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INJECTIVITY IN EQUATIONAL CLASSES OF ALGEBRAS

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1. Introduction. The concept of injectivity in classes of algebras can be traced back to Baer's initial results for Abelian groups and modules in [**1**]. The first results in non-module types of algebras appeared when Halmos [**14**] described the injective Boolean algebras using Sikorski's lemma on extensions of Boolean homomorphisms [**19**]. In recent years, there have been several results (see references) describing the injective algebras in other particular equational classes of algebras.

In [**10**], Eckmann and Schopf introduced the fundamental notion of essential extension and gave the basic relations that this concept had with injectivity in the equational class of all modules over a given ring. They developed the notion of an injective hull (or envelope) which provided every module with a minimal injective extension or equivalently, a maximal essential extension. In [**6**] and [**9**], it was noted that these relationships hold in any equational class with enough injectives.

In this paper, the problem of enough injectives in an equational class is reduced to properties of the subdirectly irreducible algebras in the class. Using this approach, a general existence theorem is proven which gives as corollaries many of the known results. In particular, it shows that non-pathological, equationally complete equational classes in which every algebra has a congruence lattice which is distributive (such a class will be called congruence distributive) has enough injectives. Modulo the results for Boolean algebras in [**8**], the injective algebras and the passage to injective hulls are described.

2. Preliminaries. All universal algebraic concepts may be found in [**12**], and all categorical results in [**18**]. As usual, an equational class will be considered as a category with all homomorphisms between algebras in the class. We will use upper case letters, A, to denote algebras, and $|A|$ to denote their underlying sets.

Let $\mathfrak{K}$ be an equational class. An algebra $Q \in \mathfrak{K}$ is called injective if for every $B \in \mathfrak{K}$, every subalgebra A of B (written $A \leqq B$) and every $f \colon A \to Q$, there exists (a homomorphism) $g \colon B \to Q$ extending f (i.e., $g|A = f$). A monomorphism $f \colon A \rightarrowtail B$ is called essential if for any $C \in \mathfrak{K}$ and any $g \colon B \to C$, gf is injective if and only if g is. Since $\mathfrak{K}$ is assumed to be an equational class, this is equivalent to

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the fact: the only congruence relation, θ, on B which separates (i.e., does not identify any pair of distinct points in) A is Δ_B, the identity relation on B. If $A \leqq B$, and the natural embedding is essential, we write $A \leqq_E B$ and say that A is a large subalgebra of B. An algebra $A \in \mathfrak{K}$ is said to have no proper essential extensions (in $\mathfrak{K}$!) if every essential monomorphism (in $\mathfrak{K}$) with domain A is an isomorphism. $A \in \mathfrak{K}$ is called an absolute sub-retract (in $\mathfrak{K}$!) if every monomorphism in $\mathfrak{K}$, with domain A, splits (i.e., $f: A \rightarrowtail B \in \mathfrak{K}$ implies there exists $g: B \twoheadrightarrow A$ with $gf = 1_A$). In an equational class these two properties are equivalent since by a standard Zorn's Lemma argument on the set of all congruence relations, θ, on B with $(f \times f)^{-1}(\theta) = \Delta_A$, every monomorphism can be continued to an essential monomorphism (i.e., if $f: A \rightarrowtail B$, there exists $g: B \twoheadrightarrow C$ such that $gf: A \rightarrowtail C$ is essential). An injective hull of A in $\mathfrak{K}$ is an injective essential extension (i.e., $f: A \rightarrowtail B$ is essential and $B \in \mathfrak{K}$ is injective). Finally, $\mathfrak{K}$ is said to have enough injectives if every algebra in $\mathfrak{K}$ has an injective extension (in $\mathfrak{K}$).

The following results can be found in [**6**].

THEOREM 2.1. *Let $\mathfrak{K}$ have enough injectives. Then for $A \in \mathfrak{K}$, t.f.a.e.:*
(1) *A is injective.*
(2) *A is an absolute subretract.*
(3) *A has no proper essential extensions.*

THEOREM 2.2. *For an equational class $\mathfrak{K}$, t.f.a.e.:*
(1) *$\mathfrak{K}$ has enough injectives.*
(2) *Every $A \in \mathfrak{K}$ has an injective hull.*
(3) *Every $A \in \mathfrak{K}$ has a representative set of essential extensions and in $\mathfrak{K}$, qua category, pushouts preserve monomorphisms.*

Since every homomorphism can be factored into the composition of a monomorphism and a surjective homomorphism, the pushout criterion in condition (3) of 2.2 is equivalent (in an equational class) to the conjunction of the following two properties.

Definition. A class, $\mathfrak{K}$, of algebra is said to satisfy:
(1) the (weak) amalgamation property if (AP): $A, B_i \in \mathfrak{K}$ $(i = 1, 2)$, $f_i: A \rightarrowtail B_i$ $(i = 1, 2)$ imply there exists $C \in \mathfrak{K}$ and $g_i: B_i \rightarrowtail C$ $(i = 1, 2)$ with $g_1 f_1 = g_2 f_2$;
(2) the congruence extension property if (CEP): $A, B \in \mathfrak{K}$ and $A \leqq B$ imply that every congruence on A is the restriction of some congruence on B.

THEOREM 2.3. *Let $\mathfrak{K}$ be an equational class with enough injectives, and let $f: A \rightarrowtail B$ be a monomorphism in $\mathfrak{K}$. Then t.f.a.e.:*
(1) *B is an injective hull of A (with respect to f).*
(2) *B is a maximal essential extension of A (with respect to f) (i.e., f is essential and if $g: B \rightarrowtail C$ is such that gf is essential, then g is an isomorphism).*
(3) *B is a minimal injective extension (with respect to f) (i.e., B is injective and no proper subalgebra of B that contains* $\mathrm{Im}(f)$ *is also injective).*

3. Injectivity and subdirectly irreducible algebras. Since every equational class, $\mathfrak{K}$, is uniquely determined by $\mathfrak{K}_{SI}$, its class of subdirectly irreducible algebras, it would be of interest to relate injectivity to the subdirectly irreducibles.

LEMMA 3.1. *Every essential extension of a subdirectly irreducible algebra is again subdirectly irreducible. Conversely, if an equational class, $\mathfrak{K}$, satisfies* (CEP), *then every large subalgebra of a subdirectly irreducible in $\mathfrak{K}$ is also in $\mathfrak{K}_{SI}$.*

Proof. The first statement was shown in [**9**]. Conversely, if $\mathfrak{K}$ satisfies (CEP), $B \in \mathfrak{K}_{SI}$, and $A \leqq_E B$, let $(\theta_i)_{i\in I}$ be a family of congruences on A whose meet ($=$ intersection) is Δ_A. There exists a family $(\psi_i)_{i\in I}$ of congruences on B such that $\psi_i|A = \theta_i$ $(i \in I)$. Now

$$\left(\bigcap_{i\in I} \theta_i\right)\Bigg| A = \bigcap_{i\in I} (\psi_i|A) = \bigcap_{i\in I} \theta_i = \Delta_A.$$

Since $A \leqq_E B$, we must have

$$\bigcap_{i\in I} \psi_i = \Delta_B$$

and since $B \in \mathfrak{K}_{SI}$, $\psi_i = \Delta_B$ for some $i \in I$. Therefore for this $i \in I$,

$$\theta_i = \Delta_B|A = \Delta_A.$$

(See also [**13**].)

Therefore, if $\mathfrak{K}$ has enough injectives, the injective hull of every $A \in \mathfrak{K}_{SI}$ will also be in $\mathfrak{K}_{SI}$, and in this sense $\mathfrak{K}$ will have "enough injective subdirectly irreducibles". Conversely, if every $S \in \mathfrak{K}_{SI}$ has an extension $T \in \mathfrak{K}_{SI}$ which is injective in $\mathfrak{K}$, then by Birkhoff's Subdirect Representation Theorem, $\mathfrak{K}$ will have enough injectives. This gives the following and at times more applicable result:

THEOREM 3.2. *An equational class has enough injectives if and only if it has enough injective subdirectly irreducibles.*

In particular cases where we have an explicit description of $\mathfrak{K}_{SI}$, this is a more viable procedure for determined when $\mathfrak{K}$ has enough injectives. This is essentially the method used in [**3**; **8**; **9**].

Before proceeding with the main existence theorem, we must first examine an interesting pathology that occurs when the equational class under consideration has no nullary operations defined in its type.

If $\mathfrak{K}$ is such an equational class, then the empty algebra, $\emptyset$, is in $\mathfrak{K}$ and clearly the empty monomorphism $f: \emptyset \rightarrowtail \{x\}$ is essential. In fact, any singleton algebra in $\mathfrak{K}$ is an injective hull of $\emptyset$ with respect to the empty map. Moreover, if $Q \in \mathfrak{K}$ is injective, the empty map $g: \emptyset \rightarrow Q$ must extend to $\bar{g}: \{x\} \rightarrowtail Q$ (i.e., $\bar{g} \circ f = g$) and therefore every injective in $\mathfrak{K}$ has a one element subalgebra.

If we then define "Boolean Algebras" as algebras of type (2, 2, 1) with the usual operations of join, meet, and complementation (to the usual distributive lattice and DeMorgan laws add $xvx' = yvy'$), this equational class has only trivial injectives even though it differs only slightly from Boolean algebras (defined with 0 and/or 1 as nullary operations). In order to avoid this pathology, we will need the following condition.

Definition. An algebra A is called $\emptyset$-regular if $\emptyset \leqq A$ (i.e., no nullary operations) implies A has at least one one-element subalgebra.

4. The main theorem. General existence theorems for enough injectives of an algebraic nature (as opposed to Banaschewski's condition in 2.2) may not be possible. However, the power of Jónsson's Lemma and its consequences (see [**15**]) allow some hope if one considers only congruence distributive equational classes.

Definition. An algebra A is called self-injective if it is $\emptyset$-regular and any homomorphism of a subalgebra of A into A, extends to an endomorphism of A.

Clearly, if A is injective in any equational class (of its type), then A will be self injective but in general, self-injectivity is independent of equational class considerations.

THEOREM 4.1. *Let $\mathfrak{K}$ be a congruence distributive equational subclass of an equational class $\mathfrak{L}$. Furthermore, assume $\mathfrak{K} = \mathbf{SP}(S)$, where S is a finite subdirectly irreducible algebra whose non-empty subalgebras are either injective in $\mathfrak{L}$ or in $\mathfrak{K}_{SI}$. Then t.f.a.e.:*

(1) *$\mathfrak{K}$ has enough injectives.*

(2) *$\mathfrak{K}$ satisfies* (AP) *and* (CEP) (*i.e., pushouts preserve monomorphisms*).

(3) *S is self-injective.*

Proof. Since S is finite and $\mathfrak{K} = \mathbf{SP}(S)$, $\mathfrak{K}_{SI} \subseteq \mathbf{S}(S)$ and S has no proper essential extensions in $\mathfrak{K}$ by 3.1. Therefore, even without congruence distributivity, (1) is equivalent to (2) by 2.1, 3.2, and the fact that in any category where pushouts preserve monomorphisms, the injective objects are exactly the absolute subretracts (see, for example, [**5**]). Clearly then, (1) implies (3) and we need only show that (3) implies (1).

It will suffice to show that S is injective as $\mathfrak{K} = \mathbf{SP}(S)$ and any product of injectives is again injective. Since S is $\emptyset$-regular and again since $\mathfrak{K} = \mathbf{SP}(S)$, we need only complete commutatively diagrams of the form in Figure (i) where A is non-empty.

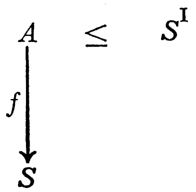

Figure (i)

Let $B = \mathrm{Im}(f) \leqq S$. If B is injective in $\mathfrak{L}$, we are finished. If $B \in \mathfrak{K}_{SI}$, then by Jónssons's Lemma, there exists an ultrafilter U on I such that $\theta_U|A \subseteq \mathrm{Ker}(f)$, the congruence on A induced by f. Since S is finite, every ultraproduct of S is isomorphic to S and therefore there exists $g: S^I \twoheadrightarrow S$ such that $\mathrm{Ker}(g) = \theta_U$.

Consider now Figure (ii) where $A' = g[A]$, $g' = g|A$, and $f': A \to S$ is the canonical homomorphism defined by the fact that:

$$\mathrm{Ker}(g') = (\mathrm{Ker}(g))|A = \theta_U|A \subseteq \mathrm{Ker}(f).$$

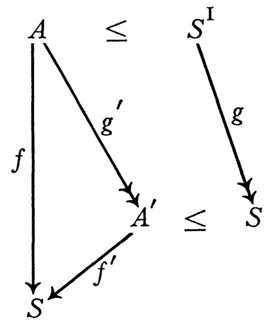

Figure (ii)

Since S is self-injective, there exists $h: S \to S$ such that $h|A' = f'$. Clearly, then, hg is our required extension and S is injective.

Let $\mathfrak{R}(p^n)$ be the equational class of rings generated by $\mathrm{GF}(p^n)$, the Galois field of order p^n (p prime, $n \geqq 1$).

COROLLARY 4.2. *For every prime p and $n \geqq 1$, $\mathfrak{R}(p^n)$ has enough injectives.*

$\mathfrak{R}(p^n)$ is congruence distributive by [**17**]. Moreover, the subrings of $\mathrm{GF}(p^n)$ are either $\{0\}$ or of the form $\mathrm{GF}(p^k)$; for, $k|n$ and for each of these there is a unique homomorphism into $\mathrm{GF}(p^n)$ which extends to the identity on $\mathrm{GF}(p^n)$.

(This result was first proved by Banaschewski using more ring-theoretical methods.)

Let

$$\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \ldots \subset \mathfrak{B}_n \subset \mathfrak{B}_{n+1} \subset \ldots \subset \mathfrak{B}_\infty$$

be the system of all equational subclasses of distributive lattices with pseudo-complementation (see [**16**] for terminology). Then the injectives in $\mathfrak{B}_\infty$ are exactly the complete Boolean algebras. (That these are injective follows from a Glinenko-Stone Theorem argument as used in [**4**] for Heyting algebras. That no non-Boolean algebra in $\mathfrak{B}$ is injective follows from the fact that every such algebra has $\bar{B}_1$, the three element chain, as a subalgebra and $\bar{B}_1$ has arbitrarily large essential extensions, namely $\bar{B}$ for each Boolean algebra B.)

COROLLARY 4.3. *The only non-trivial equational subclasses of $\mathfrak{B}$ that have enough injectives are $\mathfrak{B}_0$ (Boolean algebras) $\mathfrak{B}_1$ (Stone Lattices* [**3**]) *and $\mathfrak{B}_2$.*

By [**13**] and the second condition in 4.1, these are the only proper equational subclasses that satisfy (AP). ($\mathfrak{B}_\infty$ satisfies (CEP).)

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Note. The original proof of this result was based on a lemma that $\bar{B}_n$ is self-injective if and only if $n \leqq 2$. The appearance of [**13**] allows this simpler argument.

Let

$$\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \ldots \subset \mathfrak{A}_n \subset \mathfrak{A}_{n+1} \subset \ldots \subset CA_1$$

be the system of all (non-trivial) equational subclasses of cylindrical algebras of dimension 1. S. Comer has shown that CA_1 has only trivial injectives. Let A_n be the n-atom Boolean algebra with the trivial cylindrification: $c_1(0) = 0$ and $c_1(x) = 1$ if $x \neq 0$. Then $\mathfrak{A}_n = \mathbf{SP}(A_n)$ and:

COROLLARY 4.4. *The only non-trivial equational subclasses of CA_1 that have enough injectives are $\mathfrak{A}_0$, $\mathfrak{A}_1$, and $\mathfrak{A}_2$.*

For, if $n \geqq 3$, and $A_2 = \{0, a, a', 1\} \leqq A_n$ with a an atom of A_n, then $f: A_2 \to A_n$ by $f(a) = a'$ and $f(a') = a$ cannot be extended to an endomorphism. Therefore, A_n is self-injective if and only if $n \leqq 2$.

The next result is also an immediate corollary but because of its (almost) purely algebraic nature it seems to deserve a more prestigous title.

THEOREM 4.5. *Every equationally complete, congruence distributive equational class that contains a non-trivial $\emptyset$-regular finite algebra has enough injectives.*

Proof. If $\mathfrak{K}$ is such a class and $S \in \mathfrak{K}_{SI}$ is a homomorphic image of the non-trivial finite $\emptyset$-regular algebra, then S is finite and $\emptyset$-regular. By [**15**], 3.5, and equational completeness, it follows that S is the only (up to isomorphism) subdirectly irreducible algebra in $\mathfrak{K}$. Therefore, $\mathfrak{K} = \mathbf{SP}(S)$. Since S is finite and every non-trivial subalgebra of S must have S as a homomorphic image, S has at most trivial subalgebras. Therefore, S is self-injective and $\mathfrak{K}$ has enough injectives.

Note. It also follows from the finiteness of S that S is simple.

COROLLARY 4.6. (*Bounded*) *distributive lattices have enough injectives* [**2**; **8**].

COROLLARY 4.7. *Every equational class generated by a primal algebra has enough injectives.*

5. Equationally complete equational classes. In [**8**], Banaschewski and Bruns showed that the injective hull of a Boolean algebra is its MacNeille (or Normal) completion. Modulo these results, an explicit description of the injective algebras and the passage to injective hulls can be given for those equationally complete, congruence distributive equational classes considered in 4.5.

Let $\mathfrak{K} = \mathbf{SP}(S)$ be an equationally complete, congruence distributive equational class with S a finite $\emptyset$-regular subdirectly irreducible. Then as in the proof of 4.5, S is simple, has at most trivial subalgebras, and is up to isomorphism the only subdirectly irreducible in $\mathfrak{K}$.

If at least two distinct elements of S are images of nullary operations of τ, the type of S, then S has no proper subalgebras and only the identity endomorphism. Otherwise, let T be the set of all elements of S that are not images of nullary operations in τ and define a new type $\bar{\tau} = \tau \,\dot{\cup}\, (\lambda_t)_{t \in T}$ where $\lambda_t = 0$ for all $t \in T$ (without loss of generality the domain of τ and T are disjoint). Let $\bar{S}$ be the algebra of type $\bar{\tau}$ obtained from S by the adjunction of the nullary operations $f_{\lambda_t}(\emptyset) = t$ ($t \in T$) and let $\bar{\mathfrak{K}}$ be the equational class of type $\bar{\tau}$ generated by $\bar{S}$.

Now clearly $\bar{\mathfrak{K}} = \mathbf{SP}(\bar{S})$ is equationally complete and congruence distributive. By 4.5, $\bar{\mathfrak{K}}$ has enough injectives. We want a relation between the injectives in $\mathfrak{K}$ and those in $\bar{\mathfrak{K}}$.

THEOREM 5.1. *The injective algebras in $\bar{\mathfrak{K}}$ are exactly the injective algebras in $\mathfrak{K}$ with the extra nullary operations suitably defined.*

Proof. Every non-trivial injective in $\bar{\mathfrak{K}}$ is a $\bar{\tau}$-retract of some power of $\bar{S}$. By forgetting the added nullary operations, this algebra becomes a τ-retract of the same power of S and so is injective in $\mathfrak{K}$.

Conversely, if Q is a non-trivial injective in $\mathfrak{K}$, Q is a τ-retract of some power of S. Therefore, there exists $f: S^I \twoheadrightarrow Q$. Since S is simple and $\mathfrak{K}$ is congruence distributive, it follows from Jónsson's Lemma that $\operatorname{Ker} f$ is induced by a filter F_f on I. That is

$$f(\alpha) = f(\beta) \text{ if and only if } \mathrm{Eq}(\alpha, \beta) = \{i \in I: \alpha(i) = \beta(i)\} \in F_f.$$

If $\Delta: S \rightarrow S^I$ is the embedding of S into the constant functions of S^I, it follows that $f \circ \Delta$ is a monomorphism. Therefore, for each $t \in T$ define

$$f_{\lambda_t}{}^{Q}(\emptyset) = f\Delta(t) = f(f_{\lambda_t}{}^{S}(\emptyset)).$$

Let $\bar{Q}$ be the algebra obtained from Q by the adjunction of these extra nullary operations $(f_{\lambda_t}{}^{Q})_{t \in T}$. It follows easily that $\bar{Q}$ is injective in $\bar{\mathfrak{K}}$.

Without loss of generality, we will assume throughout the remainder of this section that S has no proper subalgebras. This will imply that S has only the identity endomorphism and that for each set I, $\mathrm{Hom}(S^I, S)$ is naturally isomorphic to $\Omega(I)$, the set of ultrafilters on I.

Let $\mathfrak{B}$ be the equational class of Boolean algebras (defined with 0).

Definition. $U: \mathfrak{B} \rightarrow \mathfrak{K}$ is the functor given by:

(a) For $B \in \mathfrak{B}$, $U(B) = S[B]$, the Boolean extension of S by B. (See [**11**], or [**12**, pp. 147–149].)

(b) For $f: A \rightarrow B$, $Uf: UA \rightarrow UB$ is defined by the function $\alpha \mapsto f \circ \alpha$.

That U is a functor is implicit in [**12**, pp. 146–151]. Moreover, U preserves products, equalizers (= monomorphisms in $\mathfrak{B}$), and coequalizers (since every finite Boolean algebra is projective). Since $\mathfrak{B}$, qua category, is locally small and has a cogenerator **2**, by general category theory there exists an adjoint functor $F: \mathfrak{K} \rightarrow \mathfrak{B}$. We wish to give an explicit description of an adjoint.

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Definition. $F: \mathfrak{K} \to \mathfrak{B}$ is the functor defined by:

(a) For $A \in \mathfrak{K}$, $F(A)$ is the field of subsets of $\mathrm{Hom}_{\mathfrak{K}}(A, S)$ generated by the sets

$$X_A(a, M) = \{f \in \mathrm{Hom}_{\mathfrak{K}}(A, S): f(a) \in M\} \quad (a \in |A|, M \subseteq |S|).$$

(b) For $f: A \to B$, $Ff: FA \to FB$ is determined by the restriction to $F(A)$ of the map $f^{\#}: P(\mathrm{Hom}_{\mathfrak{K}}(A, S)) \to P(\mathrm{Hom}_{\mathfrak{K}}(B, S))$ that takes

$$X \mapsto \{g \in \mathrm{Hom}_{\mathfrak{K}}(B, S): g \circ f \in X\}.$$

It follows easily that $Fg(X_A(a, M)) = X_B(g(a), M)$ and since S is injective, F preserves monomorphisms.

Definition. For each $A \in \mathfrak{K}$, $\eta A: A \to UFA$ is the homomorphism defined by:

$$\eta A(a)(s) = X_A(a, \{s\}) \quad (a \in |A|, s \in |S|).$$

Clearly, ηA is a well-defined homomorphism. Moreover, since $\mathrm{Hom}_{\mathfrak{K}}(A, S)$ separates the points of A, ηA is a monomorphism for each $A \in \mathfrak{K}$.

Definition. For $B \in \mathfrak{B}$, $\epsilon B: FUB \to B$ is the (Boolean) homomorphism defined by:

$$\epsilon B(X_{UB}(\alpha, M)) = \bigvee\{\alpha(s): s \in M\} \quad (\alpha \in |UB|, M \subseteq |S|).$$

This function on the generators of $F(UB)$ extends to a homomorphism since if

$$c = \bigwedge_i^{1,n} \bigvee \{\alpha_i(s): s \in M_i\} > 0 \text{ for } \alpha_i \in |UB|, M_i \subseteq |S|, i = 1, 2, \ldots, n,$$

then there exists $f: B \twoheadrightarrow 2$ such that $f(c) = 1$. Therefore, for each $i = 1, 2, \ldots, n, f(\bigvee\{\alpha_i(s): s \in M_i\}) = 1$ and there exists (a necessarily unique) $s_i \in M_i$ such that $f(\alpha_i(s_i)) = 1$. Consider then

$$\sigma \circ Uf: U(B) \to U(2) \to S$$

where σ is the natural isomorphism between $U(2)$ and S. We must have $\sigma(\alpha_i) = s_i \in M_i$ for each $i = 1, 2, \ldots, n$ and therefore

$$\sigma \circ Uf \in \bigcap_i^{i,n} X_{UB}(\alpha_i, M_i);$$

i.e., this set is not 0_{FUB}.

Since S contains at least two distinct elements, say $s \neq t$, for each $b \in B$ we can define $\alpha_b \in UB$ by:

$$\alpha_b(x) = \begin{cases} b, & x = s \\ b', & x = t \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\epsilon B(X_{UB}(\alpha_b, \{s\})) = b$ and ϵB is surjective.

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THEOREM 5.2. *The functors*

$$\mathfrak{B} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{K}$$

are adjoint covariant functors with front (respectively, back) adjunction

$$\eta: I_{\mathfrak{K}} \rightarrow UF(\epsilon: FU \rightarrow I_{\mathfrak{B}}).$$

The proof is completely computational and is obtained by proving that each of the diagrams in Figure (iii) commute.

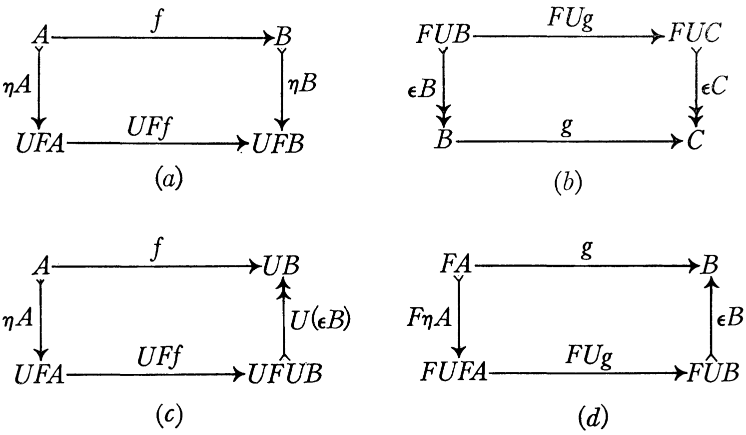

Figure (iii)

LEMMA 5.3. ϵ: $FU \rightarrow I_{\mathfrak{B}}$ *is a natural equivalence.*

Proof. Since every Boolean algebra is a subalgebra of a power set algebra $P(I)$ for some set I, and since U and F preserve monomorphisms, it is sufficient in lieu of Figure (iii) (b) to show that $\epsilon P(I)$ is a monomorphism (hence isomorphism) for each set I.

Now by [**12**, p. 147], there exists a canonical isomorphism:

$$f: S^I \rightarrowtail U(P(I))$$

given by: $f(\alpha)(s) = \mathrm{Eq}(\alpha, \Delta(s)) = \{i \in I: \alpha(i) = s\}$.

Therefore, we need only show that:

$$\epsilon P(I) \circ Ff: F(S^I) \rightarrow FU(P(I)) \rightarrow P(I)$$

is a monomorphism, where

$$(\epsilon P(I) \circ Ff)(X_{F(S^I)}(\alpha, M)) = \bigcup_{s \in M} f(\alpha)(s) = \bigcup_{s \in M} Eq(\alpha, \Delta(s)).$$

Now every homomorphism g: $S^I \rightarrow S$ is determined uniquely by an ultrafilter $U_g \in \Omega(I)$. This is given by:

$$g(\alpha) = s \text{ if and only if } \mathrm{Eq}(\alpha, \Delta(s)) \in U_g.$$

Therefore, for $\alpha_j \in |S^I|$, $M_j \subseteq |S|$, $j = 1, 2, \ldots, n$, $\bigcap_j^{1,n} X_{S^I}(\alpha_j, M_j) \neq \emptyset$ if and only if there exists $g \in \mathrm{Hom}_{\mathfrak{K}}(S^I, S)$ such that $g(\alpha_j) \in M_j (j = 1, 2, \ldots, n)$, if and only if there exists $U \in \Omega(I)$ such that

$$\bigcap_j^{1,n} \bigcup_{s \in M_j} Eq(\alpha_j, \Delta(s)) \in U,$$

if and only if

$$\bigcap_j^{1,n} \bigcup_{s \in M_j} Eq(\alpha_j, \Delta(s)) \neq \emptyset.$$

COROLLARY 5.4. *U is a full and faithful functor (i.e., the correspondence $\mathrm{Hom}_{\mathfrak{B}}(A, B) \to \mathrm{Hom}_{\mathfrak{K}}(UA, UB)$ is bijective).*

COROLLARY 5.5. *ηS^I is an isomorphism for each set I.*

Let $A = S^I$, $B = P(I)$, and $f: S^I \rightarrowtail U(P(I))$ be the canonical isomorphism in Figure (iii) (c).

COROLLARY 5.6. *For every $C \in \mathbf{HP}(S)$, ηC is an isomorphism.*

For, $A = S^I$, $B = C$, and $f: S^I \to C$ in Figure (iii) (a). $\eta C \circ f = UF(f) \circ \eta S^I$, which is surjective. Therefore, ηC is surjective.

COROLLARY 5.7. *A homomorphic image of any $U(B) \in \mathfrak{K}$ is again (up to isomorphism) the U-image of a Boolean homomorphic image of B.*

THEOREM 5.8. *The injective algebras in $\mathfrak{K}$ are up to isomorphism the Boolean extensions of S by complete (= injective) Boolean algebras.*

Proof. Since F preserves monomorphisms, by [**6**] U preserves injectives. Conversely, if $Q \in \mathfrak{K}$ is a non-trivial injective in $\mathfrak{K}$, then for some set I we have:

$$Q \overset{f}{\rightarrowtail} S^I \overset{g}{\twoheadrightarrow} Q$$

with $gf = 1_Q$.

By applying F, we have FQ is a retract of $F(S^I)$ which is isomorphic to $P(I)$, a complete Boolean algebra. From 5.6 it follows that $Q \rightarrowtail\!\!\!\!\twoheadrightarrow UFQ$ and FQ is a complete Boolean algebra.

COROLLARY 5.9. *The finite injectives in $\mathfrak{K}$ are exactly the finite powers of S.*

For, $U(B) = S[B]$ is finite if and only if B is finite.

COROLLARY 5.10. *ηA is essential for each $A \in \mathfrak{K}$.*

If $f: A \rightarrowtail U(B)$ is an injective hull of A, then $f = U(\epsilon B) \circ UF(f) \circ \eta A$ is essential. Since all are monomorphisms and $\mathfrak{K}$ satisfies (CEP), it follows that ηA is essential.

THEOREM 5.11. *For each $A \in \mathfrak{K}$ and $B \in \mathfrak{B}$, $f: A \rightarrowtail UB$ is an injective hull in $\mathfrak{K}$ if and only if $\epsilon B \circ Ff: FA \rightarrowtail B$ is an injective hull in $\mathfrak{B}$.*

Proof. Since B is injective in $\mathfrak{B}$ if and only if UB is injective in $\mathfrak{K}$, we need only show that f is essential in $\mathfrak{K}$ if and only if Ff (or equivalently, $\epsilon B \circ Ff$) is essential in $\mathfrak{B}$.

Take $g: B \to C$ such that $g \circ \epsilon B \circ Ff$ is a monomorphism. By applying U we obtain

$$U(g) \circ f = Ug \circ U(\epsilon B) \circ UF(f) \circ \eta A$$

is a monomorphism, since U preserves monomorphisms. If f is essential, then Ug is a monomorphism. Hence, g is also, as U is faithful.

Now take $g: UB \to C$ such that gf is a monomorphism. By applying F, $F(gf) = Fg \circ Ff$ is a monomorphism. If Ff is essential, then Fg is a monomorphism. Hence, g is also, as $\eta C \circ g = UFg \circ \eta UB$ is a monomorphism.

COROLLARY 5.12. *The injective hull of each* $A \in \mathfrak{K}$ *is given by* $Ug \circ \eta A: A \rightarrowtail UFA \rightarrowtail FB$ *where* $g: FA \rightarrowtail B$ *is the McNeille completion of* FA.

COROLLARY 5.13. *The injective hulls of an algebra* $A \in \mathfrak{K}$ *are unique up to unique isomorphism over* A.

For, this property holds in $\mathfrak{B}$ by [**8**].

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