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EFFICIENCY OF A UNIVERSITY TIMETABLE. AN APPLICATION OF ENTROPY OF CHOICE.

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In a large university with a wide range of student options it is often impossible to cater for all allowable student courses of study within the constraints of a practical timetable, and a decision must be made to exclude some options. The entropy of choice available to students is used to develop a measure of timetable efficiency which balances the desirability of having both popular and rarer options available against the need to deny some students of their chosen courses. This efficiency gives a way for ranking the merit of different timetable exclusions. A simple numerical example is used for illustrations.

1. Introduction

The academic rules of a university or college may allow a student to undertake a very wide range of studies at any particular stage. In many universities including that of the author a wide choice is regarded as a desirable feature; the wider the choice the better the opportunities for both breadth of study or for specialization in a variety of areas. However, in practice limitations are imposed by the facilities available (e.g. staff, rooms) and the limited number of available timetable hours

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in the week, so that not every option can be made available. In any case there is no need to provide for unwanted options. A practical timetable will accommodate some allowable programs and exclude others. The purpose of this note is to suggest a measure of the efficiency of a timetable in catering for the wishes of the students. A choice can then be made in deciding preferences for the deletion of student programs because of facilities constraints. A measure of efficiency can be obtained by considering the entropy of choice available to each student. The use of the entropy measure tends to favour both a wider available choice for students, and the provision of rarer options.

2. *Entropy and efficiency of a timetable.*

We consider all distinct programs P_i, $i = 1, \ldots, k$, allowable under the rules of the institution. Let p_i denote the probability that a student would want to select program P_i. Then using the entropy concept of information theory [1] or statistical mechanics [2] as an additive measure of the flexibility of program choice we define,

$$S = -\sum_i p_i \log p_i \, , \quad \sum_i p_i = 1 \, , \tag{2.1}$$

as the entropy of an ideal timetable. It is supposed that each program is catered for in just one way only, otherwise a further addition is required (see §4). It is seen that unwanted programs for which $p_i = 0$ do not contribute with the interpretation $p_i \log p_i = 0$ and may be deleted from consideration.

The practical timetable will cater for a subset of all such allowable programs. We call such programs feasible. We suppose for the present that the timetable caters for each feasible program in one way only. Denote by $\sum_i{}'$ the sum over i corresponding to feasible programs and $\sum_i{}''$ for infeasible or excluded programs. Clearly,

$$\sum_i \equiv \sum_i{}' + \sum_i{}'' \, .$$

Then,

$$S = s' + s'' \, , \tag{2.2}$$

where

$$s' = -\sum_i{}' p_i \log p_i \geqslant 0 ,$$

$$s'' = -\sum_i{}'' p_i \log p_i \geqslant 0 .$$

Thus s'' denotes a deficiency in the entropy sum (2.1) that an ideal timetable would have, and it is tempting to regard the ratio s'/S as the efficiency of the timetable. A simple example will show this definition to be unsatisfactory. Consider the following scheme of five allowable programs.

$$(2.3) \qquad \left\{ \begin{array}{cccccc} & P_1 & P_2 & P_3 & P_4 & P_5 \\ p_i = & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} . \end{array} \right.$$

Then, from (2.1),

$$S = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{2}{16}\log 16$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\log 2 + \frac{3}{8}\log 2 + \frac{1}{2}\log 2$$

$$= (15/8)\log 2 .$$

Omission of any of $P_1, P_2, P_4 \cup P_5$ results in the same deficiency s'' and hence yields the same value of s'. Any reasonable subjective judgement would rank the inclusion of P_1 as feasible as more important than inclusion of P_2 .

However, a modification which takes account of, (i) changes to the probabilities of those programs not excluded once the others have been excluded, (ii) the fraction of the population not catered for at all, will right the situation. The entropy of choice in (2.1) relates to the entropy <u>per student</u>. If students are excluded (assuming they do not convert to some other program) the student population is reduced by a factor $(1 - \sum_i{}'' p_i) = \sum_i{}' p_i$. Furthermore the probabilities relating to choices of feasible programs for this fraction are modified to

$$(2.4) \qquad p_i' = \frac{p_i}{(1-\sum_i{}'' p_i)} = \frac{p_i}{\sum_i{}' p_i}$$

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so that

$$\sum_i{}' p_i' = 1 \ .$$

The entropy of choice of each student choosing a feasible program is

$$\sigma' = -\sum_i{}' p_i' \ \log p_i' \ . \tag{2.5}$$

Thus the overall entropy of choice per student, for all students, including those excluded is,

$$S' = (1-\sum_j{}'' p_j)\ \sigma' \tag{2.6}$$

$$= -\ (1-\sum_j{}'' p_j)\ \sum_i{}' p_i' \ \log p_i' \quad \text{from (2.5)}$$

$$= -\ (1-\sum{}'' p_j)\ \sum_i{}' (p_i/(1-\sum_j{}'' p_j))\log(p_i/(1-\sum_j{}'' p_j)) \quad \text{from (2.4)}$$

$$= s' + (1-\sum_j{}'' p_j)\ \log\ (1-\sum_j{}'' p_j) \tag{2.7}$$

$$= s' + (\sum_j{}' p_j)\ \log\ (\sum_j{}' p_j)\ . \tag{2.8}$$

Then from (2.2),

$$S - S' = s'' - (1-\sum_j{}'' p_j)\ \log\ (1-\sum_j{}'' p_j)\ . \tag{2.9}$$

We regard $S - S'$ as the entropy deficiency or defect of the timetable, and define the efficiency η of the timetable by,

$$\eta = S'/S\ .$$

Evidently $\sum_j{}' p_j \leqslant 1$, so from (2.8) $S' \leqslant s' \leqslant S$ which ensures that $\eta \leqslant 1$. We also see from (2.7) that the difference between S' and s' may well be insignificant if $\sum_i{}'' p_i \ll 1$, that is, if only unpopular programs are excluded.

Let us return to the example (2.3) and calculate the entropy defect (2.9) and the resulting efficiency η for the omission of different program combinations.

(a) Exclusion of P_1 alone gives a defect, $-(\frac{1}{2}\log\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}) = \log 2$,

$$\eta = \frac{7}{15} \text{ or } 46.67\% .$$

(b) Exclusion of P_2 alone gives a defect, $-(\frac{1}{4}\log\frac{1}{4}+\frac{3}{4}\log\frac{3}{4})$

$$= 2\log 2 - (\tfrac{3}{4})\log 3 < \log 2$$

$$\doteq 0.56234 ,$$

$$\eta = \frac{2\log 3}{5\log 2} - \frac{1}{15} \text{ or } 56.73\% .$$

(c) Exclusion of P_4 and P_5 gives a defect, $-(\frac{2}{16}\log\frac{1}{16}+\frac{7}{8}\log\frac{7}{8})$

$$= \frac{25}{8}\log 2 - \frac{7}{8}\log 7$$

$$\doteq 0.46341 ,$$

$$\eta = \frac{7\log 7}{15\log 2} - \frac{2}{3} \text{ or } 64.34\%$$

(d) Exclusion of P_4 or P_5 gives a defect, $-(\frac{1}{16}\log\frac{1}{16}+\frac{15}{16}\log\frac{15}{16})$

$$= 4\log 2 - \frac{15}{16}\log 15$$

$$\doteq 0.23379 ,$$

$$\eta = \frac{\log 15}{2\log 2} - \frac{17}{15} \text{ or } 82.01\% .$$

(e) Exclusion of P_3 alone gives a defect, $-(\frac{1}{8}\log\frac{1}{8}+\frac{7}{8}\log\frac{7}{8})$

$$= 3\log 2 - \frac{7}{8}\log 7$$

$$\doteq 0.37677 ,$$

$$\eta = \frac{7\log 7}{15\log 2} - \frac{3}{5} \text{ or } 71.01\%$$

The exclusions in (a), (b), (c) previously rated the same using s' rather than S' are now ordered more in accordance with intuition. The case (d) illustrates another aspect; defects due to disjoint exclusions are no longer strictly additive and as a result the defect in (c) is less than twice the defect in (d). That this is a general property can easily be shown from (2.9) using convexity properties of the function $x \log x$. The exclusions in (c) and (e) bar the same number of

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students yet, (c) is ranked lower because it reduces diversity more.

3. Combinatorial approach

The entropy and efficiency arrived at from information considerations can also be obtained from statistical weight arguments of statistical mechanics [2].

Suppose we are able to put n_i students into each program P_i. The statistical weight or the number of ways this can be done is

(3.1) $$\Delta\Gamma = \frac{(\sum_i n_i)!}{\prod_i (n_i!)} (G_i)^{n_i}$$

where G_i denotes the number of distinct ways program P_i may be taken. In §2, and for the present here also, we suppose $G_i \equiv 1$. (In the terminology of statistical mechanics we have used Maxwell-Boltzmann statistics; students are distinguishable!)

The appropriate additive measure (for consecutive independent schemes) is the entropy $\log \Delta\Gamma$. From (3.1),

(3.2) $$\log \Delta\Gamma = \log(N!) - \sum_i \log(n_i!)$$

where

(3.3) $$N = \sum_i n_i \; .$$

A practical timetable that excludes students who opt for infeasible programs will have

(3.4) $$\log \Delta\Gamma' = \log(N'!) - \sum_i{}' \log(n_i!)$$

where

(3.5) $N' = \sum_i{}' n_i = N - N''$ (number of students in feasible programs)

with $N'' = \sum_i{}'' n_i$ (number of students excluded by infeasible programs).

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We now suppose that for every $n_i \neq 0$, n_i is sufficiently large that

$$\log(n_i!) \doteq n_i \log n_i - n_i \ .$$

Then, from (3.2) and (3.3),

$$\log \Delta\Gamma \doteq -N \sum_i (n_i/N) \log (n_i/N) \ . \tag{3.6}$$

Similarly from (3.4) and (3.5)

$$\log \Delta\Gamma' \doteq -N' {\sum_i}' (n_i/N') \log (n_i/N')$$

$$= N \left[-{\sum_i}' (n_i/N) \log(n_i/N) + {\sum_i}' (n_i/N) \log (N'/N) \right]$$

$$= N \left[-{\sum_i}' (n_i/N) \log(n_i/N) + {\sum_i}' (n_i/N) \log \left({\sum_j}' n_j/N \right) \right], \text{ using (3.5)}. \tag{3.7}$$

By making the substitutions

$$p_i = n_i/N$$

and $$S = (1/N) \log \Delta\Gamma \ , \ S' = (1/N) \log \Delta\Gamma'$$

for the entropies per offering student, (3.6) and (3.7) correspond exactly to the expressions (2.1) and (2.8) of §2 confirming the approach adopted there.

4. Further remarks

If in (3.1) we allow for the possibility that $G_i > 1$, and assuming each alternative is to be equally populated, $\log \Delta\Gamma$ is increased by $N \sum_i (n_i/N) \log G_i$. Similarly $\log \Delta\Gamma'$ is increased by $N {\sum_i}' (n_i/N) \log G_i'$, where $G_i' \leqslant G_i$ denotes the number of alternatives for program P_i in the actual timetable. This is accommodated in the formulation of §2 by distributing the total probability for program P_i equally into each alternative. Any preferences for different alternatives would entail unequal allocations of this total probability.

It could be argued that students excluded from their first choice would make some other less than ideal choice but still continue. If such students choose feasible programs with the same relative probabilities

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as those catered for (i.e. with the probabilities p_i' of (2.4)), the factor $(1 - \sum_j'' p_j)$ in (2.6) should be omitted. Then the corresponding entropy is

$$S^* = \sigma' = \frac{s'}{\sum_i' p_i} + \log\left(\sum_i' p_i\right)$$

and the efficiency is

$$\eta^* = S^*/S .$$

Although efficiency and entropy have been discussed from the viewpoint of a university timetable, it is thought that similar considerations would apply to other scheduling or resources allocation problems where restrictions on free choices might be necessary. The methods adopted appear to give measures, or at least rankings, in agreement with subjective assessments.

References

[1] A.I. Khinchin, *Mathematical foundations of information theory* (Dover Publications, New York, 1957).

[2] L.D. Landau and E.M. Lifshitz, *Statistical Physics*, (Pergamon Press, London, 1958).

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