$\frac{1}{2}(b \sim c)$. The in-centre I is the centroid of masses proportional to

$$4(b+c-a) \text{ at } D,\ 4(c+a-b) \text{ at } E,\ 4(a+b-c) \text{ at } F,$$

while the Nine-Point-centre is the centroid of masses proportional to

$$4a^2(b^2+c^2-a^2) \text{ at } D,\ 4b^2(c^2+a^2-b^2) \text{ at } E,\ 4c^2(a^2+b^2-c^2) \text{ at } F.$$

Hence

$$\textstyle\sum (b+c-a)(b-c)^2 = 2NI.\ 8s. \quad \text{(perp. from } I \text{ on radical axis)}$$
$$\textstyle\sum a^2(b^2+c^2-a^2)(b-c)^2 = 2NI.\ 64\Delta \quad (\ldots\ldots\ldots\ldots N \ldots\ldots\ldots\ldots\ldots\ldots)$$

or the perps. from I and N are in the ratio $64\Delta : 8\,abcs$ or $r : \frac{1}{2}R$.

Thus the radical axis of the in- and Nine-Point-circles divides externally the join of the centres in the ratio of the radii, and consequently the circles touch each other.

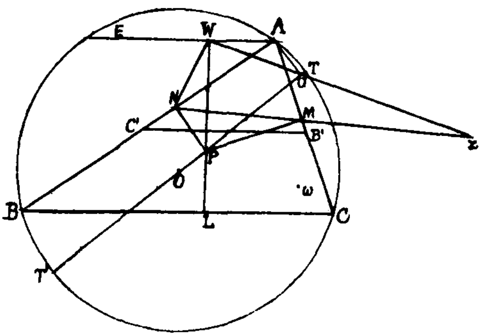
Note that $\sum (b+c-a)(b-c)^2$

$$= 2\{a^3+b^3+c^3+3abc-ab^2-ac^2-bc^2-ba^2-ca^2-cb^2\}$$
$$= 4\Delta(R-2r),$$

and that R is always greater than $2r$, except when $a=b=c$.

R. F. Davis.

Geometrical Note on the Orthopole.



Lemma.—If A, U are given fixed points; AC, AB, AE given fixed straight lines through A; and a variable circle through A, U intersects these straight lines in M, N, W respectively; then the locus x of the point of intersection of MN, UW will be a straight line parallel to AE.

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For the triangle NWx is of fixed species

$$\text{as angle } NWx\,(NWU) = NAU = \text{constant},$$
$$\text{and angle } WNx\,(WNM) = WPM = \text{constant};$$

also U divides the base Wx in constant ratio as the angle

$$WNU = WPU = \text{constant}.$$

But the locus of W is the straight line AE; and since U is a fixed point and $Ux : UW$ is a fixed ratio, therefore the locus of x is a straight line parallel to AE.

Let now ABC be a triangle whose circumcentre is O, having a fixed circum-diameter TOT'. From any variable point P on TOT' let fall the perpendiculars PL, PM, PN on the sides.

It is required to show that the circle through L, M, N (the pedal circle of P) will pass through a fixed point ω (on the N.P.C., the orthopole of TOT').

Draw AU at right angles to TOT' and let LP produced meet a parallel AE to BC in W. Then the circle upon AP as diameter will pass through M, N, U, W. Produce MN, UW to meet in x. Then (by the Lemma) since A, U are fixed points, and AC, AB, AE fixed straight lines, the locus of x is a straight line parallel to AE, or BC. By making P coincide with O it is seen that this locus is the straight line $B'C'$ bisecting the sides.

Since L is the image of W in $B'C'$ and WU intersects $B'C'$ in x, it follows that Lx passes through the image ω of U in $B'C'$. Then $x\omega \,.\, xL = xU \,.\, xW = xM \,.\, xN$ and the circle LMN will pass through ω.

Also ω lies on the N.P.C. by supposing P to coincide with O.

In Dr Coolidge's "Treatise on the Circle and Sphere" this theorem is attributed (p. 52) to Fontené (1905).

As pointed out by the late Mr W. Gallatly in his "Modern Geometry of the Triangle" it leads at once to a proof of Feuerbach's theorem. The pedal circle of a point S intersects the N.P.C. in a point which depends entirely on the direction of OS. Similarly for another point S'. When S, S' are isogonal conjugates their joint pedal circle intersects the N.P.C. in points which depend on the directions of OS, OS'. If S, S' coalesce at I (the in-centre), or are in line with O, then these two directions coincide and the circles touch.

R. F. DAVIS.

Geometrical Proofs of the Trigonometrical Ratios of 2 θ and 3 θ.

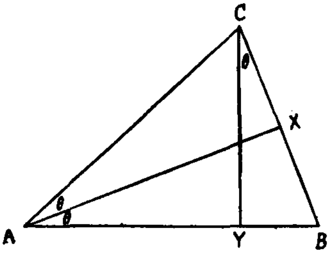

Fig. 1.

1. *Ratios of* 2 θ.

$\angle BAX = \angle XAC = \theta$.

CB is drawn perpendicular to AX, and CY perpendicular to AB: then $\angle YCB = \theta$.

$$\sin 2\theta = \frac{YC}{AC} = \frac{YC}{BC} \cdot \frac{BC}{AC}$$
$$= \frac{YC}{BC} \cdot \frac{2XC}{AC}$$
$$= \cos\theta \, . \, 2\sin\theta$$
$$= 2\sin\theta \, . \, \cos\theta.$$

$$\cos 2\theta = \frac{AY}{AC} = \frac{AB - YB}{AC} = 1 - \frac{YB}{AC}$$
$$= 1 - \frac{YB}{BC} \cdot \frac{BC}{AC}$$
$$= 1 - \frac{YB}{BC} \cdot \frac{2XC}{AC}$$
$$= 1 - \sin\theta \, . \, 2\sin\theta$$
$$= 1 - 2\sin^2\theta.$$

The other forms for $\cos 2\theta$ and that for $\tan 2\theta$ can readily be deduced by transformation.

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