5
Quantum electrodynamics

The single most important field theory is electromagnetism. It is responsible for atomic structure and for the great diversity of materials around us: solids, liquids, and gases. The development of nonrelativistic many-body theory was stimulated primarily by solid state and condensed matter physics, where the potentials used all derive from electromagnetism. This compels us to study quantum electrodynamics at high temperatures and densities where the motion of the electrons becomes relativistic. In metals, the density of plasma electrons rarely exceeds a few electrons per cubic angstrom. This means that the Fermi momentum, $k_{\rm F} = (3\pi^2 n_e)^{1/3}$, is of order 10 keV at most. Unfortunately, it is difficult to test relativistic many-body theory in the basement of the physics building in table-top experiments! Our attention must then be directed toward astrophysical and cosmological applications. Dense astrophysical objects, such as white dwarf stars, will be considered in Chapter 16.

There is another reason for developing the theory of QED at high temperature and density, and that is the extension to a nonabelian gauge theory, quantum chromodynamics (QCD). We may be able to study QCD at high energy density in terrestrial experiments by colliding energetic heavy nuclei (see Chapter 14).

5.1 Quantizing the electromagnetic field

First, let us consider the electromagnetic field in the absence of charged particles. From classical physics we can write down a field strength tensor as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.1}$$

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where A_μ is the vector potential. The electric and magnetic fields are

$$
\begin{aligned}
E_i &= -F_{0i} = F_{i0} \\
B_i &= \tfrac{1}{2}\epsilon_{ijk}F_{jk} \quad \text{or} \quad \mathbf{B} = \nabla \times \mathbf{A}
\end{aligned}
\tag{5.2}
$$

The Lagrangian density is

$$
\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.3}
$$

Notice that $F_{\mu\nu}$ is invariant under the local (or x-dependent) transformation

$$
A_\mu(\mathbf{x}, t) \to A_\mu(\mathbf{x}, t) - \partial_\mu \alpha(\mathbf{x}, t) \tag{5.4}
$$

where $\alpha(\mathbf{x}, t)$ is some smooth function of x_μ. Since the field strength tensor is invariant under this transformation, so are the electric and magnetic fields, and so is the Lagrangian. This is called a U(1) gauge symmetry.

To quantize the theory and to compute a partition function, we need a Hamiltonian formulation. In order to do this, we must agree on a gauge to work in. A convenient gauge for this purpose is the axial gauge

$$
A_3(\mathbf{x}, t) = 0 \tag{5.5}
$$

Actually (5.5) does not entirely fix the gauge, as any gauge function $\alpha(x, y, t)$ that is independent of z leaves (5.5) unchanged. We shall fix this residual gauge freedom later.

The conjugate momenta are defined by

$$
\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F_{0\mu} \tag{5.6}
$$

(This should not viewed be as a tensor equation but as true component by component.) Since $F_{\mu\nu}$ is antisymmetric in its Lorentz indices, it follows that $\pi_0 = 0$. Thus A_0 is not a dynamical field, it is a dependent field. The independent fields are A_1 and A_2 with conjugate momenta

$$
\begin{aligned}
\pi_1 &= F_{01} = -E_1 = \partial_0 A_1 - \partial_1 A_0 \\
\pi_2 &= F_{02} = -E_2 = \partial_0 A_2 - \partial_2 A_0
\end{aligned}
\tag{5.7}
$$

These two independent fields actually correspond to the two polarization degrees of freedom of free radiation.

The z component of the electric field is

$$
E_3 = F_{30} = \partial_3 A_0 \tag{5.8}
$$

Since $A_3 = 0$ there is no momentum conjugate to A_3; hence E_3, like A_0, must be a dependent field. We can determine E_3 by an application of

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Gauss's law, which, in the absence of charged particles, is

$$\nabla \cdot \mathbf{E} = 0 \tag{5.9}$$

Thus,

$$E_3(x,y,z,t) = \int_{z_0}^{z} dz' [\partial_1 \pi_1(x,y,z',t) + \partial_2 \pi_2(x,y,z',t)] + P(x,y,t) \tag{5.10}$$

and

$$A_0(x,y,z,t) = \int_{z_0}^{z} dz'' E_3(x,y,z'',t) + Q(x,y,t) \tag{5.11}$$

Here, P and Q are smooth functions of x, y, and t. The gauge is not completely fixed until these two functions are specified. They may be determined by specifying the values of A_0 and E_3 at $z = z_0$ for all x, y, and t.

The Hamiltonian may now be found from the Lagrangian in the canonical way (see (2.23)). Dropping surface terms we find the well-known result

$$\mathcal{H} = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2 = \tfrac{1}{2}\pi_1^2 + \tfrac{1}{2}\pi_2^2 + \tfrac{1}{2}E_3^2(\pi_1,\pi_2) + \tfrac{1}{2}\mathbf{B}^2 \tag{5.12}$$

The partition function is

$$Z = \int [d\pi_1][d\pi_2] \int_{\text{periodic}} [dA_1][dA_2] \times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi_1 \frac{\partial A_1}{\partial \tau} + i\pi_2 \frac{\partial A_2}{\partial \tau} - \mathcal{H}\right)\right] \tag{5.13}$$

Since we have a free-field theory, we should be able to calculate Z exactly. However, in the present form this is not easy since it is a rather complicated function of π_1 and π_2.

To put (5.12) and (5.13) in a more manageable form we insert the identity

$$1 = \int [d\pi_3] \delta(\pi_3 + E_3(\pi_1,\pi_2)) \tag{5.14}$$

and replace E_3 with $-\pi_3$ in the integrand. Note that, despite the suggestive notation, π_3 is not the conjugate momentum of any field; (5.14) is simply the condition on E_3 that ensures Gauss's law. Now

$$\delta(\pi_3 + E_3(\pi_1,\pi_2)) = \delta(\nabla \cdot \boldsymbol{\pi}) \det\left(\frac{\partial(\nabla \cdot \boldsymbol{\pi})}{\partial \pi_3}\right) \tag{5.15}$$

Furthermore,

$$\det\left(\frac{\partial(\nabla \cdot \boldsymbol{\pi})}{\partial \pi_3}\right) = \det(\partial_3) \tag{5.16}$$

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Thus far we have

$$Z = \int [d\pi_1][d\pi_2][d\pi_3] \int_{\text{periodic}} [dA_1][dA_2]\delta(\nabla \cdot \boldsymbol{\pi}) \det(\partial_3)$$
$$\times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi_1 \frac{\partial A_1}{\partial \tau} + i\pi_2 \frac{\partial A_2}{\partial \tau} - \tfrac{1}{2}\boldsymbol{\pi}^2 - \tfrac{1}{2}\mathbf{B}^2\right)\right] \quad (5.17)$$

The constraint of Gauss's law can be implemented alternatively by using the integral representation of the delta function. In vacuum field theory we would write

$$\delta(\nabla \cdot \boldsymbol{\pi}) = \int [dA_0] \exp\left(i \int d^4x \, A_0 \, \nabla \cdot \boldsymbol{\pi}\right) \quad (5.18)$$

where A_0 is some auxiliary field, or a Lagrange multiplier field. At finite temperature we make the replacement $t \to -i\tau$ and now also $A_0 \to iA_0$. Thus

$$\delta(\nabla \cdot \boldsymbol{\pi}) = \int [dA_0] \exp\left(i \int_0^\beta d\tau \int d^3x \, A_0 \, \nabla \cdot \boldsymbol{\pi}\right) \quad (5.19)$$

Using this representation to implement Gauss's law, we may integrate over $\boldsymbol{\pi}$ directly:

$$Z = \int [d\pi_1][d\pi_2][d\pi_3] \int [dA_0][dA_1][dA_2] \det(\partial_3)$$
$$\times \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi_1 \frac{\partial A_1}{\partial \tau} + i\pi_2 \frac{\partial A_2}{\partial \tau} - i\nabla A_0 \cdot \boldsymbol{\pi} - \tfrac{1}{2}\boldsymbol{\pi}^2 - \tfrac{1}{2}\mathbf{B}^2\right)\right]$$
$$= \int [dA_0][dA_1][dA_2] \det(\partial_3)$$
$$\times \exp\left\{\int_0^\beta d\tau \int d^3x \left[\tfrac{1}{2}\left(i\frac{\partial \mathbf{A}}{\partial \tau} - i\nabla A_0\right)^2 - \tfrac{1}{2}\mathbf{B}^2\right]\right\} \quad (5.20)$$

where $\mathbf{A} = (A_1, A_2, 0)$ and we have ignored an irrelevant overall normalization constant. Notice that the argument of the exponential is

$$\tfrac{1}{2}\mathbf{E}^2 - \tfrac{1}{2}\mathbf{B}^2 = \mathcal{L} \quad (5.21)$$

Making this identification and inserting the factor

$$1 = \int [dA_3]\delta(A_3) \quad (5.22)$$

we arrive at

$$Z = \int_{\text{periodic}} [dA^\mu]\delta(A_3) \det(\partial_3) \exp\left(\int_0^\beta d\tau \int d^3x \, \mathcal{L}\right) \quad (5.23)$$

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The axial gauge is not necessarily a convenient gauge to use for practical computations. Furthermore, it is not immediately apparent that (5.23) is a gauge-invariant expression for Z. Take an arbitrary gauge specified by $F = 0$, where F is some function of A^μ and its derivatives. For the axial gauge above, $F = A_3$. For this gauge, (5.23) becomes

$$Z = \int_{\text{periodic}} [dA^\mu]\delta(F) \det\left(\frac{\partial F}{\partial \alpha}\right) \exp\left(\int_0^\beta d\tau \int d^3x\, \mathcal{L}\right) \tag{5.24}$$

Equation (5.24) is manifestly gauge invariant: $\mathcal{L}$ is invariant, the gauge-fixing factor times the Jacobian of the transformation $\delta(F)\det(\partial F/\partial\alpha)$ is invariant, and the integration is over all four components of the vector potential. Equation (5.24) reduces to (5.23) in the case of the axial gauge $A_3 = 0$. We know this is correct since it was derived from first principles in the Hamiltonian formulation of the gauge theory, $Z = \text{Tr}\ \mathrm{e}^{-\beta H}$.

5.2 Blackbody radiation

It is important to verify that (5.24) describes blackbody radiation with two polarization degrees of freedom. We shall do this in two different gauges, the axial gauge $A_3 = 0$ and the covariant Feynman gauge.

In the axial gauge, we rewrite (5.20) as

$$Z = \int [dA_0][dA_1][dA_2] \det(\partial_3)\, \mathrm{e}^{S_0} \tag{5.25}$$

where

$$S_0 = \tfrac{1}{2} \int d\tau \int d^3x\, (A_0, A_1, A_2) \times \begin{pmatrix} \nabla^2 & -\partial_1\partial_\tau & -\partial_2\partial_\tau \\ -\partial_1\partial_\tau & \partial_2^2 + \partial_3^2 + \partial_\tau^2 & -\partial_1\partial_2 \\ -\partial_2\partial_\tau & -\partial_1\partial_2 & \partial_1^2 + \partial_3^2 + \partial_\tau^2 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}$$

We may express the determinant of ∂_3 as a functional integral over a complex ghost field C, which is a Grassmann field with spin 0:

$$\det(\partial_3) = \int [d\bar{C}][dC] \exp\left(\int_0^\beta d\tau \int d^3x\, \bar{C}\partial_3 C\right) \tag{5.26}$$

(This is (2.82) generalized to an infinite number of degrees of freedom.) These ghost fields C and $\bar{C}$ are not physical fields since they do not appear in the Hamiltonian. Furthermore, since they are anticommuting scalar fields they violate the spin-statistics theorem. They are simply a convenient functional integral representation of the determinant of an

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operator. The great usefulness of these fictitious ghost fields will be in nonabelian gauge theories.

In frequency–momentum space the partition function is expressed as

$$\ln Z = \ln\det(\beta p_3) - \tfrac{1}{2}\ln\det D \tag{5.27}$$

where

$$D = \beta^2 \begin{pmatrix} \mathbf{p}^2 & -\omega_n p_1 & -\omega_n p_2 \\ -\omega_n p_1 & \omega_n^2 + p_2^2 + p_3^2 & -p_1 p_2 \\ -\omega_n p_2 & -p_1 p_2 & \omega_n^2 + p_1^2 + p_3^2 \end{pmatrix}$$

Carrying out the determinantal operation,

$$\begin{aligned} \ln Z &= \tfrac{1}{2}\mathrm{Tr}\ln\left(\beta^2 p_3^2\right) - \tfrac{1}{2}\,\mathrm{Tr}\ln\left[\beta^6 p_3^2\left(\omega_n^2 + \mathbf{p}^2\right)^2\right] \\ &= \ln\left\{\prod_n \prod_{\mathbf{p}} \left[\beta^2(\omega_n^2 + \mathbf{p}^2)\right]^{-1}\right\} \\ &= 2V \int \frac{d^3p}{(2\pi)^3}\left[-\tfrac{1}{2}\beta\omega - \ln(1 - \mathrm{e}^{-\beta\omega})\right] \end{aligned} \tag{5.28}$$

Here, $\omega = |\mathbf{p}|$. Comparison with (2.40) shows that (5.28) describes massless bosons with two spin degrees of freedom in thermal equilibrium; in other words, blackbody radiation.

A family of covariant gauges is given by the condition

$$F = \partial^\mu A_\mu - f(\mathbf{x},\tau) = 0 \tag{5.29}$$

where f is an arbitrary function. Under a gauge transformation,

$$F = \partial^\mu(A_\mu - \partial_\mu\alpha) - f = \partial^\mu A_\mu - f - \partial^2\alpha \tag{5.30}$$

and $\partial F/\partial\alpha = -\partial^2$. Inserting into (5.24) yields

$$Z = \int [dA_\mu]\,\det(-\partial^2)\,\delta(\partial^\mu A_\mu - f)\exp\left(\int_0^\beta d\tau \int d^3x\,\mathcal{L}\right) \tag{5.31}$$

The physics contained in Z is unchanged if we first multiply by

$$\exp\left(-\frac{1}{2\rho}\int d\tau\int d^3x\, f^2\right)$$

and then do a functional integration over f,

$$Z = \int [dA_\mu]\,\det(-\partial^2)\exp\left(\int d\tau\int d^3x\,\mathcal{L}_{\mathrm{eff}}\right) \tag{5.32}$$

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where

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\rho}(\partial^\mu A_\mu)^2$$

is the effective Lagrangian, including the gauge-fixing term, and ρ is any real number. The Feynman gauge corresponds to the choice $\rho = 1$ and the Landau gauge to $\rho = 0$.

The partition function should be independent of α and should be the same as in the axial gauge. For simplicity, we examine Z in the Feynman gauge. Then,

$$\int d\tau \int d^3x\, \mathcal{L}_{\text{eff}} = \tfrac{1}{2} \int d\tau \int d^3x\, A_\mu \big(\partial_\tau^2 + \nabla^2\big) A_\mu \tag{5.33}$$

where the summation over μ is Euclidean because in (5.19) we let $A_0 \to iA_0$. We again employ a ghost field to write

$$\det(-\partial^2) = \int [d\bar{C}][dC] \exp\left(\int d\tau \int d^3x\, (\partial^\mu \bar{C})(\partial_\mu C)\right) \tag{5.34}$$

Combining (5.32) with (5.34), we get

$$\ln Z = 2\big(\tfrac{1}{2}\big)\, \mathrm{Tr} \ln\big[\beta^2 \left(\omega_n^2 + \mathbf{p}^2\right)\big] + 4\big(-\tfrac{1}{2}\big)\, \mathrm{Tr} \ln \left[\beta^2 \left(\omega_n^2 + \mathbf{p}^2\right)\right] \tag{5.35}$$

The four degrees of freedom of the A_μ field combine with the two degrees of freedom of the C (ghost) field, which contribute with the opposite sign, to produce just the correct number of physical degrees of freedom. The complex ghost field cancels the unphysical degrees of freedom of the longitudinal and timelike photons. Equation (5.35) is the same as (5.28).

5.3 Diagrammatic expansion

Photons interact with fermions (to be specific, we shall consider electrons) with the interaction Lagrangian

$$\mathcal{L}_{\text{I}} = -e\bar{\psi} \not{A} \psi \tag{5.36}$$

where e is the electronic charge. By far the most frequently used gauges are the covariant gauges. The partition function is

$$Z = \int [d\bar{C}][dC][dA_\mu][d\bar{\psi}][d\psi] \exp\left(\int d\tau \int d^3x\, \mathcal{L}\right) \tag{5.37}$$

where

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{I}}$$

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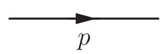
and

$$\mathcal{L}_0 = \bar{\psi}(i\,\partial\!\!\!/ - m + \mu\gamma^0)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
$$- \frac{1}{2\rho}(\partial^\mu A_\mu)^2 + (\partial^\mu \bar{C})(\partial_\mu C)$$

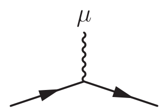
The ghost field does not interact with any other field but serves only to cancel two of the four gauge-field degrees of freedom in the ideal gas term.

The partition function and other physical quantities of interest may be formally expanded in a power series in $\mathcal{L}_\mathrm{I}$ or e. The diagrammatic rules closely parallel those discussed in Chapter 3. The bare propagators and vertex are:

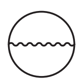
fermion
$$\mathcal{G}_0 = \frac{1}{p\!\!\!/ - m} =$$

$$p_0 = (2n+1)\pi T i + \mu \qquad (5.38)$$

photon
$$\mathcal{D}_0^{\mu\nu} = \frac{1}{p^2}[g^{\mu\nu} - (1-\rho)p^\mu p^\nu/p^2] =$$

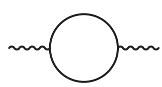
$$p_0 = 2n\pi T i$$

vertex
$$-e\gamma^\mu =$$


As an example, the lowest-order correction to the ideal gas of photons, electrons, and positrons is

$$\ln Z_2 = -\tfrac{1}{2} \qquad (5.39)$$


The photon self-energy at one loop is

$$\Pi_{\mu\nu} = \mathcal{D}^{-1}_{\mu\nu} - \mathcal{D}_{0\,\mu\nu}^{-1} = \qquad (5.40)$$


5.4 Photon self-energy

The photon self-energy is related to the inverse of the full and bare propagators by

$$\Pi_{\mu\nu} = \mathcal{D}^{-1}_{\mu\nu} - \mathcal{D}_{0\,\mu\nu}^{-1} \qquad (5.41)$$

The inverse propagator is related to the propagator by

$$\mathcal{D}^{\mu\alpha}\mathcal{D}^{-1}_{\alpha\nu} = g^\mu{}_\nu \qquad (5.42)$$

The propagator and the self-energy satisfy certain fundamental constraints. To discover them, it is convenient to work with $k_0 = 2n\pi T i$ analytically continued to arbitrary complex values. (Recall our analysis of Section 3.4. This continuation will be taken up again in Chapter 6.) Let

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k^μ be the four-momentum of the photon. Current conservation requires that $\Pi_{\mu\nu}$ be transverse,

$$k^\mu \Pi_{\mu\nu} = 0 \tag{5.43}$$

and gauge invariance requires that

$$k^\mu k^\nu \mathcal{D}_{\mu\nu} = \rho \tag{5.44}$$

in a covariant gauge specified by ρ. Both these constraints hold at $T > 0$, $\mu \neq 0$, as well as in the vacuum. The interested reader is referred to Fradkin [1] for a proof of (5.43). The proof of (5.44) will now be outlined.

Consider making the gauge transformation $A_\mu \to A_\mu - \partial_\mu \alpha$, $\psi \to \mathrm{e}^{ie\alpha}\psi$ in the partition function as expressed in (5.37). All terms are manifestly independent of α apart from the gauge-fixing term, which becomes

$$-\frac{1}{2\rho}(\partial^\mu A_\mu - f)^2$$

where

$$f = \partial^2 \alpha$$

By construction, the partition function is gauge invariant. Therefore, if we functionally differentiate $\ln Z$ with respect to f any number of times, we must get zero. In particular,

$$\begin{aligned}
\frac{\delta \ln Z}{\delta f(\mathbf{x},\tau)} &= \frac{\langle \partial^\mu A_\mu(\mathbf{x},\tau)\rangle}{\rho} - \frac{f(\mathbf{x},\tau)}{\rho} = 0 \\
\frac{\delta^2 \ln Z}{\delta f(\mathbf{x},\tau)\delta f(\mathbf{x}',\tau')} &= \frac{\langle \partial^\mu A_\mu(\mathbf{x},\tau)\partial^\nu A_\nu(\mathbf{x}',\tau')\rangle}{\rho^2} - \frac{\langle \partial^\mu A_\mu(\mathbf{x},\tau)\rangle f(\mathbf{x}',\tau')}{\rho^2} \\
&\quad - \frac{\delta(\tau-\tau')\delta(\mathbf{x}-\mathbf{x}')}{\rho} = 0
\end{aligned} \tag{5.45}$$

Evaluating (5.45) at $f = 0$ and taking the Fourier transform, we obtain (5.44). A constraint on the thermal average of a product of N vector potentials is likewise obtained by differentiating N times $\ln Z$ with respect to f, and then setting $f = 0$.

The propagator, its inverse, and the self-energy, are all symmetric second-rank tensors. Assuming rotational invariance (which would not be correct for a solid) the most general tensor of this type is a linear combination of $g_{\mu\nu}$, $k_\mu k_\nu$, $u_\mu u_\nu$, and $k_\mu u_\nu + k_\nu u_\mu$. Here $u_\mu = (1, 0, 0, 0)$ specifies the rest frame of the many-body system. Taking into account

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(5.41) to (5.44) we obtain the general forms

$$
\begin{aligned}
\Pi^{\mu\nu} &= G P_{\mathrm{T}}^{\mu\nu} + F P_{\mathrm{L}}^{\mu\nu} \\
\mathcal{D}^{\mu\nu} &= \frac{1}{G-k^2} P_{\mathrm{T}}^{\mu\nu} + \frac{1}{F-k^2} P_{\mathrm{L}}^{\mu\nu} + \frac{\rho}{k^2}\frac{k^\mu k^\nu}{k^2} \\
(\mathcal{D}^{-1})^{\mu\nu} &= (G-k^2) P_{\mathrm{T}}^{\mu\nu} + (F-k^2) P_{\mathrm{L}}^{\mu\nu} + \frac{k^\mu k^\nu}{\rho}
\end{aligned}
\tag{5.46}
$$

The quantities F and G are scalar functions of k^0 and $|\mathbf{k}|$. The two projection operators are four-dimensionally transverse, but one is also three-dimensionally transverse (P_{T}) while the other is three-dimensionally longitudinal (P_{L}):

$$
\begin{aligned}
P_{\mathrm{T}}^{00} &= P_{\mathrm{T}}^{0i} = P_{\mathrm{T}}^{i0} = 0 \\
P_{\mathrm{T}}^{ij} &= \delta^{ij} - k^i k^j/\mathbf{k}^2 \\
P_{\mathrm{L}}^{\mu\nu} &= k^\mu k^\nu/k^2 - g^{\mu\nu} - P_{\mathrm{T}}^{\mu\nu}
\end{aligned}
\tag{5.47}
$$

These have the properties

$$
\begin{aligned}
P_{\mathrm{L}}^{\mu\sigma} P_{\mathrm{L}\sigma\nu} &= -P_{\mathrm{L}\nu}^{\mu} \\
P_{\mathrm{T}}^{\mu\sigma} P_{\mathrm{T}\sigma\nu} &= -P_{\mathrm{T}\nu}^{\mu} \\
k_\mu P_{\mathrm{T}}^{\mu\nu} &= k_\mu P_{\mathrm{L}}^{\mu\nu} = 0 \\
P_{\mathrm{L}}^{\mu\sigma} P_{\mathrm{T}\sigma\nu} &= 0 \\
P_{\mathrm{L}\mu}^{\mu} &= -1 \\
P_{\mathrm{T}\mu}^{\mu} &= -2
\end{aligned}
\tag{5.48}
$$

In the vacuum there is no preferred rest frame, so the vector u_μ cannot play any role (it is not defined). Also, in the vacuum $\Pi^{\mu\nu}$ must be proportional to $g^{\mu\nu} - k^\mu k^\nu/k^2$; hence $F = G$. Furthermore, G can only depend on k^2. At finite temperature and density, however, F and G can depend on $k^0 = u \cdot k$ and $|\mathbf{k}| = \sqrt{(u\cdot k)^2 - k^2}$ separately, owing to the lack of Lorentz invariance.

Let us evaluate the photon self-energy at the one-loop level. From (5.40),

$$
\Pi^{\mu\nu} = e^2 T \sum_l \int \frac{d^3p}{(2\pi)^3} \operatorname{Tr}\left(\gamma^\nu \frac{1}{\not p - m}\, \gamma^\mu\, \frac{1}{\not p + \not k - m}\right) \tag{5.49}
$$

Here $p^0 = (2l+1)\pi T i + \mu$ and $k^0 = 2n\pi T i$. We can always write $\Pi^{\mu\nu} = \Pi_{\mathrm{vac}}^{\mu\nu} + \Pi_{\mathrm{mat}}^{\mu\nu}$, where

$$
\Pi_{\mathrm{vac}}^{\mu\nu} = \lim_{\substack{T\to 0\\ \mu\to 0}} \Pi^{\mu\nu} \tag{5.50}
$$

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is the vacuum self-energy and $\Pi^{\mu\nu}_{\text{mat}}$ is the remainder due to the presence of matter. The vacuum part is discussed in many textbooks on field theory, such as Peskin and Schroeder [2]. The matter part is readily evaluated:

$$\Pi^{00}_{\text{mat}} = -\frac{e^2}{\pi^2}\,\text{Re}\int_0^\infty \frac{dp\,p^2}{E_p} N_{\text{F}}(p)\left[1 + \frac{4E_p k^0 - 4E_{\text{P}}^2 - k^2}{4p\omega}\ln\left(\frac{R_+}{R_-}\right)\right]$$

$$\Pi^{\mu}_{\text{mat}\,\mu} = -2\frac{e^2}{\pi^2}\,\text{Re}\int_0^\infty \frac{dp\,p^2}{E_p} N_{\text{F}}(p)\left[1 - \frac{2m^2 + k^2}{4p\omega}\ln\left(\frac{R_+}{R_-}\right)\right] \tag{5.51}$$

Here

$$\omega = |\mathbf{k}| \qquad k^2 = k_0^2 - \omega^2 \qquad E_p = \sqrt{\mathbf{p}^2 + m^2}$$

$$N_{\text{F}}(p) = \frac{1}{\text{e}^{\beta(E_p-\mu)} + 1} + \frac{1}{\text{e}^{\beta(E_p+\mu)} + 1}$$

$$R_\pm = k^2 - 2k_0 E_p \pm 2p\omega$$

Also, the reader should note that here we define the action of the operator Re as follows: $\text{Re}\, f(k^0) = \frac{1}{2}[f(k^0) + f(-k^0)]$.

Various limits of (5.51) are of physical interest. They correspond to the screening of electric and magnetic fields and plasma oscillations. These topics are discussed in Chapter 6 in particular, in the context of linear response theory.

5.5 Loop corrections to $\ln Z$

5.5.1 Two loops

The lowest-order correction to $\ln Z$ due to interactions is the two-loop diagram seen in (5.39). There are two methods of doing explicit calculations with such diagrams. In the traditional method the frequency sums are performed directly. Another method uses analytic continuation and contour integrals, as discussed in Chapter 3. Both methods must of course give the same answer, but usually the contour integral method is much easier.

From (5.39), we have in the Feynman gauge the exchange contribution

$$\frac{\ln Z_{\text{ex}}}{\beta V} = -\frac{1}{2}e^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}(2\pi)^3\delta(\mathbf{p} - \mathbf{q} - \mathbf{k}) \times T^3 \sum_{n_p,n_q,n_k} \beta\delta_{n_p,n_q+n_k}\frac{\text{Tr}[\gamma^\mu(\not{p} + m)\gamma_\mu(\not{q} + m)]}{k^2(p^2 - m^2)(q^2 - m^2)} \tag{5.52}$$

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The trace is readily carried out. Apart from integration over three-momenta this becomes

$$-8T^3 \sum_{n_p,n_q,n_k} \beta\delta_{n_p,n_q+n_k} \frac{2m^2 - p\cdot q}{k^2(p^2-m^2)(q^2-m^2)} \tag{5.53}$$

The Kronecker delta may be written as

$$\begin{aligned} \beta\delta_{n_p,n_q+n_k} &= \int_0^\beta d\theta \exp[\theta(p^0 - q^0 - k^0)] \\ &= \frac{\exp\left[\beta(p^0-q^0-k^0)\right] - 1}{p^0 - q^0 - k^0} \end{aligned} \tag{5.54}$$

where $p^0 = (2n_p+1)\pi T i + \mu$, $q^0 = (2n_q+1)\pi T i + \mu$, and $k^0 = 2n_k \pi T i$. Since q^0 and k^0 enter the argument of the exponential with minus signs we multiply by $-\exp[\beta(k^0+q^0-\mu)]$, which is unity when evaluated on the integers. This procedure ensures that the integrands of the contour integrals fall off exponentially before the θ integration is performed, so that one never need worry about contributions from contours distorted out to infinity. This procedure also guarantees that the normal vacuum is recovered in the limit of zero temperature and chemical potential (see the discussion in the papers of Norton and Cornwall [3] and Kapusta [4]). With this analytic continuation of the Kronecker delta, (5.53) becomes

$$-8T\sum_{n_k}\frac{1}{k^2}\,T\sum_{n_p}\frac{1}{p^2-m^2}\,T\sum_{n_q}\frac{1}{q^2-m^2}I(p^0,q^0,k^0) \tag{5.55}$$

where

$$I(p^0,q^0,k^0) = \frac{2m^2 - p\cdot q}{p^0-q^0-k^0}\{\exp[\beta(k^0+q^0-\mu)] - \exp[\beta(p^0-\mu)]\}$$

Notice that I has no singularities. Hence, each of the sums may be converted to a contour integral via (3.40) and (3.71), and these contour integrations may be performed simultaneously and independently. For example,

$$\begin{aligned} & T\sum_{n_p}\frac{1}{p^2-m^2}I(p^0,q^0,k^0) \\ &= I(E_p,q^0,k^0)\frac{N_{\mathrm{F}}^-(p)}{2E_p} + I(-E_p,q^0,k^0)\frac{N_{\mathrm{F}}^+(p)-1}{2E_p} \\ & T\sum_{n_k}\frac{1}{k^2}I(p^0,q^0,k^0) \\ &= -I(p^0,q^0,\omega)\frac{N_{\mathrm{B}}(k)}{2\omega} - I(p^0,q^0,-\omega)\frac{N_{\mathrm{B}}(k)+1}{2\omega} \end{aligned} \tag{5.56}$$

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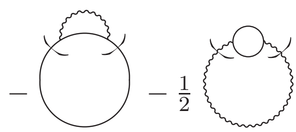
where the fermion and boson occupation numbers are

$$N_{\mathrm{F}}^{\pm}(p) = \frac{1}{\exp[\beta(E_p \pm \mu)] + 1}$$
$$N_{\mathrm{B}}(k) = \frac{1}{\exp(\beta\omega) - 1} \tag{5.57}$$

and $\omega = |\mathbf{k}|$, $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

As is evident, the contour integration method has two obvious advantages over the direct summation method. First, the contour integrals may be evaluated independently of each other whereas the direct summations must be done in consecutive order. This is a great algebraic simplification, which becomes more pronounced as the complexity of the diagram increases. Second, the contour integration puts each particle on its mass shell automatically.

When (5.56) is used to evaluate (5.55), one finds terms that are quadratic in the occupation numbers, terms that are linear, and terms that are independent of the occupation numbers. Those that are independent represent the energy shift of the vacuum and are not of interest to us. Those that are linear are canceled by the fermion and photon vacuum self-energy renormalizations. These are represented as



the angled parentheses indicating that the $T = \mu = 0$ limit of the subgraph is to be taken (cf. (3.49)). Putting all the above together we find the two-loop result:

$$\begin{aligned}\frac{\ln Z_{\mathrm{ex}}}{\beta V} = &-\frac{1}{6}e^2T^2 \int \frac{d^3p}{(2\pi)^3}\frac{N_{\mathrm{F}}(p)}{E_p} - \frac{1}{2}e^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{E_pE_q} \\ &\times\left\{\left(1 + \frac{2m^2}{(E_p - E_q)^2 - (\mathbf{p}-\mathbf{q})^2}\right)[N_{\mathrm{F}}^-(p)N_{\mathrm{F}}^-(q) + N_{\mathrm{F}}^+(p)N_{\mathrm{F}}^+(q)]\right. \\ &+\left.\left(1 + \frac{2m^2}{(E_p + E_q)^2 - (\mathbf{p}-\mathbf{q})^2}\right)[N_{\mathrm{F}}^-(p)N_{\mathrm{F}}^+(q) + N_{\mathrm{F}}^+(p)N_{\mathrm{F}}^-(q)]\right\}\end{aligned} \tag{5.58}$$

where $N_{\mathrm{F}} = N_{\mathrm{F}}^+ + N_{\mathrm{F}}^-$. This is referred to as the exchange term because in the $T = 0$ limit it arises from the exchange of the three-momenta of a pair of fermions in the Fermi sea. Various limits of the exchange term are of interest, and so are listed below; note that the Fermi momentum is

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$p_{\rm F} = \sqrt{\mu^2 - m^2}$ when $|\mu| > m$:

$$\frac{\ln Z_{\rm ex}}{\beta V} = -\frac{e^2}{(2\pi)^4}\left\{\frac{3}{2}\left[\mu p_{\rm F} - m^2 \ln\left(\frac{\mu + p_{\rm F}}{m}\right)\right]^2 - p_{\rm F}^4\right\} \quad (T = 0) \tag{5.59}$$

$$\frac{\ln Z_{\rm ex}}{\beta V} = -\frac{e^2}{288}\left(5T^4 + \frac{18}{\pi^2}\mu^2 T^2 + \frac{9}{\pi^4}\mu^4\right) \quad (m = 0) \tag{5.60}$$

$$\frac{\ln Z_{\rm ex}}{\beta V} = \frac{e^2}{2(2\pi)^3} m^2 T^2 {\rm e}^{2(\mu - m)/T} \quad (T \ll m - \mu \ll m) \tag{5.61}$$

Equation (5.59) will be useful in our discussion of white dwarf stars. Equation (5.60) will reappear in QCD plasma. Equation (5.61) modifies the classical ideal gas equation of state to $P = nT - e^2 n^2/8mT$.

5.5.2 Ring diagrams

The next order to contribute is not e^4 as naively expected but e^3 when $T > 0$ and $e^4 \ln e^2$ when $T = 0$ but $\mu \neq 0$. These arise from the set of ring diagrams shown in (3.54), where the photon self-energy is given to lowest order by (5.40),

$$\frac{\ln Z_{\rm ring}}{\beta V} = -\tfrac{1}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} {\rm Tr}\left\{\ln\left[1 + \mathcal{D}_0(k)\Pi(k)\right] - \mathcal{D}_0(k)\Pi(k)\right\} \tag{5.62}$$

Making use of the explicit forms of $\mathcal{D}_0$ and Π as given by (5.46), we may carry out the trace operation to obtain

$$\begin{aligned}\frac{\ln Z_{\rm ring}}{\beta V} = -\tfrac{1}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \Bigg\{ 2\left[\ln\left(1 - \frac{G(n,\omega)}{k^2}\right) + \frac{G(n,\omega)}{k^2}\right] \\ + \ln\left(1 - \frac{F(n,\omega)}{k^2}\right) + \frac{F(n,\omega)}{k^2}\Bigg\}\end{aligned} \tag{5.63}$$

Note that F and G are functions of n (since $k_0 = 2\pi n T i$) and $\omega = |\mathbf{k}|$. The terms involving G have a coefficient of 2 relative to the terms involving F. The reason is that there are two transverse degrees of freedom but only one longitudinal degree of freedom ($P^\mu_{T\mu} = -2$, $P^\mu_{L\mu} = -1$). Note that the expressions (5.62) and (5.63) are manifestly gauge invariant since the ρ-dependent part of $\mathcal{D}_0$ vanishes, as a consequence of current conservation, when it multiplies Π.

Since $-k^2 = \omega^2 + 4\pi^2 T^2 n^2$, the logarithms may be expanded to second order in F and G to give an e^4 contribution, as long as $n \neq 0$. If either $F(n = 0, \omega \to 0)$ or $G(n = 0, \omega \to 0)$ does not vanish then expansions of the logarithms do not converge. To isolate this potential infrared

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divergence, we write

$$-\tfrac{1}{2}T\int\frac{d^3k}{(2\pi)^3}\left[2\ln\left(1+\frac{G(0,0)}{\omega^2}\right)-\frac{2G(0,0)}{\omega^2}+\ln\left(1+\frac{F(0,0)}{\omega^2}\right)-\frac{F(0,0}{\omega^2}\right] \tag{5.64}$$

The remaining terms, which are explicitly of order e^4 and which have no infrared divergence, are

$$\frac{1}{4}T\int\frac{d^3k}{(2\pi)^3}\left\{\sum_{n\neq 0}\left[2\left(\frac{G(n,\omega)}{k^2}\right)^2+\left(\frac{F(n,\omega}{k^2}\right)^2\right]+2\left(\frac{G(0,\omega)}{\omega^2}\right)^2\right.$$
$$\left.-2\left(\frac{G(0,0)}{\omega^2}\right)^2+\left(\frac{F(0,\omega)}{\omega^2}\right)^2-\left(\frac{F(0,0)}{\omega^2}\right)^2\right\} \tag{5.65}$$

Upon examination of (5.46) and (5.51) we find that $G(n=0,\omega\to 0)=0$ but

$$F(n=0,\omega\to 0)=\frac{e^2}{\pi^2}\int_0^\infty\frac{dp}{E_p}\,(p^2+E_p^2)N_{\mathrm{F}}(p) \tag{5.66}$$

After integrating over k in (5.64), we find the order-e^3 contribution,

$$\frac{\ln Z_{\mathrm{ring}}}{\beta V}=\frac{T}{12\pi}F^{3/2}(0,0) \tag{5.67}$$

This result, nonanalytic in $\alpha=e^2/4\pi$, is precisely analogous to our result in Chapter 3 for the massless $\lambda\phi^4$ theory. The nonanalyticity here arises because interactions at finite temperature and density generate a static electric screening mass for the photon.

There are several interesting limits of $F(n=0,\omega\to 0)$. In the ultrarelativistic limit ($m=0$),

$$F(0,0)=e^2\left(\frac{T^2}{3}+\frac{\mu^2}{\pi^2}\right) \tag{5.68}$$

In the nonrelativistic limit and with classical statistics,

$$F(0,0)=\frac{2e^2}{T}\left(\frac{mT}{2\pi}\right)^{3/2}\mathrm{e}^{(\mu-m)/T} \tag{5.69}$$

which, when inserted in (5.67), is the well-known Debye–Hückel formula.

At zero temperature, the discrete frequency of the photon becomes continuous and the $n=0$ mode cannot be isolated. From (3.40),

$$\lim_{T\to 0}T\sum_n=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_4 \tag{5.70}$$

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At $T = 0$ it is convenient to work in Euclidean space with $k_4 = -ik_0$ and with $\bar{k}^2 = k_4^2 + \mathbf{k}^2 = -k^2 \geq 0$. Both F and G are functions of $|\bar{k}|$ and ϕ, where $\tan\phi = |\mathbf{k}|/k_4$. Then (5.63) becomes

$$\frac{\ln Z_{\rm ring}}{\beta V} = -\frac{1}{(2\pi)^3}\int_0^\infty d\bar{k}^2\,\bar{k}^2\int_0^{\pi/2} d\phi\sin^2\phi \\ \times\left\{2\left[\ln\left(1+\frac{G(\bar{k}^2,\phi)}{\bar{k}^2}\right)-\frac{G(\bar{k}^2,\phi)}{\bar{k}^2}\right]\right. \\ \left.+\ln\left(1+\frac{F(\bar{k}^2,\phi)}{\bar{k}^2}\right)-\frac{F(\bar{k}^2,\phi)}{\bar{k}^2}\right\} \tag{5.71}$$

The potential infrared divergence in (5.71) may be isolated by setting $\bar{k} = 0$ whenever possible in the integrand:

$$-\frac{1}{(2\pi)^3}\int_0^\infty d\bar{k}^2\,\bar{k}^2\int_0^{\pi/2} d\phi\sin^2\phi\left\{2\ln\left(1+\frac{G(0,\phi)}{\bar{k}^2}\right)+\ln\left(1+\frac{F(0,\phi)}{\bar{k}^2}\right)\right. \\ \left.-\frac{2G(0,\phi)+F(0,\phi)}{\bar{k}^2}+\frac{1}{2\bar{k}^2}\frac{1}{\bar{k}^2+\mu^2}[F^2(0,\phi)+2G^2(0,\phi)]\right\} \tag{5.72}$$

Notice the term $1/(\bar{k}^2+\mu^2)$. The choice of μ^2 is arbitrary; any choice independent of e^2 will give the same coefficient of $e^4\ln e^2$. After integrating over $\bar{k}^2$, (5.72) becomes

$$-\frac{1}{2(2\pi)^3}\int_0^{\pi/2} d\phi\sin^2\phi\left\{F^2(0,\phi)\left[\ln\left(\frac{F(0,\phi)}{\mu^2}\right)-\frac{1}{2}\right]\right. \\ \left.+2G^2(0,\phi)\left[\ln\left(\frac{G(0,\phi)}{\mu^2}\right)-\frac{1}{2}\right]\right\} \tag{5.73}$$

The explicit forms of $F(0,\phi)$ and $G(0,\phi)$ may be substituted in (5.73) and the integration performed. A lengthy analysis yields

$$\frac{\ln Z_{\rm ring}}{\beta V} = -\frac{e^4\ln e^2}{128\pi^6}\left[(6-4\ln 2)\mu p_{\rm F}^3 - 5\mu^2 p_{\rm F}^2 + 4\mu^3 p_{\rm F}\ln\left(\frac{\mu+p_{\rm F}}{\mu}\right)\right. \\ +6\mu m^2 p_{\rm F}\ln\left(\frac{\mu+p_{\rm F}}{2^{5/3}\mu}\right) - m^2(4\mu^2+m^2)\ln^2\left(\frac{\mu+p_{\rm F}}{m}\right) \\ \left.+m^2\mu(4\mu^2+m^2)\frac{I(a)}{p_{\rm F}}\right] \tag{5.74}$$

where

$$I(a) = \int_0^\infty \frac{dx}{a^2x^2-1}\ln\left(\frac{x+1}{x-1}\right) \qquad a = \frac{\mu}{p_{\rm F}}$$

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The ultrarelativistic limit is

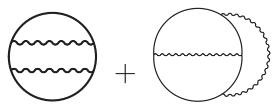
$$\frac{\ln Z_{\text{ring}}}{\beta V} = -\frac{e^4 \ln e^2}{128\pi^6}\mu^4 \tag{5.75}$$

and the nonrelativistic limit is

$$\frac{\ln Z_{\text{ring}}}{\beta V} = -\frac{e^4 \ln e^2}{48\pi^6}(1 - \ln 2)\mu p_{\text{F}}^3 \tag{5.76}$$

5.5.3 Three loops at finite density

The three-loop diagrams not included in the ring sum are



(5.77)

The evaluation of these diagrams is technically quite involved because of overlapping ultraviolet divergences. For further discussion, see Freedman and McLerran [5] and Baluni [6].

The result of evaluating (5.77) together with the order-e^4 contribution from the sum of ring diagrams is

$$P = \frac{\mu^4}{12\pi^2}\left[1 - \frac{3}{2}\frac{\alpha(M)}{\pi} - \frac{3}{2}\left(\frac{\alpha(M)}{\pi}\right)^2 \ln\left(\frac{\alpha(M)}{\pi}\right) - \frac{1}{2}\left(\frac{\alpha(M)}{\pi}\right)^2 \ln\left(\frac{\mu^2}{M^2}\right) + (2.118\,19)\left(\frac{\alpha(M)}{\pi}\right)^2\right] \tag{5.78}$$

Certain integrals had to be done numerically in producing this result, giving the number in the coefficient of α^2. The photon wavefunction renormalization constant $\mathcal{Z}_3$ was defined at a Euclidean subtraction point $\bar{k}^2 = M^2$. Equivalently, the photon self-energy was renormalized in such a way that $F(\bar{k}^2 = M^2,\ \mu = 0) = G(\bar{k}^2 = M^2,\ \mu = 0) = 0$.

The choice of subtraction energy M is completely arbitrary. In (5.78) notice that a logarithm of μ/M appears. At higher orders of α, higher powers of the logarithm will appear. Therefore, to reduce the importance of higher-order terms at high density we are free to choose $M = \mu$. (The optimum choice of the constant of proportionality between M and μ is not known.) Then (5.78) becomes

$$P = \frac{\mu^4}{12\pi^2}\left[1 - \frac{3}{2}\frac{\alpha(\mu)}{\pi} - \frac{3}{2}\left(\frac{\alpha(\mu)}{\pi}\right)^2 \ln\left(\frac{\alpha(\mu)}{\pi}\right) + (2.118\,19)\left(\frac{\alpha(\mu)}{\pi}\right)^2\right] \tag{5.79}$$

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The next question is: What is $\alpha(\mu)$? From our knowledge of the renormalization group we know that $\alpha(\mu)$ must satisfy a renormalization-group equation. To lowest order,

$$M\frac{d\alpha}{dM} = c_0\alpha^2 \tag{5.80}$$

In massless QED the constant c_0 is computed to be $2\pi/3$. Realizing that we have chosen $M = \mu$ to suppress large logarithms at high density, we find that the renormalization-group running coupling is

$$\alpha\left(\frac{\mu}{\mu_0}\right) = \frac{\alpha(1)}{1 - [2\alpha(1)/3\pi)]\ln(\mu/\mu_0)} \tag{5.81}$$

Here μ_0 is some reference scale and $\alpha(1)$ is the value of the coupling at that scale. Just as in (4.24), (4.25) for the massless $\lambda\phi^4$ theory, we can combine $\alpha(1)$ and μ_0 into one constant Λ. Then

$$\alpha\left(\frac{\mu}{\Lambda}\right) = \frac{3\pi}{2\ln(\Lambda/\mu)} \tag{5.82}$$

Here Λ is the intrinsic energy scale of massless QED. This theory is not asymptotically free. Therefore, when $\mu \ll \Lambda$ the coupling $\alpha(\mu/\Lambda)$ is very small. The perturbative expansion of the partition function for a cold high-density electron gas converges rapidly until $\mu \simeq \Lambda$ is reached. This limitation is not of practical significance because the intrinsic energy scale $\Lambda \sim m_e \mathrm{e}^{137}$ is astronomically large ($m_e = 0.511$ MeV).

It is apparent that the perturbation series for P in (5.79) is rapidly convergent at non-astronomically-large densities because $\alpha/\pi \simeq 2.3 \times 10^{-3}$.

5.5.4 Three loops at finite temperature

The pressure for finite temperature QED has been calculated for $\mu = 0$ up to order e^5. We first show results up to e^4. See Corianò and Parwani [7] for the details (especially on the delicate handling of the singularities). The usual zero-temperature ultraviolet singularities are regularized through dimensional regularization, ensuring that the physical result is gauge invariant. Evaluating the diagrams (5.77) at finite temperature for N_f electron flavors (physical QED corresponds to $N_\mathrm{f} = 1$), with the appropriate counterterms, yields

$$\begin{aligned}\frac{P}{T^4} = {} & \frac{\pi^2}{45}\left(1 + \frac{7}{4}N_\mathrm{f}\right) - \frac{5e^2 N_\mathrm{f}}{288} \\ & + \frac{e^3}{12\pi}\left(\frac{N_\mathrm{f}}{3}\right)^{3/2} + \frac{e^4 N_\mathrm{f}}{\pi^6}(0.4056 \pm 0.0030) \\ & - e^4 N_\mathrm{f}^2\left[\frac{0.4667 \pm 0.0020}{\pi^6} + \frac{5}{1728\pi^2}\ln\left(\frac{T}{M}\right)\right]\end{aligned} \tag{5.83}$$

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The above also includes the e^4 contribution for the set of ring diagrams discussed earlier. As before the uncertainties in the quantities are due to the numerical evaluation of some integrals. Note that the coupling in the expression for the pressure is to be evaluated at some renormalization scale M. This scale M may be chosen on physical grounds: for example, setting $M = T$ will eliminate the logarithm at this and higher orders. An alternative procedure is to use renormalization-group arguments to relate the coupling e at some scale M to that at another scale set by the temperature. Doing this, one may write

$$e^2(T) = e^2 \left[1 + \frac{e^2 N_{\rm f}}{6\pi^2} \ln\left(\frac{T}{M}\right)\right] + \mathcal{O}(e^6) \tag{5.84}$$

where e is the coupling in (5.83). Defining $\alpha(T) = e^2(T)/4\pi$, (5.83) can be written as

$$\begin{aligned}\frac{P}{T^4} = \frac{\pi^2}{45}\left(1 + \frac{7}{4}N_{\rm f}\right) - \frac{5\pi^2}{72}\frac{\alpha(T)N_{\rm f}}{\pi} + \frac{2\pi^2}{9\sqrt{3}}\left(\frac{\alpha(T)N_{\rm f}}{\pi}\right)^{3/2} \\ + \left(\frac{0.658 \pm 0.006}{N_{\rm f}} - 0.757 \pm 0.004\right)\left(\frac{\alpha(T)N_{\rm f}}{\pi}\right)^2 + \mathcal{O}\left(\alpha(T)^{5/2}\right)\end{aligned} \tag{5.85}$$

The logarithm in (5.83) has disappeared and has been absorbed into the renormalization-group redefinition of the coupling constant.

The order-e^5 contribution is then obtained by resumming the boson propagators in (5.77) through a ring insertion, as discussed previously. The details appear in Parwani and Corianò [8]; the result is

$$\frac{P_5}{T^4} = \left(\frac{\alpha(T)N_{\rm f}}{\pi}\right)^{5/2}\left(\frac{\pi^2\left[1 - \gamma_{\rm E} - \ln(4/\pi)\right]}{9\sqrt{3}} - \frac{\pi^2}{2N_{\rm f}\sqrt{3}}\right) \tag{5.86}$$

where $\gamma_{\rm E}$ is Euler's constant.

5.6 Exercises

5.1 Prove (5.12).

5.2 Derive the blackbody radiation formula from (5.32) for arbitrary ρ.

5.3 Discuss what happens when the nonlinear gauge $F = A^\mu A_\mu - f(\mathbf{x}, \tau) = 0$ is chosen.

5.4 Derive the free-photon propagator given by (5.38).

5.5 Obtain the general forms given in (5.46) for the in-medium photon propagator and its inverse.

5.6 Repeat the calculation in the text for $\ln Z_{\rm ex}$ but with an arbitrary covariant gauge parameter ρ. Is the result independent of ρ?

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5.7 Using (5.51) and (5.46), find the limits of F and G when $k^2 = k_0^2 - \mathbf{k}^2 = 0$.
5.8 Determine the combinatoric factors for the two diagrams of (5.77).
5.9 Derive (5.63) from (5.62).
5.10 Calculate the relative contributions to the pressure in QED at finite temperature and zero electron mass from orders 0, 2, 3, 4 and 5 in e for arbitrary N_f.

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