Properties of some Groups of Wallace Lines.

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1. Take the point P in the arc AB, and let $\angle$ ABP $= \theta$, then the trilinear equation of the Wallace line of P (p say) is

$$a\alpha\cos(C-\theta)\cos(B+\theta)\sin\theta - b\beta\cos\theta\sin(C-\theta)\cos(B+\theta) + c\gamma\cos(C-\theta)\sin(B-\theta)\cos\theta = 0 \qquad \text{(A)}$$

2. (a) *The lines AP, BQ, CR drawn parallel to the sides BC, CA, AB of the triangle ABC.*

In this case p, q, r, by [§ 17] * are readily seen to be concurrent. To find the point, we must solve the equations

for $P(\theta = C - B)$, for $Q(\theta = A)$,

$$a\alpha\cos B\cos C\sin(B-C) - b\beta\cos(B-C)\sin B\cos C + c\gamma\cos B\sin C\cos(B-C) = 0,$$

$$-a\alpha\cos(C-A)\cos A\sin A + b\beta\cos A\sin(C-A)\cos C + c\gamma\cos(C-A)\sin C\cos A = 0.$$

The point is given by

$$\alpha/\cos^2A\cos(B-C) = \beta/\cos^2B\cos(C-A) = \gamma/\cos^2C\cos(A-B). \dagger$$

3. (b) *PQ drawn parallel to AB.*

Then $P(\theta)$, $Q(C-\theta)$ have for Wallace lines, viz., p, equation (A), and q,

$$-a\alpha\cos\theta\cos(A+\theta)\sin(C-\theta) + b\beta\cos(C-\theta)\sin\theta\cos(A+\theta) + c\gamma\cos\theta\sin(A+\theta)\cos(C-\theta) = 0.$$

These lines intersect in c' (say) given by

$$\frac{\alpha}{\cos B} = \frac{\beta}{\cos A} = \frac{\gamma\cos(C-\theta)\cos\theta}{\cos(A+\theta)\cos(B+\theta)},$$

hence Aa', Bb', Cc' meet in the orthocentre of ABC, or the loci of a', b', c' are the respective perpendiculars from A, B, C on the opposite sides.

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* The references in [ ] are to a paper in vol. iii. (pp. 77–93) by Mr J. Alison, entitled *The so-called Simson Line.* To [§§ 19, 27] should be added the further reference *Ed. Times* Reprint, vol. xxxix., p. 121.

† *Cf. Educational Times* Reprint, vol. xxxviii., p. 29.

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4. Take AP = BQ = BR = CS = CT = AV and suppose they subtend the angle θ at the circumference and let the Wallace lines of P, Q, R, S, T, V, *i.e.*, p, q, r, s, t, v intersect in a', b', c', d', e', f', then [§ 17] angles at

$a', c', e' = 2\theta$, acute angles at $b' = \mathrm{C} - 2\theta$, at $d' = \mathrm{A} - 2\theta$, and at $f'\mathrm{B} - 2\theta$

Then r, s ; q, t intersect on $\perp^r$ from A,

t, v ; p, s „ „ „ „ B,

and p, q ; r, v „ „ „ „ C ;

and the intercepts on the respective perpendiculars are

$$a\sin 2\theta, \quad b\sin 2\theta, \quad c\sin 2\theta.$$

5. I get these results from the equations referred to BC, BA as axes. They are

$$p. \quad y\cos(\mathrm{B}+\theta) + x\cos\theta = 2\mathrm{R}\cos\theta\cos(\mathrm{B}+\theta)\sin(\mathrm{C}-\theta) \qquad \text{(i.)}$$

$$q. \quad y\cos(\mathrm{A}+\theta) - x\cos(\mathrm{C}-\theta) = 2\mathrm{R}\cos(\mathrm{C}-\theta)\cos(\mathrm{A}+\theta)\sin\theta \qquad \text{(ii.)}$$

$$r. \quad -y\cos(\mathrm{A}-\theta) + x\cos(\mathrm{C}+\theta) = 2\mathrm{R}\cos(\mathrm{C}+\theta)\cos(\mathrm{A}-\theta)\sin\theta \qquad \text{(iii.)}$$

$$s. \quad y\cos\theta + x\cos(\mathrm{B}+\theta) = 2\mathrm{R}\cos(\mathrm{B}+\theta)\cos\theta\sin(\mathrm{A}-\theta) \qquad \text{(iv.)}$$

t. and v. may be similarly obtained.

p. and s. intersect in $y\sin\mathrm{B} = 2\mathrm{R}\cos\theta\cos(\mathrm{B}+\theta)\cos\mathrm{C}$,

$$x\sin\mathrm{B} = 2\mathrm{R}\cos\theta\cos(\mathrm{B}+\theta)\cos\mathrm{A},$$

hence they intersect on the $\perp^r$ from B at m (say),

i.e. $\perp^r$ from m on BC is $2\mathrm{R}\cos\theta\cos(\mathrm{B}+\theta)\cos\mathrm{C}$,

hence $m\mathrm{E} = 2\mathrm{R}[\sin\mathrm{A}\sin\mathrm{C} - \cos\theta\cos(\mathrm{B}+\theta)]$.

6. Again r, s intersect in d' so that

$$d'\mathrm{D} = 2\mathrm{R}\cos(\mathrm{B}+\theta)\cos(\mathrm{C}+\theta) ;$$

hence $d'\mathrm{D}\cos(\mathrm{A}+\theta) = f'\mathrm{E}\cos(\mathrm{B}+\theta) = b'\mathrm{F}\cos(\mathrm{C}+\theta)$,

and also mf'

$= 2\mathrm{R}[\sin\mathrm{A}\sin\mathrm{C} - \cos\theta\cos(\mathrm{B}+\theta) - \cos(\mathrm{C}+\theta)\cos(\mathrm{A}+\theta)]$

$= \mathrm{R}[\cos(\mathrm{C}-\mathrm{A}) + \cos\mathrm{B} - \cos\mathrm{B} - \cos(\mathrm{B}+2\theta) - \cos(\mathrm{C}+\mathrm{A}+2\theta) - \cos(\mathrm{C}-\mathrm{A})]$

$= b\sin 2\theta$.

Hence $ld' : mf' : nb' = a : b : c$.

Again $\qquad mH = mE - EH = 2R\sin\theta\sin(B+\theta)$,

i.e. $\quad Hl : Hm : Hn = \sin(A+\theta) : \sin(B+\theta) : \sin(C+\theta)$;

and $\qquad Hf' = 2R\sin\theta\sin(B-\theta)$,

$$\therefore Hd' : Hf' : Hb' = \sin(A-\theta) : \sin(B-\theta) : \sin(C-\theta).$$

7. By the cosine method we readily get

$$b'm = a\sin\theta \text{ and } \therefore b'm : d'n : f'l = a : b : c;$$

also $\qquad b'l = b\sin\theta$ and $\therefore f'n : b'l : d'm = a : b : c$.

The areas of $mb'lf'$ and $md'nf'$ are respectively

$$R^2\sin^2\theta[2\sin A\sin B\sin C(1+2\cos 2\theta) - (\cos 2C - \cos 2A)\sin 2\theta]$$

and $R^2\sin^2\theta[2\sin A\sin B\sin C(1+2\cos 2\theta) + (\cos 2C - \cos 2A)\sin 2\theta]$,

hence area of hexagon $\qquad = 4\sin A\sin B\sin C R^2\sin^2\theta(1+2\cos 2\theta)$.

8. We may note that the angles are

$$\left.\begin{aligned} Hb'l &= Hf'l = A+\theta,\\ Hb'm &= Hd'm = B+\theta,\\ Hd'n &= Hf'n = C+\theta,\end{aligned}\right\} \text{ and } \left\{\begin{aligned} Hmd' &= Hnd' = A-\theta,\\ Hlf' &= Hnf' = B-\theta,\\ Hlb' &= Hmb' = C-\theta.\end{aligned}\right.$$

9. The intersection of p, v, *i.e.*, a' lies on the line

$$y/b\cos(B-C) + x/b\cos C = 1,$$

which cuts BC at a distance from B, towards C, $=$ CD, and so for the analogous lines.

10. PQRS is a square, and p, q, r, s are the Wallace lines of P, Q, R, S.

$$p, q \text{ meet in } a; \quad q, r \text{ in } b; \quad r, s \text{ in } c; \quad \text{and } s, p \text{ in } d.$$

Then since P, R; Q, S are diametral points, b, d are on the nine-point circle of the triangle, [§ 19] and, from the symmetry of the figure, bd is a diameter.

p, r; q, s; cut at right angles [§ 19] and the angles formed by p, q; r, s; respectively are half right angles [§ 17].

Hence a, b, c, d are concyclic.

Now $\qquad \angle acb = \angle adb = \angle gfk \quad \therefore gf \parallel ac$,

and as g is mid point of ak $\therefore f$ is mid point of ck,

$$\therefore df \text{ bisects } ck \text{ at right angles},$$

and $ac = 2fg = \rho\sqrt{2}$, where $\rho =$ radius of nine-point circle.

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Again since $\angle\, bed =$ right angle $= 2^{ce} \angle\, bcd,$

and ac is a diameter, $\therefore\ e$ is centre of circle $abcd$.

Hence $eg,\ em \parallel$ to $bc,\ cd$.

We have $a\mathrm{N}^2 + c\mathrm{N}^2 = 2\mathrm{N}e^2 + 2ea^2 = 6\rho^2$

11. Let ab, bc, cd, da be represented by a', b', c', d' respectively,

and take $\angle\, cad = \phi,$

then $c' = 2\sqrt{2}\rho\sin\phi, \qquad d' = 2\sqrt{2}\rho\cos\phi,$

$$a' = 2\rho(\sin\phi - \cos\phi), \quad b' = 2\rho(\sin\phi + \cos\phi).$$

From the figure we see $a'^2 + b'^2 = ac^2 = c'^2 + d'^2$,

and rectangle $bgdl = \frac{1}{2}a'b'$, rectangle $bfdm = \frac{1}{2}c'd'$,

$\therefore$ sum of rectangles $= abcd$;

$$a\mathrm{N}^2 = d'^2 + \rho^2 - 2d'\rho\cos\left(\phi - \frac{\pi}{4}\right) = \rho^2[3 + 2(\cos 2\phi - \sin 2\phi)],$$

$$c\mathrm{N}^2 = \rho^2[3 - 2(\cos 2\phi - \sin 2\phi)].$$

12. Let h be the mid point of the third diagonal kn,

then $bh = \frac{1}{2}kn = dh,$

$$= \tfrac{1}{2}\sqrt{a'^2 + b'^2} = ae = \rho\sqrt{2},$$

hence, since we know that $e\mathrm{N}h$ is a straight line, h is on the Nine-point circle.

13. PQRSTV is a regular six-side in the circle and the Wallace lines are indicated by p', q', r', s', t', v' respectively.

The angles [§ 17] at f, b, d are each 30°,

and at a, c, e, are supplements of 30°.

By [§ 19] p', s'; q', t'; r', v' intersect at right angles in g, h, k respectively, on the nine-point circle.

From the symmetry of the figure ghk is an equilateral triangle, or we may prove this by finding the projections of gh on BC and at right angles thereto.

The projections will be found to be respectively

$$\frac{\mathrm{R}\sqrt{3}}{2}\cos\left(\mathrm{B} - \mathrm{C} + 4\theta + \frac{\pi}{3}\right), \quad \frac{\mathrm{R}\sqrt{3}}{2}\sin\left(\mathrm{B} - \mathrm{C} + 4\theta + \frac{\pi}{3}\right),$$

hence $gh = \mathrm{R}\sqrt{3}/2 = \rho\sqrt{3}.$

14. Since $\angle kqh = \angle kgh = 60°$, $\therefore qh = qb = qs$, and $qh \parallel fg$. Now $\angle clh = \angle gkh$ $\therefore$ l is mid point of wd, and similarly m of bz, and n of fx.

The $\angle cfg = \angle cky = \angle qhy = \angle hgf$,

$$\therefore cf \text{ is bisected in } h \text{ and } = 2\rho\sqrt{3}.$$

Also q is mid point of ct.

15. It is seen that the arcs ph, qy, rk are equal, and if we suppose the $\angle$ they subtend at the circumference to be ϕ,

then $$wd = 2lh = 4\rho\cos\phi = xf = zb.$$

Since hq bisects $\angle chy$ and is $\parallel hy$

$$\therefore ch = yh \text{ and } cw = gl,$$

hence $$cd = ef = ab \text{ and } bc = de = af$$

$$\therefore bdf \text{ is equilateral, as also is } stv.$$

[Note.—The reader is requested to draw the figures. The following details, which refer to the figures which accompanied the paper, will render the text more intelligible :

In § 4, $A = 60°$, $B = 45°$, $C = 75°$, $\theta = 18°$.

$$\left.\begin{array}{l} vp \text{ meet in } a' \\ pq \;\;,,\;\; ,,\;\; b' \\ \text{and so on to } tv \text{ in } f'. \end{array}\right\} \text{angle } m = 100°,$$

In § 10, $A = 61°$, $B = 39°$, $C = 80°$, $\theta = 20°$,

bc, ad meet in k ; ab, cd in n.

the N.P. circle meets the sides again, viz.,

bc in f, cd in l, da in g, ab in m.

In § 13, $A = 60°$, $B = 48°$, $C = 72°$, $\theta = 16°$.

p', q' meet in a ; q', r' in b, and so on to v', p' in f.

$$\left.\begin{array}{ll} p' \text{ cuts N.P. circle in} & p,\ g\ ; \\ q' & m,\ h\ ; \\ r' & q,\ k\ ; \\ s' & g,\ l\ ; \\ t, & r,\ h\ ; \\ v, & n,\ k. \end{array}\right\} \quad \left.\begin{array}{ll} p' \text{ cuts } r' & \text{in } t, \\ r' & t' \text{ in } s, \\ t' & p' \text{ in } v. \end{array}\right\}$$

H, O, N are the orthocentre, circumcentre, and N.P. centre.]

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