Canad. Math. Bull. Vol. **18** (4), 1975

ISOMORPHIC GROUP RINGS

BY
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Let R and S be rings with 1, G a group and RG and SG the corresponding group rings. In this paper, we study the problem of when $RG \simeq SG$ implies $R \simeq S$. This problem was previously investigated in [8] for the case where G is assumed to be infinite cyclic. The corresponding question for polynomial rings, namely, when does $R[x] \simeq S[x]$ imply $R \simeq S$, has been considered by several authors, particularly Coleman and Enochs [3]. Recently Hochster [5] found two nonisomorphic integral domains M and N with $M[x] \simeq N[x]$. However, if $\langle x \rangle$ is an infinite cyclic group, it is still not known whether $R\langle x \rangle \simeq S\langle x \rangle$ implies $R \simeq S$ even when R and S are integral domains.

In the first two sections, we develop some necessary background material which may be of interest on its own. In section one, we determine the units of RG where R is commutative and G is right ordered. Section two studies the R-automorphisms of the group ring $R\langle x \rangle$, where $\langle x \rangle$ is an infinite cyclic group. We determine necessary and sufficient conditions that $x \to \sum a_i x^i$ induces an R-automorphism of $R\langle x \rangle$. The corresponding results for polynomial rings were obtained by Gilmer [4] when R is commutative and by Coleman and Enochs [3] in general.

Section three takes up the problem $R\langle x \rangle \simeq S\langle x \rangle$ where $\langle x \rangle$ is infinite cyclic. We show that if R and S have no idempotent $\neq 0, 1$, then $R\langle x \rangle \simeq S\langle x \rangle$ implies R and S are subisomorphic, that is, R can be embedded in S and S can be embedded in R. We then extend this result to commutative Noetherian rings. This is interesting because no corresponding result is known for polynomial rings, even over integral domains.

In section four, we prove several results about $RG \simeq SG$ for G more general than infinite cyclic. For example, we show that if R and S are commutative regular rings and G belongs to a class C which contains all ordered groups, and if $\sigma: RG \to SG$ is a homomorphism, then $\sigma(R) \subseteq S$. We also show that similar results hold for commutative local rings. Corresponding results for polynomial rings have been obtained by Jacobson [6].

This work was completed during the summer of 1973 at the University of Manitoba. The author would like to thank the Department of Mathematics at the University of Manitoba for its generous support.

Received by the editors September 17, 1973 and, in revised form, May 22, 1974.
This research supported in part by National Research Council of Canada Grant A-8775.

https://doi.org/10.4153/CMB-1975-101-3 Published online by Cambridge University Press

1. **Units.** In this section, we find the units of the group ring RG where R is a commutative ring with 1 and G is right-ordered. Let $U(RG)$ denote the units of RG.

PROPOSITION 1.1. *Let R be a commutative ring with 1 and let G be right-ordered. Then the following are equivalent:*

(i) $U(RG)=\{\sum \alpha_g g \mid$ *there exist* β_g *in* R *with* $\sum \alpha_g\beta_{g^{-1}}=1$ *and* $\alpha_g\beta_h=0$ *whenever* $gh\neq 1\}$.

(ii) *R has no non-zero nilpotent elements.*

Proof. Assume (i) holds and let $n \in R$ be nilpotent. Choose $g \in G$, $g\neq 1$. Then $1+ng$ is a unit in RG and, if $n\neq 0$, $1+ng$ does not satisfy condition (i). Hence $n=0$ and (ii) holds.

Conversely, assume (ii) holds and let $yz=1$ where $y=\sum_{i=1}^{r} \alpha_i g_i$ and $z=\sum_{i=1}^{s} \beta_i h_i$. We will show that $\alpha_i\beta_j=0$ whenever $g_ih_j\neq 1$. The other statement follows immediately.

Suppose that $g_1<\cdots<g_r$ and $h_1<\cdots<h_s$. For any fixed j, we know that $g_1h_j<g_2h_j<\cdots<g_rh_j$. Choose j_1, $1\leq j_1\leq s$, with $g_rh_{j_1}$ maximal in $\{g_rh_j\}_j$. Notice that from the choice of j_1 the term $g_rh_{j_1}$ does not occur again in the set of products g_ih_k. We want to show $\alpha_i\beta_j=0$ whenever $g_ih_j>1$. If $g_rh_{j_1}\leq 1$ there is nothing to show. If $g_rh_{j_1}>1$ we have $\alpha_r\beta_{j_1}=0$ from the above. Hence we conclude that $\alpha_r\beta_{j_1}=0$. Assume that we know that $\alpha_l\beta_m=0$ whenever $g_lh_m>g_{i_1}h_{k_1}=g_{i_2}h_{k_2}=\cdots=g_{i_p}h_{k_p}>1$ (the $g_{i_s}h_{k_s}$ being a complete list of products equal to $g_{i_1}h_{k_1}$). We see that $\alpha_{i_1}\beta_{k_1}+\cdots+\alpha_{i_p}\beta_{k_p}=0$ and we may assume that $i_1<i_2<\cdots<i_p$. Multiplying on the right by α_{i_p}, we obtain $\alpha_{i_1}\beta_{k_1}\alpha_{i_p}+\cdots+\alpha_{i_p}\beta_{k_p}\alpha_{i_p}=0$. For $t<p$, $g_{i_p}h_{k_t}>g_{i_t}h_{k_t}$. Hence, by induction, $\alpha_{i_p}b_{k_t}=0$. We conclude that $\alpha_{i_p}b_{k_p}\alpha_{i_p}=0$, hence $(\alpha_{i_p}b_{k_p})^2=0$ and $\alpha_{i_p}b_{k_p}=0$ using (ii). Working back, we obtain $\alpha_{i_t}b_{k_t}=0$ for $1\leq t\leq p$. Therefore, we have shown that $\alpha_i\beta_j=0$ whenever $g_ih_j>1$.

An identical argument to that given above, starting with $g_1h_{j_1}$ minimal in $\{g_1h_j\}_j$ shows that $\alpha_i\beta_j=0$ whenever $g_ih_j<1$. This completes the proof.

We now prove the general case.

THEOREM 1.2. *Let R be a commutative ring with 1 and let G be right-ordered. Then $\sum \alpha_g g$ is a unit in RG if and only if there exist β_g in R such that $\sum \alpha_g\beta_{g^{-1}}=1$ and $\alpha_g\beta_h$ is nilpotent whenever $gh\neq 1$.*

Proof. First assume $\sum \alpha_g g$ is a unit in RG. Let $\beta(R)$ denote the prime radical of R. Passing from RG to $(R/\beta(R))G$, $\sum \bar{\alpha}_g g$ is a unit in $(R/\beta(R))G$. Proposition 1.1 then says that there exist $\bar{\beta}_g$ in $(R/\beta(R))G$ such that $\sum \bar{\alpha}_g\bar{\beta}_{g^{-1}}=\bar{1}$ and $\bar{\alpha}_g\bar{\beta}_h=0$ whenever $gh\neq 1$. Hence $\sum \alpha_g\beta_{g^{-1}}=1+n$ where $n \in B(R)$ and $\alpha_g\beta_h$ is nilpotent whenever $gh\neq 1$. If $n^k=0$, we see that $\sum \alpha_g\beta_{g^{-1}}(1-n+n^2\cdots \pm n^{k-1})=1$ and $\alpha_g\beta_h(1-n+n^2\cdots \pm n^{k-1})$ is nilpotent whenever $gh\neq 1$. Hence $\sum \alpha_g g$ satisfies the required conditions.

Conversely, if $\sum \alpha_g g$ satisfies the conditions, then $\sum \bar{\alpha}_g g$ is a unit in $(R/\beta(R))G$. Since $\beta(R)G$ is nil, we conclude that $\sum \alpha_g g$ is a unit in RG.

We state two special cases of this theorem which will be required later.

COROLLARY 1.3. *Let R be commutative with no idempotents $\neq 0, 1$. Let G be right-ordered. Then $\sum \alpha_g g$ is a unit in RG if and only if α_h is a unit for some h and all other α_g's are nilpotent.*

Proof. By Theorem 1.2, there exist β_g in R such that $\sum \alpha_g \beta_{g-1} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$. Hence $\alpha_g \beta_{g-1} \alpha_g = \alpha_g + n$ where n is nilpotent. Therefore $\alpha_g \beta_{g-1}$ is an idempotent modulo $\beta(R)$, and we conclude that $\alpha_g \beta_{g-1} \in \beta(R)$ or $\alpha_g \beta_{g-1} - 1 \in \beta(R)$ since idempotents can be lifted modulo $\beta(R)$.

If $\alpha_g \beta_{g-1} \in \beta(R)$, then $\alpha_g \in \beta(R)$ by the above. If $\alpha_g \beta_{g-1} - 1 \in \beta(R)$ and $k \neq g$, then $\alpha_k \beta_{g-1} \in \beta(R)$ implies that $\alpha_k \in \beta(R)$. Hence α_h is a unit for some h and if $g \neq h$, α_g must be nilpotent.

COROLLARY 1.4 [7]. *Let R be commutative with no nilpotent elements $\neq 0$ and no idempotents $\neq 0, 1$. Then the only units in RG are of the form ug where u is a unit of R and g is in G.*

2. **Automorphisms.** We will present necessary and sufficient conditions for a map $x \to \sum a_i x^i$ to induce an R-automorphism of $R\langle x \rangle$, where $\langle x \rangle$ is an infinite cyclic group. It is interesting to compare this result with the corresponding theorem for polynomial rings obtained by Gilmer [4] for R commutative and by Coleman and Enochs [3] in general.

Let $Z(R)$ denote the centre of R.

THEOREM 2.1. *Let R be a ring with 1. Then $x \to \sum a_i x^i$ induces an R-automorphism of $R\langle x \rangle$ if and only if the following two conditions hold:*

(i) *$\sum a_i x^i$ is a unit in $Z(R)\langle x \rangle$.*

(ii) *If $i \neq 1, -1$, then a_i is nilpotent.*

NOTE. Using Theorem 1.2, (i) can be replaced by the condition that there exists $\sum b_i x^i$ in $Z(R)\langle x \rangle$ with $\sum a_i b_{-i} = 1$ and such that $a_i b_j$ is nilpotent whenever $i + j \neq 0$.

Proof of Theorem. First assume $x \to \sum a_i x^i$ induces an R-automorphism of $R\langle x \rangle$. Then since x is a unit in $Z(R\langle x \rangle) = Z(R)\langle x \rangle$, $\sum a_i x^i$ must be a unit there also. It remains to prove (ii).

Let $(\sum a_i x^i)^{-1} = \sum b_i x^i$. We know from Theorem 1.2 that $\sum a_i b_{-i} = 1$ and that $a_i b_j$ is nilpotent whenever $i + j \neq 0$.

Note that a_i, b_i all lie in $Z(R)$. If P is a prime ideal of $Z(R)$, then passing to $(Z(R)/P)\langle x \rangle$ we get that $\sum \bar{a}_i x^i$ is a unit in $(Z(R)/P)\langle x \rangle$. Corollary 1.4 says that for some j, $\bar{a}_j$ is a unit and, for all $i \neq j$, $\bar{a}_i = 0$. Hence we conclude that exactly one a_i and exactly one b_j do not lie in P. In particular, if $i \neq j$, $a_i a_j$ and $b_i b_j$ are nilpotent.

Since $x \to \sum a_i x^i$ induces an R-automorphism of $R\langle x\rangle$, there exist $c_j \in Z(R)$ such that $x = \sum c_j(\sum a_i x^i)^j$. Equating coefficients of x and using the above remarks and the fact that $(\sum a_i x^i)^{-j} = (\sum b_i x^i)^j$, we get that $1 = c_1 a_1 + c_{-1} b_1 + n$ where n is nilpotent. Hence a_1 and b_1 do not lie in any common prime ideal of $Z(R)$.

Now $\sum a_i b_{-i} = 1$ together with the fact that $a_i b_j$ is nilpotent if $i+j \neq 0$, implies that $b_1 a_{-1} b_1 = b_1 + m$ where m is nilpotent. We conclude that if P is a prime ideal of $Z(R)$ and $a_{-1} \in P$, then $b_1 \in P$ also. This, together with the last paragraph, leads to the conclusion that a_1 and a_{-1} lie in no common prime ideals of $Z(R)$. However, we remarked earlier that exactly one a_i does not lie in any given prime ideal. Hence, if $i \neq 1, -1$, then a_i must be nilpotent.

Conversely, assume that $\sum a_i x^i$ satisfies the two conditions. We will first prove that $x \to \sum a_i x^i$ induces a $Z(R)$-algebra automorphism of $Z(R)\langle x\rangle$. To do this, we must show that the induced map is $1-1$ and onto.

First assume that there exist $c_j \in Z(R)$ with $\sum c_j(\sum a_i x^i)^j = 0$. As before, let $(\sum a_i x^i)^{-1} = \sum b_i x^i$. If P is a prime ideal of $Z(R)$, passing to $(Z(R)/P)G$ we get $\sum \bar{c}_j(\sum \bar{a}_i x^i)^j = 0$. We know by condition (ii) that all $\bar{a}_i$ except for $\bar{a}_1$ or $\bar{a}_{-1}$ must be zero, and condition (i) and Corollary 1.4 say that either $\bar{a}_1$ or $\bar{a}_{-1}$ must also equal zero. Say $\bar{a}_1 \neq 0$. Then $\bar{b}_{-1} \neq 0$, $\bar{a}_1 \bar{b}_{-1} = 1$ and all other $\bar{a}_i$ and $\bar{b}_i$ equal zero. Hence, for each $j > 0$, $\bar{c}_j \bar{a}_1^j = 0$ and hence $\bar{c}_j = 0$ since $\bar{a}_1$ is a unit in $Z(R)/P$ by above. For each $j < 0$, $\bar{c}_j \bar{b}_{-1}^{-j} = 0$ and $\bar{c}_j = 0$. $\bar{c}_0 = 0$ is obtained directly. Therefore, each c_j is nilpotent in $Z(R)$.

Let T be the ideal of $Z(R)$ generated by $\{c_j\} \cup \{a_i : a_i \text{ is nilpotent}\} \cup \{b_i : b_i \text{ is nilpotent}\} \cup \{a_i a_j : i \neq j\} \cup \{b_i b_j : i \neq j\}$. Then T is a nilpotent ideal of $Z(R)$.

Say that each c_j is in T^k but that some c_i does not belong to T^{k+1}. Then mod $T^{k+1}\langle x\rangle$, we have $\sum \bar{c}_j(\sum \bar{a}_i x^i)^j = 0$, i.e.

$$\bar{c}_0 + \sum_1^t (\bar{c}_j \bar{a}_{-1}^j x^{-j} + \bar{c}_j \bar{a}_1^j x^j) + \sum_{-s}^{-1} (\bar{c}_j \bar{b}_{-1}^{-j} x^j + \bar{c}_j \bar{b}_1^{-j} x^{-j}) = 0$$

since all other products will equal zero in $(Z(R))/T^{k+1}$. Equating coefficients of x^j, $j > 0$, we get

$$\bar{c}_j \bar{a}_1^j + \bar{c}_{-j} \bar{b}_1^j = 0.$$

Hence $\bar{c}_j \bar{a}_1^j \bar{b}_{-1} + \bar{c}_{-j} \bar{b}_1^j \bar{b}_{-1} = 0$. But $c_{-j} b_1 b_{-1}$ is in T^{k+1}. Hence $\bar{c}_j \bar{a}_1^j \bar{b}_{-1} = 0$.

Now since $\sum \bar{a}_i \bar{b}_{-i} = 1$ and $\bar{c}_j \bar{a}_i = 0$ if $i \neq 1, -1$, we conclude that $0 = \bar{c}_j \bar{a}_1 \bar{b}_{-1} = \bar{c}_j \bar{a}_1^{j-1}(\bar{a}_1 \bar{b}_{-1}) = \bar{c}_j \bar{a}_1^{j-1}(1 - \bar{a}_{-1} \bar{b}_1) = \bar{c}_j \bar{a}_1^{j-1}$ since $a_1 a_{-1}$ is in T. Continuing this argument, we get $\bar{c}_j \bar{a}_1 = 0$.

Similarly, equating coefficients of x^{-j} for $j > 0$ gives $\bar{c}_j \bar{a}_{-1}^j + \bar{c}_{-j} \bar{b}_{-1}^j = 0$.

An identical argument to that given above yields $\bar{c}_j \bar{a}_{-1} = 0$. Hence we have $\bar{c}_j \bar{a}_1 = 0$ and $\bar{c}_j \bar{a}_{-1} = 0$, and therefore $\bar{c}_j = \bar{c}_j(\sum \bar{a}_i \bar{b}_{-i}) = 0$. Hence $c_j \in T^{k+1}$. In this way, we conclude that all c_j's are in arbitrarily large powers of T, and hence equal zero.

Therefore the $Z(R)$-homomorphism induced by $x \to \sum a_i x^i$ is $1-1$. To show that it is onto, we must find $c_j \in Z(R)$ such that $x = \sum c_j(\sum a_i x^i)^j$. First notice that if

https://doi.org/10.4153/CMB-1975-101-3 Published online by Cambridge University Press

$(\sum a_i x^i)^{-1} = \sum b_i x^i$, then $b_{-1}(\sum a_i x^i) + a_{-1}(\sum b_i x^i) = x + n_1$ where n_1 is nilpotent. Say $n_1 = \sum d_i x^i$ and let T be the ideal of $Z(R)$ generated by the d_i. Then T is nilpotent.

Now $x + n_1 - \sum d_i(x+n_1)^i = x + n_2$ where $n_2 \in T^2\langle x\rangle$. Continuing in this way, and using the fact that T is nilpotent, we get x as a linear combination of powers of $\sum a_i x^i$.

Thus $x \to \sum a_i x^i$ induces a $Z(R)$-algebra automorphism of $Z(R)\langle x\rangle$. It is easy to see that we can lift such an automorphism to $R\langle x\rangle$. Define $\sigma: R\langle x\rangle \to R\langle x\rangle$ by $\sigma(x) = \sum a_i x^i$ and $\sigma(r) = r$. Clearly σ is onto and invertible, hence an automorphism of $R\langle x\rangle$.

3. **Infinite cyclic groups.** We now proceed to the study of isomorphic group rings. In this section, we will consider the case where the group $\langle x\rangle$ is infinite cyclic. All rings are assumed to have 1.

Before beginning we need a lemma.

LEMMA 3.1. *Any central idempotent of $R\langle x\rangle$ lies in R.*

Proof. Since central idempotents of a group ring have finite support group [2], the result is immediate.

Recall that two rings R and S are subisomorphic if R can be embedded in S and S can be embedded in R.

THEOREM 3.2. *Let R, S have no idempotents $\neq 0$, 1. Then $R\langle x\rangle \simeq S\langle x\rangle$ implies R and S are subisomorphic.*

Proof. Let $\theta: R\langle x\rangle \to S\langle x\rangle$ be an isomorphism. We may set $R\langle x\rangle = T\langle u\rangle$ where $T = \theta^{-1}(S)$ and $u = \theta^{-1}(x)$. Since u is a unit in $Z(R\langle x\rangle) = Z(R)\langle x\rangle$, Corollary 1.3 says that $u = vx^r + \sum a_i x^i$ where v is a unit in $Z(R)$ and a_i is nilpotent in $Z(R)$ for all i.

Hence $R\langle x\rangle = T\langle vx^r + \sum a_i x^i\rangle$.

First consider the case where $r=0$. Then we have $R\langle x\rangle = T\langle v + \sum a_i x^i\rangle$. Now x is a unit in $Z(T)\langle v + \sum a_i x^i\rangle$, so we must have

$$x = m(v + \sum a_i x^i)^l + \sum b_j(v + \sum a_i x^i)^j$$

where m is a unit in $Z(T) \subseteq Z(R)\langle x\rangle$ and b_j is nilpotent in $Z(T)$ for all j. Here we are using the fact that $Z(T)$ has no idempotents $\neq 0$, 1, which is true using Lemma 3.1 since $Z(T) \subseteq Z(R\langle x\rangle)$. We shorten this by writing $x = mv^l + \sum c_i x^i$ where m is a unit in $Z(R)\langle x\rangle$ and c_i is nilpotent and central for all i.

Since m is a unit, we must have

$$m = wx^s + \sum d_i x^i$$

for some w a unit in $Z(R)$ and d_i nilpotent in $Z(R)$. Hence

$$x = (wx^s + \sum d_i x^i)v^l + \sum c_i x^i.$$

Since c_i and d_i are nilpotent, we conclude that $s=1$. Therefore we have $m=wx+\sum d_ix^i$ is a unit in $Z(T)$.

Using Theorem 2.1, we obtain $T\langle v+\sum a_ix^i\rangle=T\langle(wx+\sum d_ix^i)(v+\sum a_ix^i)\rangle=T\langle wvx+\sum e_ix^i\rangle$ with e_i nilpotent in $Z(R)$.

Also $R\langle x\rangle=R\langle wvx\rangle$ since w, v are units in $Z(R)$
$=R\langle wvx+\sum e_iw^{-i}v^{-i}(wxv)^i\rangle$ again using Theorem 2.1. Hence we have

$$R\langle wvx+\sum e_ix^i\rangle = T\langle wvx+\sum e_ix^i\rangle.$$

Recall that the augmentation ideal $\Delta_R(G)$ of a group ring RG is the ideal generated by $\{g-1:g\in G\}$ and $(RG/\Delta_R(G))\simeq R$. In our case, the augmentation ideal is the ideal generated by $wvx+\sum e_ix^i-1$, and factoring this out from both sides yields $R\simeq T$. Since $T=\theta^{-1}(S)$, we have $R\simeq S$.

Next we assume that $r\neq 0$, i.e.

$R\langle x\rangle=T\langle vx^r+\sum a_ix^i\rangle$, $r\neq 0$. Now $R\langle x\rangle$ contains $R\langle vx^r+\sum a_ix^i\rangle$ as a subring, so we have $R\langle vx^r+\sum a_ix^i\rangle\subseteq T\langle vx^r+\sum a_ix^i\rangle$. To see that R can be embedded in T, we follow the diagram $R\langle vx^r+\sum a_ix^i\rangle\to^i T\langle vx^r+\sum a_ix^i\rangle\to^p (T\langle vx^r+\sum a_ix^i\rangle/I)\simeq T$ where I is the augmentation ideal of $T\langle vx^r+\sum a_ix^i\rangle$. Clearly, the augmentation ideal of $R\langle vx^r+\sum a_ix^i\rangle$ is contained in $\operatorname{Ker} p\circ i$. Conversely, assume s is in $R\cap \operatorname{Ker} p\circ i$. Then we have

$$s = (vx^r+\sum a_ix^i-1)(g(x))$$

for some $g(x)\in R\langle x\rangle$.

Let T be the ideal of R generated by $\{a_i\}$. Since the a_i are in $Z(R)$, T is nilpotent. Passing to $(R/T)\langle x\rangle$, we get $(\bar{v}x^r-1)(\bar{g}(x))=\bar{s}$ which implies that $\bar{s}=0$ and $\bar{g}(x)=0$ since $\bar{v}$ is a unit in (R/T) and $r\neq 0$. Hence $s\in T$ and $g(x)\in T\langle x\rangle$.

Passing to $(R/T^2)\langle x\rangle$, we get $(\bar{v}x^r-1)(\bar{g}(x))=\bar{s}$ since $a_ig(x)\in T^2\langle x\rangle$ for all i, and this again implies $\bar{s}=0$ and $\bar{g}(x)=0$. Continuing we get that s is in arbitrarily large powers of T and hence $s=0$.

Therefore $\operatorname{Ker} p\circ i$ is the augmentation ideal of $R\langle vx^r+\sum a_ix^i\rangle$, and weconclude that $p\circ i$ induces an embedding of R into T and hence into S. Arguing in the other direction produces an embedding of S into R. Hence R and S are subisomorphic.

Of course, the above proof requires R and S to have no idempotent $\neq 0$, 1. However, we have

PROPOSITION 3.3. *Let R and S be finite direct sums of rings with no idempotents $\neq 0$, 1. Then $R\langle x\rangle\simeq S\langle x\rangle$ implies R and S are subisomorphic.*

Proof. Let $R=R_1\oplus R_2\oplus\cdots\oplus R_m$ where each R_i has no idempotent except zero and the identity. Say $1=\ell_1+\cdots+\ell_m$. The ℓ_i are orthogonal, central idempotents of R. They are also primitive because the only idempotents of R are of the form $\sum \ell_j$ for certain j.

Say $\sigma:R\langle x\rangle\to S\langle x\rangle$ is an isomorphism. Then $R\langle x\rangle\simeq R_1\langle x\rangle\oplus\cdots\oplus R_m\langle x\rangle$ and

$\sigma(\ell_i)$ must be a primitive, orthogonal central idempotent of $S\langle x\rangle$. If $S=S_1\oplus\cdots\oplus S_n$, then we conclude that $\sigma(R_i\langle x\rangle)=S_j\langle x\rangle$ for some j. By Theorem 3.2, R_i and S_j are subisomorphic. Hence R and S are subisomorphic.

Since commutative Noetherian rings can be written as a finite direct sum of rings with no idempotents $\neq 0, 1$, we have:

COROLLARY 3.4. *Let R and S be commutative Noetherian rings. Then* $R\langle x\rangle\simeq S\langle x\rangle$ *implies R and S are subisomorphic.*

Note that in [8], certain classes of rings were found for which $R\langle x\rangle\simeq S\langle x\rangle$ implied R and S are isomorphic. These included Artinian rings.

4. **Regular and local rings.** We now consider the case where $RG\simeq SG$ with G more general than an infinite cyclic group. In particular, the class of groups we are interested in is the class C defined as follows:

G belongs to C if and only if whenever F is a field, then FG has no zero-divisors and the only units of FG are of the form ug where $u\in F$ and $g\in G$.

This class is known to contain many torsion free groups including ordered groups (Corollary 1.4) and all groups with a finite series $G=G_0\supseteq G_1\supseteq\cdots\supseteq G_n=(1)$ with $G_{i+1}\lhd G_i$ and G_i/G_{i+1} torsion-free abelian for all i [1]. It is an open question as to whether or not all torsion-free groups are in C.

Recall that a ring R is (von Neumann) regular if for any a in R, there exists b in R with $aba=a$. We prove the following:

PROPOSITION 4.1. *Let R, S be commutative regular rings with* 1 *and let G be in C. Then if* $\sigma:RG\to SG$ *is a homomorphism,* $\sigma(R)\subseteq S$.

Proof. Let $f\in SG$ satisfy $fgf=f$ and $(f-1)h(f-1)=f-1$ for some $g, h\in SG$. We will show that $f\in S$.

Let P be a prime ideal of S. Passing from SG to $(S/P)G$, we get $\bar f\bar g\bar f=\bar f$, so $\bar f(\bar g\bar f-1)=0$. Since G is in C, $(S/P)G$ has no zero-divisors, so either $\bar f=0$ or $\bar g\bar f=1$. Similarly, either $\overline{f-1}=0$ or $(\overline{f-1})(\bar h)=1$.

If $\bar f\neq 0$, then $\bar f$ is a unit in $(S/P)G$, so $\bar f=ug$ for some u a unit in S/P and g in G, since G is in C. $\overline{f-1}=0$ would then imply that $g=1$. Otherwise, $\overline{f-1}$ is a unit, which means $\overline{f-1}=vh$ for some v in S/P and $h\in G$. Hence $\bar f=1+vh=ug$ so $g=h=1$.

Therefore, if $\bar f\neq 0$, then $\bar f=u$ for some unit in S/P. We conclude that if $f=\sum \alpha_g g$ and $g\neq 1$, then α_g belongs to every prime ideal of S and hence $\alpha_g=0$ since S is regular. Therefore f is in S.

Now if r is in R, then $\sigma(r)$ satisfies the conditions on f stated at the beginning of the proof because R is regular. Hence $\sigma(r)$ is in S and $\sigma(R)\subseteq S$ as required.

We notice several corollaries of this proposition.

COROLLARY 4.2. *Let R, S be commutative regular rings with* 1, *and let G be in C. If* $\sigma:RG\to SG$ *is an isomorphism, then* $\sigma(R)=S$.

COROLLARY 4.3. *Let R, S be regular rings with 1 and let G be torsion-free abelian. If $\sigma: RG \to SG$ is an isomorphism, then $\sigma(Z(R)) = Z(S)$.*

Proof. σ restricts to an isomorphism from $Z(RG)$ to $Z(SG)$. But, by the assumption on G, $Z(RG) = Z(R)G$ and $Z(SG) = Z(S)G$.

Hence $\sigma: Z(R)G \to Z(S)G$.

Also $Z(R)$ and $Z(S)$ are regular [6]. Hence Corollary 4.2 gives the result.

We now turn our attention to local rings. Recall that a ring R with 1 is called local if the non-units of R form an ideal.

PROPOSITION 4.4. *Let R, S be commutative local rings with no non-zero nilpotent elements. Let G be ordered. If $\sigma: RG \to SG$ is a homomorphism, then $\sigma(R) \subseteq S$.*

Proof. Since R is local, for all r in R either r or $1-r$ is a unit. Hence it is enough to prove that if r is a unit in R, then $\sigma(r)$ belongs to S.

Since R and S are local, they contain no idempotents $\neq 0, 1$. Since they also have no nilpotent elements, Corollary 1.4 says that if r is a unit of R, then $\sigma(r) = ug$ for some unit u of S and some g in G. Note that if $g \neq 1$, then $g \neq g^{-1}$ since G is ordered.

Now $\sigma(r + r^{-1}) = ug + u^{-1}g^{-1}$ which is not a unit of SG unless $g = 1$ by Corollary 1.4. Hence $r + r^{-1}$ is not a unit of R. Since R is local, $1 - (r + r^{-1})$ is a unit of R. But $\sigma(1 - r - r^{-1}) = 1 - ug - u^{-1}g^{-1}$ is not a unit of SG unless $g = 1$. Hence $g = 1$ and $\sigma(R) \subseteq S$ as required.

If we assume further that G is abelian, we can drop the condition that R and S have no nilpotent elements and still conclude that $RG \simeq SG$ implies $R \simeq S$, as is proved in the following:

PROPOSITION 4.5. *Let R, S be commutative local rings. Let G be finitely generated torsion free abelian. If $\sigma: RG \to SG$ is an isomorphism, then $R \simeq S$.*

Proof. Assume $G = \langle x_1 \rangle x \cdots x \langle x_k \rangle$. It is easy to check that $\beta(RG) = \beta(R)G$ and $\beta(SG) = \beta(S)G$, where $\beta(R)$ is the prime radical of R. Hence σ induces an isomorphism $\bar{\sigma}: (R/\beta(R))G \to (S/\beta(S))G$. Proposition 4.4 then says that $\bar{\sigma}(R/\beta(R)) = S/\beta(S)$.

Hence if r is in R, then $\sigma(r) = s + n$ where s is in S and n is nilpotent in SG.

Let M be the subring of SG generated by S and $\{\sigma(x_i^{\pm 1})\}$. We will show that $M = SG$.

Let t be in SG and say $t = \theta(w)$, $w \in RG$, where $w = \sum r_i g_i$. Then $t = \theta(w) = \sum \theta(r_i)\theta(g_i) = \sum (s_i + n_i)\theta(g_i)$ where s_i is in S and n_i is nilpotent as before

$$= \sum s_i \theta(g_i) + \sum n_i \theta(g_i)$$

$$= m + n \text{ where } m \text{ is in } M \text{ and } n \text{ is nilpotent in } SG.$$

Moreover, we can assume that the coefficient of the identity in n is 0 since $S \subseteq M$. In particular, there exist elements $n_1(x_1, \ldots, x_k), \ldots n_k(x_1, \ldots, x_k)$ of $\beta(SG)$

and $m_1, \ldots, m_k$ in M such that

$$\begin{aligned} x_1 &= m_1+n_1(x_1, \ldots, x_k) \\ x_2 &= m_2+n_2(x_1, \ldots, x_k) \\ &\vdots \\ x_k &= m_k+n_k(x_1, \ldots, x_k). \end{aligned}$$

Continually substituting for $x_1, \ldots, x_k$ and using the fact that $n_1, \ldots, n_k$ have nilpotent coefficients we conclude that $x_1, x_2, \ldots, x_k$ all belong to M. Hence $M=SG$.

We have just shown that SG is generated as a ring by S and $\{\sigma(x_i^{\pm 1})\}$. In fact, we claim that $SG=S\langle\sigma(x_1), \ldots, \sigma(x_n)\rangle$. It remains only to prove that $\{\sigma(g):g \text{ in } G\}$ is independent over S.

Notice that if g_i, g_j are in G, $g_i \neq g_j$, then $\sigma(g_i)=uk_i+\sum a_ih_i$ and $\sigma(g_j)=vk_j+\sum b_ih_i$ where u and v are units in S, a_i and b_i are nilpotent in S, k_i, h_i, k_j are in G and $k_i \neq k_j$. All of these follow from Corollary 1.3, since commutative local rings have no idempotents $\neq 0$, 1, except for the last statement. To see the last statement, let r_1, r_2 be units in R and n_1, n_2 nilpotent in RG such that $\sigma(r_1+n_1)=u^{-1}$ and $\sigma(r_2+n_2)=v^{-1}$ and assume that $k_i=k_j$. Hence $\sigma((r_1+n_1)g_i-(r_2+n_2)g_j)$ is nilpotent and we conclude that $(r_1+n_1)g_i-(r_2+n_2)g_j$ has nilpotent coefficients. This clearly implies $g_i=g_j$.

Now we will prove that $\{\sigma(g):g \in G\}$ is independent over S. Assume to the contrary that $\sum s_i\sigma(g_i)=0$. Say $\sigma(g_i)=u_ik_i+\sum a_{i,j}h_{i,j}$ with u_i units in S, $a_{i,j}$ nilpotent in S, k_i and $h_{i,j}$ in G for all i, j and $k_i \neq k_j$ if $i \neq j$. Then $\sum s_i(u_ik_i+\sum a_{i,j}h_{i,j})=0$ which implies s_iu_i is nilpotent for all i. Hence s_i is nilpotent for all i. Let T be the ideal of S generated by $\{s_i\} \cup \{a_{i,j}\}$. Passing to $(S/T^2)G$, we obtain $\sum \bar{s}_i(\bar{u}_ik_i+\sum \bar{a}_{i,j}h_{i,j})=0$, hence $\sum \bar{s}_i(\bar{u}_ik_i)=0$ so each $\bar{s}_i=0$ and $s_i \in T^2$. Continuing we get that each s_i is in arbitrarily large powers of T and hence each $s_i=0$ since T is nilpotent.

Therefore $SG=S\langle\sigma(x_1), \ldots, \sigma(x_n)\rangle$.

Hence $R\langle x_1, \ldots, x_n\rangle \simeq S\langle\sigma(x_1) \cdots \sigma(x_n)\rangle$. Clearly we can modify this isomorphism so that each $\sigma(x_i)$ has content (sum of coefficients) equal to one. Therefore if $\Delta_R(G)$ and $\Delta_S(G)$ are the augmentation ideals of RG and SG respectively, we can conclude that $\sigma(\Delta_R(G))=\Delta_S(G)$.

Hence

$$R \simeq \frac{RG}{\Delta_R(G)} \simeq \frac{SG}{\Delta_S(G)} \simeq S.$$

The corresponding results for polynomial rings have been obtained by Jacobson [6], and several of the techniques in the above proof are found there.

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