ON GENERATING FUNCTIONS FOR CLASSICAL POLYNOMIALS

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1. Introduction

Recently Brown [1] gave two new classes of generating functions which include the generating functions for the polynomials of Gegenbauer, Jacobi and Laguerre. The aim of the paper is to give a new class of generating functions which includes both sets of generating functions given by Brown and provides a new class of generating functions for the polynomials of Gegenbauer, Jacobi and Laguerre.

2. Preliminaries

First of all we will prove a formal series relation with the help of a combinatorial identity, as a lemma.

LEMMA. *Given a sequence ϕ_n $(n \geqq 0)$, define the new one*

$$\psi_n = \sum_{k=0}^{n} \binom{\alpha + \beta_n}{n-k} \phi_k \qquad (n \geqq 0)$$

then

$$\sum_{n=0}^{\infty} \frac{\alpha(p+q_n)}{(\alpha+\beta_n)} \psi_n \left[\frac{x}{(1+x)^\beta}\right]^n = \alpha(1+x)^\alpha \sum_{n=0}^{\infty} \left[\frac{p+q_n}{\alpha+\beta_n} + \frac{qx}{1+(1-\beta)x}\right] \phi_n x^n, \tag{2}$$

where α, β, p and q are any complex numbers.

PROOF. The proof is based on the identity [4, 16(b) p. 169]

$$\sum_{n=0}^{\infty} \frac{\alpha(p+q_n)}{(\alpha+\beta_n)} \binom{\alpha+\beta_n}{n} \left[\frac{x}{(1+x)^\beta}\right]^n = (1+x)^\alpha \left[p + \frac{q\alpha x}{1+(1-\beta)x}\right]. \tag{3}$$

To obtain (2), simply recall (1) and write

$$\sum_{n=0}^{\infty} \frac{\alpha(p+q_n)}{(\alpha+\beta_n)} \Psi_n \left[\frac{x}{(1+x)^\beta}\right]^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\alpha(p+q_n)}{(\alpha+\beta_n)} \binom{\alpha+\beta_n}{n-k} \phi_k \left[\frac{x}{(1+x)^\beta}\right]^n$$

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$$= \sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty} \frac{\alpha+\beta_k}{\alpha+\beta_k+\beta_n}\cdot(p+q_k+q_n)\binom{\alpha+\beta_k+\beta_n}{n}\left[\frac{x}{(1+x)^\beta}\right]^n\right\}$$
$$\frac{\alpha}{\alpha+\beta_k}\phi_k\left[\frac{x}{(1+x)^\beta}\right]^k.$$

Then, using (3) with α and p replaced by $\alpha+\beta k$ and $p+qk$ respectively, we arrive at the desired result. From this lemma we get the lemma [1, § 2] by taking $p=1$, $q=\beta/\alpha$ and $p=1$, $q=0$.

3. Generating functions

Our new class of the generating functions follows readily from the above lemma.

*Theorem. *Let*

$$g_n^\alpha(x,c) = \binom{\alpha+\beta_n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{c_k x^k}{(1+\alpha+(\beta-1)n)_k}, \tag{4}$$

where the c_k are arbitrary. Then

$$\sum_{n=0}^{\infty}\frac{\alpha(p+q_n)}{\alpha+\beta_n}g_n^\alpha(x,c)t^n$$
$$= \alpha(1+v)^\alpha\left[\sum_{n=0}^{\infty}\frac{p+q_n}{\alpha+\beta_n}\frac{c_n x^n v^n}{n!}+\frac{qv}{1+(1-\beta)v}\sum_{n=0}^{\infty}\frac{c_n x^n v^n}{n!}\right] \tag{5}$$

where

$$v = (1+v)^\gamma t. \tag{6}$$

Proof. In the lemma let $\phi_n = \frac{c_n x^n}{n!}$ and observe that ψ_n becomes the polynomial $g_n^\alpha(x,c)$ defined by (4), and (5) becomes (2). This completes the proof of the theorem.

Interesting special cases follow. With $\beta=1/2$ in (5), we easily get generating functions [1,(7)] and [1,(8)] by taking $p=1$, $q=1/2\alpha$ and $p=1$, $q=0$ respectively (from (6), with $\beta=1/2$ we have $v=k/2[t\pm\sqrt{k^2+4}]$).

From (5) we easily have

$$\sum_{n=0}^{\infty}\frac{\alpha(p+q_n)}{(\alpha+\beta_n)}P_n^{(\alpha-(1-\beta)n,\gamma-\beta_n)}(x)t^n \tag{7}$$

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$$= \alpha(1+v)^{\alpha}\left[\frac{p}{\alpha}\,{}_3F_2\left(\begin{matrix}\alpha/\beta, 1+p/q, 1+\alpha+\gamma \\ 1+\alpha/\beta, p/q\end{matrix} - \left(\frac{1-x}{2}\right)v\right) + \frac{qv}{1+(1-\beta)v}\right.$$
$$\left.\left\{1+\left(\frac{1-x}{2}\right)v\right\}^{-\alpha-\gamma-1}\right],$$

where $P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n}\,{}_2F_1\left(\begin{matrix}-n, 1+\alpha+\beta+n \\ 1+\alpha\end{matrix}\ \frac{1-x}{2}\right)$ is Jacobi polynomial and $v-(1+v)^{\beta}t$. In (7) if we take $\beta = 1/2$, we get

(8)
$$\sum_{n=0}^{\infty} \frac{\alpha(p+qn)}{(\alpha+n/2)} P_n^{(\alpha-n/2,\ \gamma-n/2)}(x)t^n$$
$$= \alpha(1+u)_{\alpha}\left[\frac{p}{\alpha}\,{}_3F_2\left(\begin{matrix}2\alpha, 1+p/q, 1+\alpha+\gamma \\ 1+2\alpha, p/q\end{matrix} - \left(\frac{1-x}{2}\right)u\right) + \frac{qu}{1+u/2}\right.$$
$$\left.\left\{1+\left(\frac{1-x}{2}\right)u\right\}^{-\alpha-\gamma-1}\right],$$

where $u = \frac{t}{2}\left[t \pm \sqrt{t^2+4}\right]$. In (8) if we put $\gamma = \alpha$ we get the generating function for Gegenbauer polynomials. With $\beta = 1$, $p = 1$ and $q = 1/\alpha$ in (7) we get a generating function of Feldheim [3].

For the modified Laguerre polynomials

$$L_n^{(\alpha+\beta_n)}(x) = \sum_{k=0}^{\infty}\binom{\alpha+(\beta+1)n}{n-k}\frac{(-x)^k}{k!},$$

we have

(9)
$$\sum_{n=0}^{\infty} \frac{\alpha(p+q_n)}{(\alpha+\beta_n+n)} L_n^{(\alpha+\beta_n)}(x)t^n = \alpha(1+v)^{\alpha}$$
$$\left[\frac{p}{\alpha}\,{}_2F_2\left(\begin{matrix}\alpha/(1+\beta), 1+p/q \\ 1+\alpha/(1+\beta), p/q\end{matrix} - xv\right) + \frac{qv}{1-\beta v}e^{-xv}\right]$$

where $v = (1+v)^{\beta+1}t$. In (9) if we take $p = 1$, $q = (1+\beta)/\alpha$ we get the generating function [2, (8)].

Using the particular form of the Jacobi polynomial [5, (15)], namely,

$$P_n^{(\alpha-n,\beta-n)}(x) = \binom{n-\alpha-\beta-1}{n}\left(\frac{1-x}{2}\right)^n {}_2F_1\left(\begin{matrix}-n, -\alpha \\ -\alpha-\beta\end{matrix}\ \frac{2}{1-x}\right),$$

we easily get from (5)

(10)
$$\sum_{n=0}^{\infty} \frac{(1+\alpha+\gamma)(p+q_n)}{(1+\alpha+\gamma-\beta_n)} P_n^{(\alpha-n,\gamma-\beta_n)}(x)t^n$$
$$= (1+\alpha+\gamma)(1+w)^{-\alpha-\gamma-1}$$

$$\left[\frac{p}{(1+\alpha+\gamma)}\,{}_3F_2\left(\begin{matrix}-(1+\alpha+\gamma)/\beta, 1+p/q, -\alpha \\ 1-(1+\alpha+\gamma)/\beta, p/q\end{matrix}\ \frac{-2w}{1-x}\right)\right.$$
$$\left.-\frac{qw}{1+(1-\beta)w}\left(1+\frac{2w}{1-x}\right)^{\alpha}\right],$$

where $w = (1-x)t(1+w)^{\beta}/2$. By taking $p = 1$, $q = -\beta/(1+\alpha+\gamma)$, we get the generating function [5, (16)], which includes many particular cases as cited in paper [5].

In the last we may remark that the main result of Srivastava [5, (9)] and its generalization [5, (*)] are the direct consequence of the lemma [1, §2] which have generalized in this paper.

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