Bull. Aust. Math. Soc. **97** (2018), 453–458
doi:10.1017/S0004972718000138

INEQUALITIES ASSOCIATED WITH RATIOS OF GAMMA FUNCTIONS

JENICA CRINGANU

(Received 4 January 2018; accepted 19 January 2018; first published online 23 April 2018)

Abstract

We use properties of the gamma function to estimate the products $\prod_{k=1}^{n}(4k-3)/4k$ and $\prod_{k=1}^{n}(4k-1)/4k$, motivated by the work of Chen and Qi ['Completely monotonic function associated with the gamma function and proof of Wallis' inequality', *Tamkang J. Math.* **36**(4) (2005), 303–307] and Mortici *et al.* ['Completely monotonic functions and inequalities associated to some ratio of gamma function', *Appl. Math. Comput.* **240** (2014), 168–174].

2010 *Mathematics subject classification*: primary 33B15; secondary 11Y60, 40A05.

Keywords and phrases: gamma function, Wallis ratio, inequalities, complete monotonicity.

1. Introduction

Chen and Qi [2] proved the following inequalities for the Wallis ratio:

$$\frac{1}{\sqrt{\pi(n+a)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+b)}}, \quad \text{for all } n \ge 1, \tag{1.1}$$

with the best possible constants $a = 4/\pi - 1$ and $b = 1/4$. These inequalities are a consequence of the complete monotonicity on $(0,\infty)$ of the function

$$x \to \ln \frac{x\Gamma(x)}{\sqrt{(x+\frac{1}{4})}\Gamma(x+\frac{1}{2})},$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-x}\,dt$ for $x > 0$ is the gamma function. Mortici *et al.* [3] found the following inequalities:

$$\frac{a}{\sqrt[3]{n^2}} \le \frac{1\cdot 4\cdots(3n-2)}{3\cdot 6\cdots(3n)} < \frac{b}{\sqrt[3]{n^2}}, \qquad \frac{c}{\sqrt[3]{n}} \le \frac{2\cdot 5\cdots(3n-1)}{3\cdot 6\cdots(3n)} < \frac{d}{\sqrt[3]{n}},$$

where the constants

$$a = \frac{1}{3} \approx 0.3333, \quad b = \frac{1}{\Gamma(\frac{1}{3})} \approx 0.3732, \quad c = \frac{2}{3} \approx 0.6666, \quad d = \frac{1}{\Gamma(\frac{2}{3})} \approx 0.7384$$

are sharp. The inequalities on the left-hand sides hold with equality if and only if $n = 1$. These inequalities are a consequence of the complete monotonicity on $(0, \infty)$ of the functions

$$x \to \ln \frac{\left(\frac{1}{2\pi}\sqrt{3}\Gamma(\frac{2}{3})\right)^3}{x^2\left(\frac{\Gamma(x+\frac{1}{3})}{\Gamma(x+1)\Gamma(\frac{1}{3})}\right)^3}, \quad x \to -\ln \frac{x\left(\frac{\Gamma(x+\frac{2}{3})}{\Gamma(x+1)\Gamma(\frac{2}{3})}\right)^3}{(\frac{1}{\Gamma(\frac{2}{3})})^3}.$$

Inspired by Mortici *et al.* [3], we consider the following products for an integer $n \geq 1$:

$$P_1 = \frac{1 \cdot 5 \cdots (4n-3)}{4 \cdot 8 \cdots (4n)}, \quad P_2 = \frac{2 \cdot 6 \cdots (4n-2)}{4 \cdot 8 \cdots (4n)}, \quad P_3 = \frac{3 \cdot 7 \cdots (4n-1)}{4 \cdot 8 \cdots (4n)}.$$

Note that $P_2 = (2n-1)!!/(2n)!!$, for which we already have the estimate (1.1). Expressing P_1 and P_3 in terms of the gamma function by

$$P_1 = \frac{\Gamma(n+\frac{1}{4})}{\Gamma(n+1)\Gamma(\frac{1}{4})}, \quad P_3 = \frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{3}{4})} \tag{1.2}$$

motivates us to consider the functions

$$x \to \ln \frac{\left(\frac{1}{2\pi}\sqrt{2}\Gamma(\frac{3}{4})\right)^4}{x^3\left(\frac{\Gamma(x+\frac{1}{4})}{\Gamma(x+1)\Gamma(\frac{1}{4})}\right)^4}, \quad x \to -\ln \frac{x\left(\frac{\Gamma(x+\frac{3}{4})}{\Gamma(x+1)\Gamma(\frac{3}{4})}\right)^4}{\left(\frac{1}{\Gamma(\frac{3}{4})}\right)^4}.$$

We prove that these functions are completely monotonic on $(0, \infty)$ and, as a result, we establish sharp inequalities for P_1 and P_3.

2. The main results

The digamma function $\psi : (0, \infty) \to R$ is defined by

$$\psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

and its derivatives $\psi', \psi'', \ldots$ are the polygamma functions. We have the following integral representations:

$$\psi^{(n)}(x) = (-1)^{n-1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\, dt \tag{2.1}$$

and

$$\frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-xt}\, dt \tag{2.2}$$

for every real number $x > 0$ and integer $n \geq 1$ (see, for example, [1]).

Recall that a function $s : (0, \infty) \to R$ is completely monotonic if it is infinitely differentiable on $(0, \infty)$ and $(-1)^n s^{(n)}(x) \geq 0$, for every real $x > 0$ and integer $n \geq 0$.

https://doi.org/10.1017/S0004972718000138 Published online by Cambridge University Press

As a consequence of the Hausdorff–Bernstein–Widder theorem (see [4]), a function $s(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$s(x)=\int_0^\infty e^{-xt}\psi(t)\,dt,$$

where ψ is a nonnegative function on $(0,\infty)$ such that the integral converges for all $x>0$ (see [4]). Now we can state and prove our results.

THEOREM 2.1. *The function* $f:(0,\infty)\to R$ *given by*

$$f(x)=\ln\frac{\left(\frac{1}{2\pi}\sqrt{2}\Gamma(\frac{3}{4})\right)^4}{x^3\left(\frac{\Gamma(x+\frac{1}{4})}{\Gamma(x+1)\Gamma(\frac{1}{4})}\right)^4}$$

is completely monotonic on $(0,\infty)$.

PROOF. First observe that

$$f(x)=4\ln\left(\frac{1}{2\pi}\sqrt{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)\right)-3\ln x-4\ln\Gamma\left(x+\frac{1}{4}\right)+4\ln\Gamma(x+1).$$

By a standard calculation,

$$f''(x)=\frac{3}{x^2}-4\psi'\left(x+\frac{1}{4}\right)+4\psi'(x+1).$$

Using (2.1) and (2.2),

$$f''(x)=3\int_0^\infty te^{-xt}\,dt-4\int_0^\infty\frac{te^{-(x+\frac{1}{4})t}}{1-e^{-t}}\,dt+4\int_0^\infty\frac{te^{-(x+1)t}}{1-e^{-t}}\,dt=\int_0^\infty\frac{t\varphi(\frac{t}{4})}{e^t-1}e^{-xt}\,dt,$$

where $\varphi(t)=3e^{4t}-4e^{3t}+1$. Since $\varphi'(t)=12e^{3t}(e^t-1)>0$ for all $t>0$, it follows that φ is strictly increasing on $[0,\infty)$ and so $\varphi(t)>\varphi(0)=0$ for all $t>0$.

According to the Hausdorff–Bernstein–Widder theorem, f'' is completely monotonic, that is, $(-1)^n(f'')^{(n)}\geq 0$, or

$$(-1)^n(f)^{(n)}\geq 0,\quad \text{for all } n\geq 2. \tag{2.3}$$

In particular, $f''>0$, so that f' is strictly increasing. Since $f'(\infty)=0$, it follows that $f'<0$. Thus, f is strictly decreasing with $f(\infty)=0$ and so $f>0$.

Finally, (2.3) is true also for $n\in\{0,1\}$, so f is completely monotonic. □

THEOREM 2.2. *Define the function* $g:(0,\infty)\to R$ *by*

$$g(x)=\ln\frac{x\left(\frac{\Gamma(x+\frac{3}{4})}{\Gamma(x+1)\Gamma(\frac{3}{4})}\right)^4}{\left(\frac{1}{\Gamma(\frac{3}{4})}\right)^4}.$$

Then $-g$ *is completely monotonic on* $(0,\infty)$.

https://doi.org/10.1017/S0004972718000138 Published online by Cambridge University Press

PROOF. Observe that

$$g(x) = \ln x + 4\ln\Gamma(x+\tfrac{3}{4}) - 4\ln\Gamma(x+1)$$

and so

$$g''(x) = -\frac{1}{x^2} + 4\psi'\Big(x+\frac{3}{4}\Big) - 4\psi'(x+1).$$

Using (2.1) and (2.2),

$$g''(x) = -\int_0^\infty te^{-xt}\,dt + 4\int_0^\infty \frac{te^{-(x+\frac{3}{4})t}}{1-e^{-t}}\,dt - 4\int_0^\infty \frac{te^{-(x+1)t}}{1-e^{-t}}\,dt = \int_0^\infty \frac{t\phi(\frac{t}{4})}{e^t-1}e^{-xt}\,dt,$$

where $\phi(t) = 4e^t - e^{4t} - 3$.

Since $\phi'(t) = 4e^t(1-e^{3t}) < 0$ for all $t > 0$, we deduce that ϕ is strictly decreasing. Thus, $\phi(t) < \phi(0) = 0$ for all $t > 0$. From the Hausdorff–Bernstein–Widder theorem, $-g''$ is completely monotonic, that is, $(-1)^n(-g'')^{(n)} \geq 0$, or

$$(-1)^n(-g)^{(n)} \geq 0, \quad \text{for all } n \geq 2. \tag{2.4}$$

Since $g'' < 0$, it follows that g' is strictly decreasing. But $g'(\infty) = 0$, so $g' > 0$. Thus, g is strictly increasing with $g(\infty) = 0$ and so $g < 0$. Consequently, (2.4) is also true for $n \in \{0, 1\}$, so $-g$ is completely monotonic. □

As a consequence of the complete monotonicity of the functions f and $-g$, we can give the following sharp inequalities for P_1 and P_3.

COROLLARY 2.3. *For all integers $n \geq 1$,*

$$\frac{a}{\sqrt[4]{n^3}} \leq \frac{1\cdot 5\cdots(4n-3)}{4\cdot 8\cdots(4n)} < \frac{b}{\sqrt[4]{n^3}},$$

with the best constants $a = \frac{1}{4} = 0.25$ and $b = 1/\Gamma(\frac{1}{4}) = 0.2758\ldots$.

PROOF. Since f is completely monotonic, it is also strictly decreasing. Thus, for every integer $n \geq 1$,

$$f(\infty) < f(n) \leq f(1).$$

From (1.2) and a standard computation,

$$1 < \frac{1}{2\pi}\sqrt{2}\Gamma\Big(\frac{3}{4}\Big)\Big/\frac{1\cdot 5\cdots(4n-3)}{4\cdot 8\cdots(4n)}\sqrt[4]{n^3} \leq \exp\Big\{\frac{1}{4}f(1)\Big\}$$

or

$$\frac{\sqrt{2}\Gamma(\frac{3}{4})}{2\pi\sqrt[4]{n^3}}\exp\Big\{-\frac{1}{4}f(1)\Big\} \leq \frac{1\cdot 5\cdot\cdots\cdot(4n-3)}{4\cdot 8\cdot\cdots\cdot(4n)} < \frac{\sqrt{2}\Gamma(\frac{3}{4})}{2\pi\sqrt[4]{n^3}},$$

so that

$$\frac{1}{4\sqrt[4]{n^3}} \leq \frac{1\cdot 5\cdot\cdots\cdot(4n-3)}{4\cdot 8\cdot\cdots\cdot(4n)} < \frac{\sqrt{2}\Gamma(\frac{3}{4})}{2\pi\sqrt[4]{n^3}} = \frac{1}{\Gamma(\frac{1}{4})\sqrt[4]{n^3}},$$

which is the desired conclusion. □

https://doi.org/10.1017/S0004972718000138 Published online by Cambridge University Press

REMARK 2.4. Since $\lim_{n\to\infty} f(n) = 0$, it follows that

$$\lim_{n\to\infty} \frac{1\cdot 5\cdots(4n-3)}{4\cdot 8\cdots(4n)}\sqrt[4]{n^3} = \frac{1}{\Gamma(\frac{1}{4})}. \tag{2.5}$$

(In fact, we can see that $\lim_{n\to\infty} P_1 n^{3/4} = 1/\Gamma(\frac{1}{4})$, by simple application of the asymptotic result $\Gamma(n+a)/\Gamma(n+b) \sim n^{a-b}$ as $n \to \infty$.) The left-hand inequality of (2.5) can be improved in the following way. For $r \in (0, 1/\Gamma(\frac{1}{4}))$, there exists $n_r \in N$ such that

$$\frac{r}{\sqrt[4]{n^3}} < \frac{1\cdot 5\cdots(4n-3)}{4\cdot 8\cdots(4n)} \quad \text{for all } n \geq n_r.$$

COROLLARY 2.5. *For all integers $n \geq 1$,*

$$\frac{c}{\sqrt[4]{n}} \leq \frac{3\cdot 7\cdots(4n-1)}{4\cdot 8\cdots(4n)} < \frac{d}{\sqrt[4]{n}},$$

with the best constants $c = \frac{3}{4} = 0.75$ and $d = 1/\Gamma(\frac{3}{4}) = 0.8160\ldots$.

PROOF. Since $-g$ is completely monotonic, we deduce that g is strictly increasing. Then, for all integers $n \geq 1$,

$$g(1) \leq g(n) < g(\infty).$$

From 1.2 and a standard computation,

$$\frac{3}{4}\Gamma\Big(\frac{3}{4}\Big) \leq \sqrt[4]{n}\frac{\Gamma(n+\frac{3}{4})}{\Gamma(n+1)\Gamma(\frac{3}{4})}\Big/\frac{1}{\Gamma(\frac{3}{4})} < 1$$

or

$$\frac{3}{4\sqrt[4]{n}} \leq \frac{3\cdot 7\cdots(4n-1)}{4\cdot 8\cdots(4n)} < \frac{1}{\Gamma(\frac{3}{4})\sqrt[4]{n}},$$

which is the desired conclusion. □

REMARK 2.6. Since $\lim_{n\to\infty} g(n) = 0$, it follows that

$$\lim_{n\to\infty} \frac{3\cdot 7\cdots(4n-1)}{4\cdot 8\cdots(4n)}\sqrt[4]{n} = \frac{1}{\Gamma(\frac{3}{4})}.$$

The left-hand inequality can be improved in the following way. For $r \in (0, 1/\Gamma(\frac{3}{4}))$, there exists $n_r \in N$ such that

$$\frac{r}{\sqrt[4]{n}} < \frac{3\cdot 7\cdots(4n-1)}{4\cdot 8\cdots(4n)} \quad \text{for all } n \geq n_r.$$

REMARK 2.7. We can apply the same approach to P_2, using the function

$$x \to \ln \frac{\Big(\frac{1}{\Gamma(\frac{1}{2})}\Big)^2}{x\Big(\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)\Gamma(\frac{1}{2})}\Big)^2}.$$

https://doi.org/10.1017/S0004972718000138 Published online by Cambridge University Press

We obtain

$$\frac{e}{\sqrt{n}} \leq \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} < \frac{f}{\sqrt{n}}, \quad \text{for all } n \geq 1,$$

with the best constants $e = \frac{1}{2} = 0.5$ and $f = 1/\sqrt{\pi} = 0.5641\ldots$, but these inequalities are weaker than (1.1).

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Dover, New York, 1972).

[2] C. P. Chen and F. Qi, 'Completely monotonic function associated with the gamma function and proof of Wallis' inequality', *Tamkang J. Math.* **36**(4) (2005), 303–307.

[3] C. Mortici, V. G. Cristea and D. Lu, 'Completely monotonic functions and inequalities associated to some ratio of gamma function', *Appl. Math. Comput.* **240** (2014), 168–174.

[4] D. Widder, *The Laplace Transform* (Princeton University Press, Princeton, NJ, 1941).

JENICA CRINGANU, Department of Mathematics and Computer Science,
'Dunarea de Jos' University of Galati, Domneasca No. 111, Galati, Romania
e-mail: jcringanu@ugal.ro