrightarrow \sigma(i) = j$ and

$$\tilde{b}(i, j, 0) = \frac{d_0}{d_i d_j} b(i, j, 0) = \frac{1}{|d_j|^2} b(\sigma(j), j, 0)\delta_{i\sigma(j)} = \frac{1}{|d_j|^2 w_j} \delta_{\sigma(i)j},$$

(A3) follows immediately. q.e.d.

DEFINITION. The generalized hypergroup $(\tilde{\mathcal{K}}, \mathcal{A}_0)$ constructed as described in Lemma 2.1 will be called a *renormalization* of $(\mathcal{K}, \mathcal{A}_0)$.

LEMMA 2.2. *Let* $(\mathcal{K}, \mathcal{A}_0)$ *be a generalized hypergroup. Then there exists a renormalization* $\mathcal{K}$ *which is a normalized generalized hypergroup if and only if there exists a sequence* $d_i \in \mathbf{C}$ *satisfying* (2.1) *and*

$$d_i d_j = \sum_k b(i, j, k) d_k. \tag{2.3}$$

Proof. The generalized hypergroup $\tilde{\mathcal{K}}$ is normalized if and only if $\sum_k \tilde{b}(i, j, k) = 1$ for any i, j. In view of (2.2) one sees that the last condition is equivalent to (2.3). q.e.d.

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LEMMA 2.3. *Let $(\mathcal{K}, \mathcal{A}_0)$ be a positive generalized hypergroup. Then there exists a renormalization $\tilde{\mathcal{K}}$ which is a countable discrete hypergroup if there exists a sequence $d_i > 0$ satisfying* (2.1) *and* (2.3).

This is immediate from definition and Lemma 2.2. It is now convenient to introduce the following

DEFINITION. Let $(\mathcal{K}, \mathcal{A}_0)$ be a generalized hypergroup. A function $\chi : \mathcal{K} \to \mathbf{C}$ is called a *character* of $\mathcal{K}$ if it admits an extension to a non-zero $*$-homomorphism of $\mathcal{A}_0$ into $\mathbf{C}$. Let $\mathcal{X} = \mathcal{X}(\mathcal{K})$ denote the set of all characters of $\mathcal{K}$.

For any character χ of $\mathcal{K}$ we denote by the same symbol the (unique) extension to a $*$-homomorphism of $\mathcal{A}_0$ into $\mathbf{C}$. It is easily seen that $\chi(c_0) = 1$ for any $\chi \in \mathcal{X}$. By definition

LEMMA 2.4. *Let $(\mathcal{K}, \mathcal{A}_0)$ be a generalized hypergroup. A function $\chi : \mathcal{K} \to \mathbf{C}$ is a character if and only if the sequence $d_i = \chi(c_i)$ satisfies* (2.1) *and* (2.3).

Thus we come to

THEOREM 2.5. *Let $(\mathcal{K}, \mathcal{A}_0)$ be a positive generalized hypergroup. Then there exists a renormalization $\tilde{\mathcal{K}}$ which is a countable discrete hypergroup if there exists a character χ of $\mathcal{K}$ such that $\chi(c_i) > 0$ for all i.*

Such a character need not exist. Here is a simple example discussed by Voit [32]. Let $H_n(x)$ be the Hermite polynomials which satisfy the orthogonal relation:

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn}.$$

Then it holds that

$$H_m(x) H_n(x) = \sum_{k=0}^{m \wedge n} 2^k k! \binom{n}{k} \binom{m}{k} H_{m+n-2k}(x).$$

Hence the Hermite polynomials constitute a positive, commutative generalized hypergroup. Let χ be a character and assume that $c_n = \chi(H_n) > 0$. Then $c_1 c_n = c_{n+1} + 2n c_{n-1}$ for $n = 1, 2, \ldots$, which implies that $c_n \leq c_1 c_{n-1} \leq c_1^2 c_n / 2n$. But this is impossible as is seen by $n \to \infty$.

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Remark. A character $\chi \in \mathscr{X}$ with $\chi(c_i) > 0$ for all i is called a *dimension function* for the generalized hypergroups studied by Sunder [26].

3. Generalized hypergroups imbedded in C^*-algebras

We first note the following

LEMMA 3.1. *For a generalized hypergroup $(\mathscr{K}, \mathscr{A}_0)$ define a linear functional φ_0: $\mathscr{A}_0 \to \mathbf{C}$ by*

$$\varphi_0\left(\sum_i \alpha_i c_i\right) = \alpha_0. \tag{3.1}$$

Then, $\varphi_0(a^*a) \geq 0$ for all $a \in \mathscr{A}_0$ and $\varphi_0(a^*a) = 0$ if and only if $a = 0$.

Proof. First note that φ_0 is well defined. In fact, by (A2) every $a \in \mathscr{A}_0$ admits a unique expression of the form $a = \sum_i \alpha_i c_i$ with only finitely many non-zero $\alpha_i \in \mathbf{C}$. Then we have

$$a^*a = \sum_{i,j} \overline{\alpha_i}\alpha_j c_i^* c_j = \sum_{i,j,k} \overline{\alpha_i}\alpha_j b(\sigma(i), j, k) c_k$$

and therefore

$$\varphi_0(a^*a) = \sum_{i,j} \overline{\alpha_i}\alpha_j b(\sigma(i), j, 0) = \sum_i |\alpha_i|^2 b(\sigma(i), i, 0) = \sum_i |\alpha_i|^2 w_i^{-1}. \tag{3.2}$$

The assertion is then immediate. q.e.d.

DEFINITION. A generalized hypergroup $(\mathscr{K}, \mathscr{A}_0)$ satisfies Condition (B) if for each i there exists $\gamma_i \geq 0$ such that

$$|\varphi_0(b^* c_i b)| \leq \gamma_i \varphi_0(b^*b) \quad \text{for all } b \in \mathscr{A}_0, \tag{3.3}$$

where φ_0 is defined as in (3.1).

THEOREM 3.2. *Let $(\mathscr{K}, \mathscr{A}_0)$ be a generalized hypergroup satisfying* (B). *Then there exist a unital C^*-algebra $\mathscr{A}$, a positive functional φ on $\mathscr{A}$ and an injective $*$-homomorphism $\pi : \mathscr{A}_0 \to \mathscr{A}$ with dense image such that the following diagram commutes*

$$\begin{tikzcd} \mathscr{A}_0 \arrow[rr, "\pi"] \arrow[dr, "\varphi_0"'] & & \mathscr{A} \arrow[dl, "\varphi"] \\ & \mathbf{C} & \end{tikzcd} \tag{3.4}$$

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Here we recall that every positive linear functional on a C^*-algebra is continuous, see e.g. [21]. We begin with the following preliminary result.

LEMMA 3.3. *For a generalized hypergroup* $(\mathcal{K}, \mathcal{A}_0)$ *Condition* (B) *is equivalent to Condition* (B′)*: for each* $a \in \mathcal{A}_0$ *there exists* $\gamma(a) \geq 0$ *such that*

$$\varphi_0(b^*a^*ab) \leq \gamma(a)\varphi_0(b^*b) \quad \text{for all } b \in \mathcal{A}_0. \tag{3.5}$$

Proof. (B)$\Rightarrow$(B′). For $a = \sum_i \alpha_i c_i \in \mathcal{A}_0$ and $b \in \mathcal{A}_0$, we have

$$\begin{aligned}\varphi_0(b^*a^*ab) &= \sum_{i,j} \overline{\alpha_i}\alpha_j \varphi_0(b^*c_i^*c_jb) \\ &= \sum_{i,j,k} \overline{\alpha_i}\alpha_j b(\sigma(i), j, k)\varphi_0(b^*c_kb).\end{aligned}$$

Then by assumption (B),

$$\varphi_0(b^*a^*ab) \leq \sum_{i,j,k} |\overline{\alpha_i}\alpha_j b(\sigma(i), j, k)|\gamma_k\varphi_0(b^*b) = \gamma(a)\varphi_0(b^*b),$$

where

$$\gamma(a) = \sum_{i,j,k} |\overline{\alpha_i}\alpha_j b(\sigma(i), j, k)|\gamma_k < \infty. \tag{3.6}$$

In fact, the right hand side is a finite sum, as is easily seen from Lemma 1.2 and the fact that $\alpha_i = 0$ except finitely many i.

(B′)$\Rightarrow$(B). Let $a_1, a_2 \in \mathcal{A}_0$ and $\varepsilon = \pm 1, \pm i$. By assumption

$$\begin{aligned}\gamma(a_1 + \varepsilon a_2)\varphi_0(b^*b) &\geq \varphi_0(b^*(a_1 + \varepsilon a_2)^*(a_1 + \varepsilon a_2)b) \\ &= \varphi_0(b^*a_1^*a_1b) + \varphi_0(b^*a_2^*a_2b) \\ &\quad + \varepsilon\varphi_0(b^*a_1^*a_2b) + \bar{\varepsilon}\varphi_0(b^*a_2^*a_1b).\end{aligned}$$

Since $\varphi_0(b^*a_1^*a_1b) \geq 0$ and $\varphi_0(b^*a_2^*a_2b) \geq 0$, we have

$$\varepsilon\varphi_0(b^*a_1^*a_2b) + \bar{\varepsilon}\varphi_0(b^*a_2^*a_1b) \leq \gamma(a_1 + \varepsilon a_2)\varphi_0(b^*b), \quad \varepsilon = \pm 1, \pm i. \tag{3.7}$$

In particular, $\varepsilon\varphi_0(b^*a_1^*a_2b) + \bar{\varepsilon}\varphi_0(b^*a_2^*a_1b) \in \mathbf{R}$ for $\varepsilon = \pm 1, \pm i$, and hence,

$$\varphi_0(b^*a_1^*a_2b) = \overline{\varphi_0(b^*a_2^*a_1b)}.$$

Using this fact, one can easily see from (3.7) that

$$\begin{aligned}\pm 2\,\mathrm{Re}[\varphi_0(b^*a_1^*a_2b)] &\leq \gamma(a_1 \pm a_2)\varphi_0(b^*b), \\ \pm 2\,\mathrm{Im}[\varphi_0(b^*a_1^*a_2b)] &\leq \gamma(a_1 \mp ia_2)\varphi_0(b^*b).\end{aligned} \tag{3.8}$$

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Now put

$$\gamma(a_1, a_2) = \frac{1}{\sqrt{2}} \max_{\varepsilon=\pm1,\pm i} \gamma(a_1 + \varepsilon a_2).$$

It then follows from (3.8) that

$$|\varphi_0(b^* a_1^* a_2 b)| \leq \gamma(a_1, a_2)\varphi_0(b^* b).$$

In particular, putting $\gamma_i = \gamma(c_0, c_i)$, we obtain

$$|\varphi_0(b^* c_i b)| \leq \gamma_i \varphi_0(b^* b), \quad b \in \mathscr{A}_0,$$

as desired. q.e.d.

Proof of Theorem 3.2. Assume that $(\mathscr{K}, \mathscr{A}_0)$ satisfies (B). According to Lemma 3.1 we introduce an inner product and a norm of $\mathscr{A}_0$ by

$$\langle a, b\rangle = \varphi_0(b^* a), \quad \| a \| = \sqrt{\langle a, a\rangle}, \quad a, b \in \mathscr{A}_0.$$

Let $\mathscr{H}$ be the completion of $\mathscr{A}_0$ with respect to $\|\cdot\|$. With each $a \in \mathscr{A}_0$ we associate a linear operator $\pi(a)$ on $\mathscr{A}_0$ by

$$\pi(a)b = ab, \quad b \in \mathscr{A}_0.$$

Then, we have

$$\| \pi(a)b \|^2 = \| ab \|^2 = \langle ab, ab\rangle = \varphi_0(b^* a^* ab), \quad a, b \in \mathscr{A}_0.$$

It then follows from Lemma 3.3 that there exists $\gamma(a) \geq 0$ such that

$$\| \pi(a)b \|^2 \leq \gamma(a)\varphi_0(b^* b) = \gamma(a) \| b \|^2.$$

Hence $\pi(a)$ can be extended to a bounded operator on $\mathscr{H}$, which will be denoted by the same symbol. We have thus obtained a map $\pi : \mathscr{A}_0 \to \mathscr{B}(\mathscr{H})$ which is, as is easily verified, a $*$-homomorphism. Moreover, π is injective. In fact, $\pi(a) = 0$ implies that $0 = \pi(a)c_0 = ac_0 = a$. Finally, define $\varphi(x) = \langle xc_0, c_0\rangle$ for $x \in \mathscr{B}(\mathscr{H})$. Then φ is a continuous positive functional on $\mathscr{B}(\mathscr{H})$ with $\varphi(\pi(a)) = \varphi_0(a)$ for $a \in \mathscr{A}_0$. This proves (3.4). q.e.d.

COROLLARY 3.4. *Notations being as above, $\{\sqrt{w_i}\, c_i\}_{i=0}^{\infty}$ is a complete orthonormal basis of $\mathscr{H}$.*

Proof. In fact,

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$$\langle c_i, c_j \rangle = \varphi_0(c_j^* c_i) = b(\sigma(j), i, 0) = \delta_{ij} w_i^{-1}. \tag{3.9}$$

The completeness follows from the construction of $\mathscr{H}$. q.e.d.

4. A sufficient condition for (B)

We give a sufficient condition for (B) in terms of the structure constants.

THEOREM 4.1. *Let $\mathscr{K}$ be a generalized hypergroup with structure constants $b(i, j, k)$. If*

$$\gamma_i \equiv \frac{1}{2} \sup_j \sum_k (|\, b(\sigma(j), i, \sigma(k)) \,| + |\, b(i, j, k) \,|) < \infty \quad \textit{for all } i, \tag{4.1}$$

then Condition (B) *is satisfied.*

Proof. For $b = \sum_j \beta_j c_j \in \mathscr{A}_0$ we have

$$b^* c_i b = \sum_{j,k} \overline{\beta_j} \beta_k c_j^* c_i c_k = \sum_{j,k} \sum_{l,m} \overline{\beta_j} \beta_k b(\sigma(j), i, l) b(l, k, m) c_m.$$

Therefore,

$$\begin{aligned}
\varphi_0(b^* c_i b) &= \sum_{j,k,l} \overline{\beta_j} \beta_k b(\sigma(j), i, l) b(l, k, 0) \\
&= \sum_{j,k} \overline{\beta_j} \beta_k b(\sigma(j), i, \sigma(k)) b(\sigma(k), k, 0) \\
&= \sum_{j,k} \overline{\beta_j b(\sigma(j), j, 0)} \times \beta_k b(\sigma(k), k, 0) \times b(\sigma(j), j, 0)^{-1} b(\sigma(j), i, \sigma(k)) \\
&= \sum_{j,k} \overline{\beta_j} w_j^{-1} \times \beta_k w_k^{-1} \times w_j b(\sigma(j), i, \sigma(k)).
\end{aligned}$$

Then by the Schwarz inequality we obtain

$$\begin{aligned}
|\, \varphi_0(b^* c_i b) \,| \le & \Big(\sum_{j,k} |\, \beta_j w_j^{-1} \,|^2 w_j \,|\, b(\sigma(j), i, \sigma(k)) \,| \Big)^{1/2} \\
& \times \Big(\sum_{j,k} |\, \beta_k w_k^{-1} \,|^2 w_j \,|\, b(\sigma(j), i, \sigma(k)) \,| \Big)^{1/2}.
\end{aligned} \tag{4.2}$$

Since $w_j b(\sigma(j), i, \sigma(k)) = w_k b(i, k, j)$ by (1.4) we have

$$\begin{aligned}
& \sum_{j,k} |\, \beta_k w_k^{-1} \,|^2 w_j \,|\, b(\sigma(j), i, \sigma(k)) \,| = \sum_{j,k} |\, \beta_k w_k^{-1} \,|^2 w_k \,|\, b(i, k, j) \,| \\
& = \sum_{j,k} |\, \beta_j w_j^{-1} \,|^2 w_j \,|\, b(i, j, k)) \,| = \sum_{j,k} |\, \beta_j \,|^2 w_j^{-1} \,|\, b(i, j, k) \,|.
\end{aligned} \tag{4.3}$$

Inserting (4.3) into (4.2), we have

$$|\varphi_0(b^*c_ib)| \leq \Big(\sum_{j,k} |\beta_j|^2 w_j^{-1} |b(\sigma(j), i, \sigma(k))|\Big)^{1/2} \Big(\sum_{j,k} |\beta_j|^2 w_j^{-1} |b(i, j, k))|\Big)^{1/2}$$

$$\leq \frac{1}{2}\sum_{j,k} |\beta_j|^2 w_j^{-1}(|b(\sigma(j), i, \sigma(k))| + |b(i, j, k)|)$$

$$= \frac{1}{2}\sum_{j} |\beta_j|^2 w_j^{-1}\Big\{\sum_k |b(\sigma(j), i, \sigma(k))| + |b(i, j, k)|)\Big\}.$$

By assumption (4.1), we conclude that

$$|\varphi_0(b^*c_ib)| \leq \gamma_i \sum_j |\beta_j|^2 w_j^{-1} = \gamma_i\varphi_0(b^*b),$$

which proves (B). q.e.d.

COROLLARY 4.2. *Every countable discrete hypergroup satisfies* (B).

Proof. By definition the structure constants $b(i, j, k)$ of a countable discrete hypergroup satisfy

$$b(i, j, k) \geq 0, \quad \sum_k b(i, j, k) = 1.$$

Hence (4.1) is satisfied as $\gamma_i = 1$. q.e.d.

COROLLARY 4.3. *Let $\mathcal{K} = \{p_0 \equiv 1, p_1, p_2, \ldots\}$ be a generalized hypergroup of orthogonal polynomials on $\mathbf{R}$ with respect to a finite measure μ. Then $\mathcal{K}$ satisfies* (B) *if*

$$\sup_j \sum_{k=|i-j|}^{i+j} |b(i, j, k)| < \infty \quad \textit{for all } i. \tag{4.4}$$

In particular, if $\sup_{j,k} |b(i, j, k)| < \infty$ *for all i, Condition* (B) *is satisfied.*

Proof. Since $\sigma = \mathrm{id}$ and $b(i, j, k) = b(j, i, k)$ in the present case, γ_i defined as in (4.1) becomes

$$\gamma_i = \sup_j \sum_k |b(i, j, k)| = \sup_j \sum_{k=|i-j|}^{i+j} |b(i, j, k)|.$$

Hence (4.4) coincides with the condition in Theorem 4.1.

We next assume that $M_i \equiv \sup_{j,k} |b(i, j, k)| < \infty$ for any i. Then,

$$\sum_{k=|i-j|}^{i+j} |b(i,j,k)| \leq M_i(i+j-|i-j|+1) = M_i(\min\{2i, 2j\}+1) \leq M_i(2i+1),$$

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and therefore,

$$\gamma_i = \sup_j \sum_{k=|i-j|}^{i+j} | b(i, j, k) | \leq M_i(2i+1) < \infty$$

for all i. Then we only need to apply the first half of the assertion. q.e.d.

In fact, for a generalized hypergroup of orthogonal polynomials we have a more complete statement.

THEOREM 4.4. *Let $\mathcal{K} = \{p_0 \equiv 1, p_1, p_2, \dots\}$ be a generalized hypergroup of orthogonal polynomials on $\mathbf{R}$ with respect to a finite measure μ. Then $\mathcal{K}$ satisfies Condition* (B) *if and only if* $\mathrm{supp}(\mu)$ *is compact.*

Proof. For $a, b \in \mathcal{A}_0$ we write

$$a = \sum_i \alpha_i p_i, \quad b = \sum_j \beta_j p_j.$$

Then by definition,

$$\varphi_0(b^* a) = \sum_{i,j} \alpha_i \overline{\beta_j} b(j, i, 0).$$

In view of the simple relation (1.6) we obtain

$$\varphi_0(b^* a) = \sum_{i,j} \alpha_i \overline{\beta_j} \mu(\mathbf{R})^{-1} \langle p_i, p_j \rangle_{L^2(\mathbf{R},\mu)} = \mu(\mathbf{R})^{-1} \langle a, b \rangle_{L^2(\mathbf{R},\mu)}.$$

Hence Condition (B′) reads that for each $a \in \mathcal{A}_0$ there exists $\gamma(a) \geq 0$ such that

$$\mu(\mathbf{R})^{-1} \langle ab, ab \rangle_{L^2(\mathbf{R},\mu)} \leq \gamma(a) \mu(\mathbf{R})^{-1} \langle b, b \rangle_{L^2(\mathbf{R},\mu)}, \quad b \in \mathcal{A}_0,$$

i.e.,

$$\int_{\mathbf{R}} | a(x) b(x) |^2 \mu(dx) \leq \gamma(a) \int_{\mathbf{R}} | b(x) |^2 \mu(dx), \quad b \in \mathcal{A}_0.$$

Since $\mathcal{A}_0$ is a dense subspace of $\mathcal{H} = L^2(\mathbf{R}, \mu)$, the above condition is equivalent to that the multipilcation operator by any polynomial is bounded on $L^2(\mathbf{R}, \mu)$. Obviously, this occurs if and only if $\mathrm{supp}(\mu) \subset \mathbf{R}$ is compact. q.e.d.

For example, the Jacobi polynomials $P_n^{(\alpha,\beta)}$ defined for $\alpha, \beta > -1$ satisfy Condition (B). In fact, they are orthogonal on $[-1,1]$ with respect to the measure $\mu(dx) = (1-x)^\alpha (1+x)^\beta dx$, see Szegö [27, Chap. IV].

5. Commutative generalized hypergroups

In this section we restrict ourselves to commutative generalized hypergroups.

For a compact (always assumed to be Hausdorff) space X we denote by $C(X)$ the usual commutative C^*-algebra of continuous functions on X. The norm of $C(X)$ is denoted by $\|\cdot\|_\infty$. The dual space $C(X)'$ is identified with the space of Radon (or equivalently, $\mathbf{C}$-valued regular Borel) measures on X.

If a countable subset $\mathscr{D} = \{f_0 \equiv 1, f_1, f_2, \ldots\} \subset C(X)$ and a dense $*$-subalgebra $\mathscr{B}_0 \subset C(X)$ constitute a generalized hypergroup $(\mathscr{D}, \mathscr{B}_0)$, we refer to it as a *function realization* on X.

THEOREM 5.1. *Let $(\mathscr{K}, \mathscr{A}_0)$ be a commutative generalized hypergroup satisfying* (B) *and let φ_0 be defined as in* (3.1). *Then it is isomorphic to a function realization $(\mathscr{D}, \mathscr{B}_0)$ on a compact space X with positive Radon measure μ such that $\mu(X) = 1$ and*

$$\varphi_0(a) = \int_X \hat{a}(x)\mu(dx), \quad a \in \mathscr{A}_0, \tag{5.1}$$

where $a \mapsto \hat{a}$ stands for the isomorphism from $(\mathscr{K}, \mathscr{A}_0)$ onto $(\mathscr{D}, \mathscr{B}_0)$ Moreover, $\mathscr{D}$ is a complete orthogonal set for $L^2(X, \mu)$, and $\operatorname{supp}(\mu) = X$.

Proof. By Theorem 3.2 there exist a C^*-algebra $\mathscr{A}$, a positive functional φ on it and an injective $*$-homomorphism π such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_0 & \xrightarrow{\pi} & \mathscr{A} \\ & {}_{\varphi_0}\searrow \quad \swarrow_{\varphi} & \\ & \mathbf{C} & \end{array}$$

Since $\pi(\mathscr{A}_0) \subset \mathscr{A}$ is dense, $\mathscr{A}$ is commutative. From now on, identifying $\mathscr{A}_0$ with $\pi(\mathscr{A}_0)$, we do not use the symbol π.

Since $\mathscr{A}$ is a unital commutative C^*-algebra, by Gelfand's theorem there exists a compact space X such that $\mathscr{A} \simeq C(X)$ under the Gelfand map $a \mapsto \hat{a}$. Then φ gives a positive (hence continuous) functional on $C(X)$, namely, a positive Radon measure μ on X:

$$\varphi(a) = \int_X \hat{a}(x)\mu(dx), \quad a \in \mathscr{A}. \tag{5.2}$$

Since $\varphi_0(a) = \varphi(a)$ for $a \in \mathscr{A}_0$, (5.1) follows. Furthermore, $\varphi(c_0) = \varphi_0(c_0) = 1$ implies $\mu(X) = 1$. The isomorphism $(\mathscr{K}, \mathscr{A}_0) \simeq (\mathscr{D}, \mathscr{B}_0)$ is obtained from $\mathscr{A} \simeq$

$C(X)$.

Let $\langle \cdot, \cdot \rangle_\mu$ and $\|\cdot\|_\mu$ denote the inner product and the norm of $L^2(X, \mu)$, respectively. Then, by (5.2) we have

$$\langle \hat{a}, \hat{b} \rangle_\mu = \int_X \hat{a}(x) \overline{\hat{b}(x)} \mu(dx) = \varphi(b^* a), \quad a, b \in \mathscr{A}. \tag{5.3}$$

In particular, by (3.9) we obtain

$$\langle \hat{c}_i, \hat{c}_j \rangle_\mu = \varphi(c_j^* c_i) = \varphi_0(c_j^* c_i) = \delta_{ij} w_i^{-1}. \tag{5.4}$$

Hence $\hat{\mathscr{K}} = \mathscr{D}$ is an orthogonal set in $L^2(X, \mu)$. Since $\mathscr{B}_0 \subset C(X)$ is dense with respect to the norm $\|\cdot\|_\infty$, we see that $\hat{\mathscr{K}}$ is complete. q.e.d.

The following result has been already established during the above proof, see also Corollary 3.4.

COROLLARY 5.2. *Notations and assumptions being the same as in Theorem* 5.1, $\{\sqrt{w_i}\hat{c}_i\}_{i=0}^\infty$ *is a complete orthonormal basis of* $L^2(X, \mu)$.

COROLLARY 5.3. *The Gelfand map* $a \mapsto \hat{a}$ *yields the following isomorphisms*:

$$\begin{array}{ccccccc} \mathscr{K} & \subset & \mathscr{A}_0 & \subset & \mathscr{A} & \subset & \mathscr{H} \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ \mathscr{D} & \subset & \mathscr{B}_0 & \subset & C(X) & \subset & L^2(X, \mu) \end{array}$$

where $\mathscr{A} \simeq C(X)$ *stands for an isomorphism between* C^**-algebras and* $\mathscr{H} \simeq L^2(X, \mu)$ *is a unitary isomorphism with respect to the norms* $\|\cdot\|$ *and* $\|\cdot\|_\mu$.

We now consider a commutative positive generalized hypergroup.

THEOREM 5.4. *Let* $\mathscr{K}$ *be a commutative generalized hypergroup satisfying* (B). *If* $\mathscr{K}$ *is positive, then there exists a renormalization* $\tilde{\mathscr{K}}$ *which is a commutative hypergroup.*

For the proof we need the following result which is contained in Voit [32, Corollary 1.2].

THEOREM 5.5. *Let* μ *be a positive Radon measure on a compact space* X *with* $\mu \neq 0$. *Let* $\mathscr{F}$ *be a family of* $\mathbf{C}$*-valued continuous functions on* X *such that*

(i) $\bar{f} \in \mathscr{F}$ *for* $f \in \mathscr{F}$;

(ii) $fg \in \mathscr{F}$ *for* $f, g \in \mathscr{F}$;

(iii) $\int_X f(x)\mu(dx) \geq 0$ *for* $f \in \mathscr{F}$.

Then there exists a point $x_0 \in \operatorname{supp}(\mu)$ *such that* $f(x_0) \geq 0$ *for all* $f \in \mathscr{F}$.

Proof of Theorem 5.4. We retain the same notation as in the proof of Theorem 5.1. We put

$$\mathscr{F} = \left\{\sum_j \alpha_j \hat{c}_j \,;\, \alpha_j \geq 0\right\} \subset C(X).$$

Then by the positivity assumption on $\mathscr{K}$ and the fact that

$$\int_X \sum_j \alpha_j \hat{c}_j(x)\mu(dx) = \alpha_0$$

we see that $\mathscr{F}$ satisfies the conditions of Theorem 5.5. Hence there is a point $x_0 \in X$ with $\hat{c}_j(x_0) \geq 0$ for all j. We next prove that $\hat{c}_j(x_0) > 0$ for all j. Suppose otherwise, namely, $\hat{c}_j(x_0) = 0$ for some j. Then for all i,

$$0 = \hat{c}_i(x_0)\hat{c}_j(x_0) = \sum_k b(i, j, k)\hat{c}_k(x_0).$$

By Lemma 1.2 and the positivity of $\mathscr{K}$, for any k we may find i such that $b(i, j, k) \neq 0$ so that $\hat{c}_k(x_0) = 0$. This means that $\hat{a}(x_0) = 0$ for all $a \in \mathscr{A}_0$. But this contradicts the density of $\hat{\mathscr{A}}_0$ in $C(X)$. It follows that $\hat{c}_i(x_0) > 0$ for all i.

Define $\chi : \mathscr{A}_0 \to \mathbf{C}$ by $\chi(\hat{a}) = \hat{a}(x_0)$, $a \in \mathscr{A}_0$. Obviously, χ becomes a character of $\mathscr{K}$ such that $\chi(c_i) > 0$ for all i. It then follows from Theorem 2.5 that there exists a renormalization $\tilde{\mathscr{K}}$ which is a hypergroup. In fact, the renormalization is given by

$$\tilde{c}_i = \frac{c_i}{\hat{c}_i(x_0)}. \tag{5.5}$$

This completes the proof. q.e.d.

We now recall the following result of Szwarc [28], see also Voit [32, Lemma 2.3].

THEOREM 5.6. *Let* μ *be a positive Radon measure on a locally compact space* X. *Let* $\{f_i\}$ *be a countable orthogonal family of* $\mathbf{C}$*-valued, continuous, bounded functions in* $L^2(X, \mu)$ *such that* $\| f_i \|_{L^2(X,\mu)} \neq 0$ *for all* i *and such that* $\{f_i\}$ *is closed under com-*

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plex conjugation. Assume that for all i, j

$$f_i f_j = \sum_k b(i, j, k) f_k \quad \textit{(possibly infinite series)} \tag{5.6}$$

holds in $L^2(X, \mu)$ *with* $b(i, j, k) \geq 0$ *and* $\sum_k b(i, j, k) \leq 1$. *Then* $|f_i(x)| \leq 1$ *holds for all* i *and all* $x \in \operatorname{supp}(\mu)$.

THEOREM 5.7. *Let* $(\mathscr{K}, \mathscr{A}_0)$ *be a positive, commutative generalized hypergroup satisfying* (B) *and let* $a \mapsto \hat{a}$ *be a function realization of* $\mathscr{K}$ *on a compact space* X *with a positive Radon measure* μ. *Then there exists an* $x_0 \in X$ *such that* $\hat{c}_i(x_0) > 0$ *for all* i. *Furthermore,*

$$\hat{c}_i(x_0) = \max_{x \in X} |\hat{c}_i(x)|$$

holds for all i.

Proof. Let $\tilde{\mathscr{K}} = \{\tilde{c}_i\}$ be the renormalization of $\mathscr{K}$ described as in Theorem 5.4. Then, $\hat{\tilde{\mathscr{K}}} = \{\hat{\tilde{c}}_i\} \subset C(X)$ satisfies the condition in Theorem 5.6. In fact, (5.6) is reduced to a finite sum and $\sum_k b(i, j, k) = 1$. Hence we deduce that $|\hat{\tilde{c}}_i(x)| \leq 1$ for all i and all $x \in \operatorname{supp}(\mu) = X$. In view of (5.5) we obtain

$$|\hat{c}_i(x)| \leq \hat{c}_i(x_0), \quad x \in X.$$

The result then follows. q.e.d.

Theorem 5.7 gives a necessary condition for the positivity of a commutative generalized hypergroup satisfying (B). A good class of examples is given by generalized hypergroups of orthogonal polynomials.

COROLLARY 5.8. *Let* $\mathscr{K} = \{p_0, p_1, p_2, \ldots\}$ *be a generalized hypergroup of orthogonal polynomials with respect to a finite measure* μ *on* $[a, b]$. *Assume that* $\mathscr{K}$ *satisfies Condition* (B) *and that* $p_1(x) = cx + d$ *with* $c > 0$. *If* $\mathscr{K}$ *is positive,*

$$\max_{a \leq x \leq b} |p_i(x)| = p_i(b)$$

holds for all i.

Proof. Condition on p_1 ensures that its maximum occurs at $x = b$. Hence by Theorem 5.7 the positivity of the generalized hypergroup $\mathscr{K}$ forces all p_i to have maximum at $x = b$. q.e.d.

It seems interesting to apply the above result to determining the parameters (α, β) of the Jacobi polynomials which give rise to a positive generalized hypergroup, see Gasper [10].

6. Point measures as joint eigenvectors

Given a generalized hypergroup $(\mathscr{K}, \mathscr{A}_0)$ we consider the vector space of formal series:

$$\mathscr{A}_\infty = \left\{\xi = \sum_j \xi_j c_j\,;\, \xi_j \in \mathbf{C}\right\}. \tag{6.1}$$

For $\xi = \sum_j \xi_j c_j \in \mathscr{A}_\infty$ and $c_i \in \mathscr{K}$ define

$$c_i\xi = \sum_k \left(\sum_j \xi_j b(i, j, k)\right) c_k, \quad \xi c_i = \sum_k \left(\sum_j \xi_j b(j, i, k)\right) c_k. \tag{6.2}$$

Here we note that both $\sum_j \xi_j b(i, j, k)$ and $\sum_j \xi_j b(j, i, k)$ are finite sums by Lemma 1.2. Moreover, it is straightforward to see that (6.2) extends to bilinear maps $\mathscr{A}_0 \times \mathscr{A}_\infty \to \mathscr{A}_\infty$ and $\mathscr{A}_\infty \times \mathscr{A}_0 \to \mathscr{A}_\infty$. Namely, $\mathscr{A}_\infty$ becomes an $\mathscr{A}_0$-bimodule. The $*$-operation on $\mathscr{A}_\infty$ is simply the extention of that on $\mathscr{A}_0$:

$$\left(\sum_i \xi_i c_i\right)^* = \sum_i \overline{\xi_i} c_i^*.$$

Then, obviously

$$(a\xi)^* = \xi^* a^*, \quad (\xi a)^* = a^* \xi^*, \quad a \in \mathscr{A}_0, \quad \xi \in \mathscr{A}_\infty.$$

We say that $\xi = \sum_j \xi_j c_j \in \mathscr{A}_\infty$ is *hermitian* if $\xi = \xi^*$, or equivalently if $\xi_{\sigma(i)} = \overline{\xi_i}$ for all i.

DEFINITION. A non-zero $\xi \in \mathscr{A}_\infty$ is called a (*left generalized*) *joint eigenvector* if for each i there exists $\Lambda(c_i) \in \mathbf{C}$ such that $c_i\xi = \Lambda(c_i)\xi$. An *invariant vector* is by definition a non-zero element $\xi \in \mathscr{A}_\infty$ satisfying $c_i\xi = \xi$ for all i.

PROPOSITION 6.1. *For each joint eigenvector $\xi \in \mathscr{A}_\infty$ there exists a non-zero homomorphism $\Lambda : \mathscr{A}_0 \to \mathbf{C}$ such that*

$$a\xi = \Lambda(a)\xi, \quad a \in \mathscr{A}_0. \tag{6.3}$$

Conversely, for any non-zero homomorphism $\Lambda : \mathscr{A}_0 \to \mathbf{C}$ there is a joint eigenvector $\xi \in \mathscr{A}_\infty$ such that (6.3) *holds. In that case, any joint eigenvector is a constant multiple* *of*

$$\xi = \sum_j \Lambda(c_j^*) w_j c_j. \tag{6.4}$$

Proof. The first assertion is straightforward from the definition. Suppose that we are given a non-zero homomorphism $\Lambda: \mathscr{A}_0 \to \mathbf{C}$. Let $\xi \in \mathscr{A}_\infty$ be defined as in (6.4). Obviously, $\xi \neq 0$. We shall show that (6.3) holds. For that purpose it is sufficient to prove that $c_i\xi = \Lambda(c_i)\xi$. But this is directly verified with the help of (1.4) as follows:

$$\begin{aligned} c_i\xi &= \sum_{j,k} \Lambda(c_j^*) w_j b(i, j, k) c_k \\ &= \sum_{j,k} \Lambda(c_{\sigma(j)}) w_k b(\sigma(k), i, \sigma(j)) c_k \\ &= \sum_k \Lambda\Big(\sum_j b(\sigma(k), i, j) c_j\Big) w_k c_k \\ &= \sum_k \Lambda(c_{\sigma(k)} c_i) w_k c_k \\ &= \Lambda(c_i) \sum_k \Lambda(c_k^*) w_k c_k \\ &= \Lambda(c_i)\xi. \end{aligned}$$

Finally we prove the uniqueness. Suppose that $\xi = \sum_j \xi_j c_j \in \mathscr{A}_\infty$ satisfies (6.3). Then, in particular $c_i\xi = \Lambda(c_i)\xi$ and hence

$$\sum_{j,k} \xi_j b(i, j, k) c_k = \sum_j \Lambda(c_i) \xi_j c_j,$$

for any choice of i. Comparing the coefficients of c_0, we have

$$\Lambda(c_i)\xi_0 = \sum_j \xi_j b(i, j, 0) = \xi_{\sigma(i)} b(i, \sigma(i), 0) = w_{\sigma(i)}^{-1} \xi_{\sigma(i)},$$

namely

$$\xi_i = \Lambda(c_i^*) w_i \xi_0.$$

This means that any joint eigenvector $\xi \in \mathscr{A}_\infty$ satisfying (6.3) is uniquely determined up to constant multiple. q.e.d.

For the existence of an invariant vector we prove the following

PROPOSITION 6.2. *Let $(\mathscr{K}, \mathscr{A}_0)$ be a generalized hypergroup. Then there exists an invariant vector if and only if $\mathscr{K}$ is normalized. In that case, any invariant vector is a constant multiple of*

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(6.5) $$\lambda = \sum_j w_j c_j.$$

Proof. Let $\Lambda : \mathscr{A}_0 \to \mathbf{C}$ be the linear function uniquely defined by $\Lambda(c_i) = 1$. Since

$$\Lambda(c_i c_j) = \sum_k b(i, j, k)\Lambda(c_k) = \sum_k b(i, j, k),$$

Λ is a non-zero homomorphism if and only if $\sum_k b(i, j, k) = 1$ for any i, j. Hence by Proposition 6.1, there exists an invariant vector if and only if the case occurs. In that case, any invariant vector is a constant multiple of λ by Proposition 6.1 again. q.e.d.

From now on all the generalized hypergroups under consideration are assumed to be commutative.

PROPOSITION 6.3. *Let $(\mathscr{K}, \mathscr{A}_0)$ be a commutative generalized hypergroup and let Λ be a non-zero homomorphism from $\mathscr{A}_0$ into $\mathbf{C}$. Then Λ is a character, i.e. $\Lambda \in \mathscr{X}$, if and only if there exists a hermitian joint eigenvector $\xi \in \mathscr{A}_\infty$ with $a\xi = \Lambda(a)\xi$ for all $a \in \mathscr{A}_0$.*

Proof. Let $\xi \in \mathscr{A}_\infty$ be a joint eigenvector associated with the given Λ, see (6.4). Then $a\xi = \Lambda(a)\xi$ for $a \in \mathscr{A}_0$ and since $\mathscr{A}_0$ is commutative, we have

$$a^*\xi^* = \overline{\Lambda(a)}\xi^*.$$

If the above ξ is hermitian, we see that $\Lambda(a^*) = \overline{\Lambda(a)}$. Hence Λ is a $*$-homomorphism and, therefore, it is a character of $\mathscr{K}$.

Conversely, if Λ is a character of $\mathscr{K}$, it is extended to a non-zero $*$-homomorphism from $\mathscr{A}_0$ into $\mathbf{C}$. Then

$$a^*\xi^* = (a\xi)^* = (\Lambda(a)\xi)^* = \overline{\Lambda(a)}\xi^* = \Lambda(a^*)\xi^*.$$

In other words, ξ^* is also a joint eigenvector with $a\xi^* = \Lambda(a)\xi^*$ for $a \in \mathscr{A}_0$. By the uniqueness of a generalized joint eigenvector (Proposition 6.1) we see that $\xi^* = c\xi$ with some $c \in \mathbf{C}$. Then, at least one of the two vectors

$$\frac{\xi + \xi^*}{2} = \frac{1+c}{2}\xi, \quad \frac{\xi - \xi^*}{2i} = \frac{1-c}{2i}\xi,$$

becomes a hermitian joint eigenvector associated with the given character Λ. q.e.d.

THEOREM 6.4. *Let $(\mathscr{K}, \mathscr{A}_0)$ be a commutative generalized hypergroup satisfying* (B) *and let $(\tilde{\mathscr{K}}, \tilde{\mathscr{A}}_0)$ be the function realization on a compact space X as in Theorem* 5.1. *Then, for each $x \in X$,*

$$\chi_x(a) = \hat{a}(x), \quad a \in \mathscr{A}_0, \tag{6.6}$$

is a character of $\mathscr{K}$ and the corresponding joint eigenvectors are constant multiples of

$$\delta_x = \sum_j \overline{\hat{c}_j(x)}\, w_j c_j \in \mathscr{A}_\infty. \tag{6.7}$$

Moreover, the map $x \mapsto \chi_x$ yields an injection from X into $\mathscr{X}$.

Proof. It is easy to see that χ_x defined as in (6.6) is a non-zero $*$-homomorphism of $\mathscr{A}_0$ into $\mathbf{C}$, and hence χ_x is a character of $\mathscr{K}$. It then follows from Proposition 6.1 that

$$\sum_j \chi_x(c_j^*)\, w_j c_j = \sum_j \overline{\chi_x(c_j)}\, w_j c_j = \sum_j \overline{\hat{c}_j(x)}\, w_j c_j$$

is a joint eigenvector associated with χ_x. Since X is a compact Hausdorff space, $C(X)$ separates the points and therefore $x \mapsto \chi_x$ is injective. q.e.d.

Viewing $\mathscr{A}_0 \subset \mathscr{H} \subset \mathscr{A}_\infty$, we extend the inner product $\langle \cdot, \cdot \rangle$ of $\mathscr{H}$ to a sesquilinear form on $\mathscr{A}_0 \times \mathscr{A}_\infty$. More precisely, define

$$\langle a, \xi \rangle = \varphi_0(\xi^* a) = \sum_i a_i \bar{\xi}_i w_i^{-1}, \quad a = \sum_i \alpha_i c_i \in \mathscr{A}_0, \quad \xi = \sum_i \xi_i c_i \in \mathscr{A}_\infty. \tag{6.8}$$

With this notation, we have

PROPOSITION 6.5. $\langle a, \delta_x \rangle = \hat{a}(x)$ *holds for any $a \in \mathscr{A}_0$ and $x \in X$.*

Proof. Put $a = \sum_j \alpha_j c_j$. Then by Definition (6.7) we have

$$\langle a, \delta_x \rangle = \sum_j \alpha_i \hat{c}_j(x) = \Big(\sum_j \alpha_j c_j\Big)^{\wedge}(x) = \hat{a}(x),$$

as desired. q.e.d.

Namely, δ_x corresponds to a point measure on X concentrated at $x \in X$. Moreover, since

$$|\langle a, \delta_x \rangle| = |\hat{a}(x)| \le \|\hat{a}\|_\infty = \|a\|, \quad a \in \mathscr{A}_0,$$

δ_x extends to a continuous linear functional on $\mathscr{A}$. In other words,

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PROPOSITION 6.6. *$\delta_x \in \mathscr{A}'$ for all $x \in X$.*

Note the following diagram which extends that of Corollary 5.3:

$$\begin{array}{ccccccccc}
\mathscr{A}_0 & \subset & \mathscr{A} & \subset & \mathscr{H} & \subset & \mathscr{A}' & \subset & \mathscr{A}_\infty \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
\hat{\mathscr{A}}_0 & \subset & C(X) & \subset & L^2(X, \mu) & \subset & C(X)' & &
\end{array}$$

7. Fourier transform

We continue to assume that $(\mathscr{K}, \mathscr{A}_0)$ is a commutative generalized hypergroup satisfying Condition (B). The particular element $\lambda \in \mathscr{A}_\infty$ introduced in (6.5) is now regarded as a measure on $\mathscr{K}$. Let $\mathscr{F}_0(\mathscr{K})$ denote the space of all $\mathbf{C}$-valued functions on $\mathscr{K}$ with finite supports. For $f \in \mathscr{F}_0(\mathscr{K})$ we define

$$\langle f \rangle_\lambda = \sum_i f(c_i) w_i. \tag{7.1}$$

If $\mathscr{K}$ is normalized, it is natural to say that λ is an invariant measure on $\mathscr{K}$, see Proposition 6.2. In case of a countable discrete hypergroup λ is well known for a Haar measure, see [12].

Now we define

$$\| f \|_\lambda^2 = \langle f\bar{f} \rangle_\lambda = \sum_i | f(c_i) |^2 w_i.$$

Let $L^2(\mathscr{K}, \lambda)$ denote the Hilbert space which is the completion of $\mathscr{F}_0(\mathscr{K})$ with respect to the norm $\| \cdot \|_\lambda$ and let $\mathscr{F}_\infty(\mathscr{K})$ denote the space of all $\mathbf{C}$-valued functions on $\mathscr{K}$. Obviously,

$$\mathscr{F}_0(\mathscr{K}) \subset L^2(\mathscr{K}, \lambda) \subset \mathscr{F}_\infty(\mathscr{K}).$$

Note also that any character of $\mathscr{K}$ belongs to $\mathscr{F}_\infty(\mathscr{K})$, i.e., $\mathscr{X} \subset \mathscr{F}_\infty(\mathscr{K})$. Modelled after the Fourier transform introduced by Jewett [12] for a hypergroup, we make the following

DEFINITION. The *Fourier transform* of $f \in \mathscr{F}_0(\mathscr{K})$ is defined by

$$\mathfrak{F}f(\chi) = \langle f\chi \rangle_\lambda, \quad \chi \in \mathscr{X}. \tag{7.2}$$

With each $F \in \mathscr{F}_0(\mathscr{K})$ we associate a formal series

$$TF = \sum_i F(c_i) w_i c_i \in \mathscr{A}_\infty. \tag{7.3}$$

Then, T becomes a linear isomorphism from $\mathscr{F}_\infty(\mathscr{K})$ onto $\mathscr{A}_\infty$ and from $\mathscr{F}_0(\mathscr{K})$ onto

$\mathscr{A}_0$. Furthermore, comparing (6.8) and (7.1), we obtain

LEMMA 7.1. *$\langle Tf, TF\rangle = \langle f\bar{F}\rangle_\lambda$ for $f \in \mathscr{F}_0(\mathscr{K})$ and $F \in \mathscr{F}_0(\mathscr{K})$.*

As is mentioned in Theorem 6.4, there is a particular class of characters, namely, χ_x indexed by $x \in X$. Recall that

$$\chi_x(c_i) = \hat{c}_i(x). \tag{7.4}$$

In this connection we have

LEMMA 7.2. *Let $f \in \mathscr{F}_0(\mathscr{K})$. Then*

$$\mathfrak{F}f(\chi_x) = (Tf)\hat{}(x), \quad x \in X.$$

In particular, $x \mapsto \mathfrak{F}f(\chi_x)$ is a continuous function on X.

Proof. By Lemma 7.1,

$$\mathfrak{F}f(\chi_x) = \langle f\chi_x\rangle_\lambda = \langle Tf, T\overline{\chi_x}\rangle.$$

On the other hand, it follows from (7.3), (7.4) and Theorem 6.4 that

$$T\overline{\chi_x} = \sum_i \overline{\chi_x(c_i)}\, w_i c_i = \sum_i \overline{\hat{c}_i(x)}\, w_i c_i = \delta_x.$$

Therefore, by Proposition 6.5 we see that

$$\mathfrak{F}f(\chi_x) = \langle Tf, \delta_x\rangle = (Tf)\hat{}(x),$$

as desired. q.e.d.

According to Theorem 6.4, one may regard X as a subset of $\mathscr{X}$ by the map $x \mapsto \chi_x$. Then $\mathscr{X}$ becomes a probability space in an obvious manner, where the image measure of μ is denoted by the same symbol. Then we come to an analogue of the Plancherel theorem.

THEOREM 7.3. *The Fourier transform $\mathfrak{F}$ is extended to a unitary map from $L^2(\mathscr{K}, \lambda)$ onto $L^2(\mathscr{X}, \mu)$.*

Proof. Let $f \in \mathscr{F}_0(\mathscr{K})$. By the definition of μ on $\mathscr{X}$ we have

$$\int_{\mathscr{X}} |\mathfrak{F}f(\chi)|^2 \mu(d\chi) = \int_X |\mathfrak{F}f(\chi_x)|^2 \mu(d\chi). \tag{7.5}$$

On the other hand, since $\mathfrak{F}f(\chi_x) = (Tf)\hat{}(x)$ by Lemma 7.2 and the Gelfand map is a unitary map between $L^2(X, \mu)$ and $\mathscr{H}$ by Corollary 5.3,

$$(7.6) \qquad \int_X |\,\mathfrak{F}f(\chi_x)\,|^2\mu(dx) = \int_X |\,(Tf)\hat{}(x)\,|^2\mu(dx) = \|\,Tf\,\|^2 = \langle Tf,\, Tf\rangle.$$

Finally, applying Lemma 7.1, we have

$$(7.7) \qquad \langle Tf,\, Tf\rangle = \langle f\bar{f}\rangle_\lambda = \|\,f\,\|_\lambda^2.$$

Combining (7.5), (7.6) and (7.7), we obtain

$$\int_{\mathfrak{X}} |\,\mathfrak{F}f(\chi)\,|^2\mu(d\chi) = \|\,f\,\|_\lambda^2, \quad f \in \mathscr{F}_0(\mathscr{K}),$$

which means that $\mathfrak{F}$ is isometric. Since both $\mathscr{F}_0(\mathscr{K}) \subset L^2(\mathscr{K}, \lambda)$ and $\hat{\mathscr{A}}_0 \subset L^2(X, \mu) \cong L^2(\mathfrak{X}, \mu)$ are dense subspaces, $\mathfrak{F}$ extends to a unitary map from $L^2(\mathscr{K}, \lambda)$ onto $L^2(\mathfrak{X}, \mu)$. q.e.d.

Thus the Plancherel measure is supported on $\{\chi_x\,;\, x \in X\} \subset \mathfrak{X}$. The phenomenon that the Plancherel measure is supported by a subspace of $\mathfrak{X}$ has been already observed by Jewett [12] in case of a hypergroup. Moreover, this phenomenon is now easily understood as the Fourier transform on a generalized hypergroup is essentially the Gelfand map.

We have an inversion formula for the Fourier transform.

THEOREM 7.4. *For $f \in \mathscr{F}_0(\mathscr{K})$ it holds that*

$$f(c_i) = \int_{\mathfrak{X}} \mathfrak{F}f(\chi)\overline{\chi(c_i)}\mu(d\chi) = \int_X \mathfrak{F}f(\chi_x)\overline{\chi_x(c_i)}\mu(dx).$$

Proof. For each i define $F_i \in \mathscr{F}_\infty(\mathscr{K})$ by

$$F_i(c_j) = \delta_{ij}w_i^{-1}.$$

Note that $TF_i = c_i$. Then for $f \in \mathscr{F}_0(\mathscr{K})$ we have

$$f(c_i) = \langle fF_i\rangle_\lambda = \langle Tf,\, TF_i\rangle = \langle Tf,\, c_i\rangle.$$

Using Corollary 4.2 and Lemma 7.2, we obtain

$$f(c_i) = \int_X (Tf)\hat{}(x)\overline{\hat{c}_i(x)}\mu(dx) = \int_X \mathfrak{F}f(\chi_x)\overline{\chi_x(c_i)}\mu(dx).$$

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The assertion is now immediate q.e.d.

From the viewpoint of Fourier transform the subclass $\{\chi_x ; x \in X\}$ is more important than the whole $\mathscr{X}$. The subclass is characterized as follows, of which proof is easy by observing the functional realization.

PROPOSITION 7.5. *Let $\Lambda \in \mathscr{X}$ and $\xi \in \mathscr{A}_\infty$ be related as $a\xi = \Lambda(a)\xi$ for $a \in \mathscr{A}_0$. Then $\Lambda = \chi_x$ for some $x \in X$ if and only if $\xi \in \mathscr{A}'$.*

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