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ON INTEGRAL REPRESENTATIONS OF THE DRAZIN INVERSE IN BANACH ALGEBRAS

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Abstract The purpose of this paper is to derive an integral representation of the Drazin inverse of an element of a Banach algebra in a more general situation than previously obtained by the second author, and to give an application to the Moore–Penrose inverse in a C^*-algebra.

Keywords: Banach algebra; Drazin inverse; integral representation

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1. Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit e. In [**4**], a generalized *Drazin inverse* of an element $a \in \mathcal{A}$ was defined as $b \in \mathcal{A}$ such that

$$ab = ba, \qquad b^2a = b, \qquad a^2b = a + u, \tag{1.1}$$

where $u \in \mathcal{A}$ is quasinilpotent, that is, $\lim_{n\to\infty} \|u^n\|^{1/n} = 0$ [**4**, Definition 4.1] (see also [**3**]). This definition subsumes (for Banach algebras) the pseudo-inverse defined originally for elements of semigroups and rings [**2**], which arises when u is nilpotent. The Drazin inverse b of a is unique when it exists, and is denoted a^{D}. The *Drazin index* $i(a)$ of a is defined to be 0 if a is invertible, k if the element u in (1.1) is nilpotent of order k, and ∞ otherwise.

According to [**4**], an element $a \in \mathcal{A}$ is Drazin invertible if and only if 0 is not an accumulation point of $\sigma(a)$. This occurs if and only if there exists an idempotent $p \in \mathcal{A}$ such that [**4**, Theorem 4.2]

$$ap = pa \text{ is quasinilpotent}, \qquad a + p \text{ is invertible}; \tag{1.2}$$

p is the *spectral idempotent* of a denoted by a^π. We have

$$a^{\mathrm{D}} = (a + a^\pi)^{-1}(e - a^\pi) \quad \text{and} \quad a^\pi = e - a^{\mathrm{D}}a. \tag{1.3}$$

We also need the core-quasinilpotent decomposition of a Drazin invertible element $a \in \mathcal{A}$ introduced in [**4**] in the form $a = x + y$, where $xy = yx = 0$, x is of the Drazin index not exceeding 1, and y is quasinilpotent; x is called the *core* of a. Explicitly, $x = a(e - a^\pi)$. The importance of the core of a is reflected in the equations

$$i(x) \leqslant 1, \qquad \sigma(x) = \sigma(a), \qquad x^{\mathrm{D}} = a^{\mathrm{D}}. \tag{1.4}$$

Various representations of the Drazin inverse, mostly for matrices, appear in the literature (see, for example, [**8**, **10**, **11**]).

In [**4**], an integral representation was given for an element $a \in \mathcal{A}$ for which $\exp(ta)$ converges as $t \to \infty$. This representation turned out to be a useful tool in the theory of singular differential equations, where it was applied to derive conditions for the asymptotic convergence of solutions both in the setting of matrices [**6**] and semigroups of operators [**1**, **7**].

The purpose of the present paper is to derive an integral representation of the Drazin inverse in a more general situation than in [**4**] and give an application to the Moore–Penrose inverse in a C^*-algebra.

2. The integral representation

We say that $a \in \mathcal{A}$ is *semistable* if a is Drazin invertible with $\operatorname{ind}(a) \leqslant 1$ and the non-zero spectrum of a lies in the open left half of the complex plane. The following result is [**4**, Theorem 6.3].

Proposition 2.1. *Let $a \in \mathcal{A}$ be semistable with the spectral idempotent a^π. Then*

$$a^{\mathrm{D}} = -\int_0^\infty \exp(ta)(e - a^\pi)\,\mathrm{d}t. \tag{2.1}$$

In our first main result we show that the integral representation remains true for a with an arbitrary Drazin index.

Theorem 2.2. *Let $a \in \mathcal{A}$ be a Drazin invertible element with a finite or infinite Drazin index such that the non-zero spectrum of a lies in the open left half of the complex plane. Then equation (2.1) holds.*

Proof. The hypothesis of the Drazin invertibility of a implies that 0 is a resolvent point or an isolated spectral point of a. Let $p = a^\pi$ and let $x = a(e - p)$ be the core of a. In view of (1.4), x is semistable, and $x^{\mathrm{D}} = -\int_0^\infty \exp(tx)(e - p)\,\mathrm{d}t$ by Proposition 2.1. Furthermore,

$$\begin{aligned} \exp(tx)(e - p) &= \exp(ta(e - p))(e - p) = (p + \exp(ta)(e - p))(e - p) \\ &= \exp(ta)(e - p), \end{aligned}$$

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and
$$a^{\mathrm{D}} = x^{\mathrm{D}} = -\int_0^{\infty} \exp(tx)(e-p)\,\mathrm{d}t = -\int_0^{\infty} \exp(ta)(e-p)\,\mathrm{d}t.$$
□

The following representation is valid for elements of finite Drazin index.

Theorem 2.3. *Let $a \in \mathcal{A}$ be a Drazin invertible element with a finite Drazin index $k \geqslant 1$ such that for some $n \geqslant 1$ the non-zero spectrum of a^n lies in the open left half of the complex plane. Then, for any $m \geqslant k$,*
$$-\int_0^{\infty} \exp(ta^n)a^m\,\mathrm{d}t = (a^{\mathrm{D}})^n a^m = \begin{cases} (a^{\mathrm{D}})^{n-m} & \text{if } m < n, \\ e - a^{\pi} & \text{if } m = n, \\ x^{m-n} & \text{if } m > n. \end{cases} \tag{2.2}$$

Proof. Let $a \in \mathcal{A}$ be a Drazin invertible element with $p = a^{\pi}$. Then a^n is also Drazin invertible, and $(a^n)^{\mathrm{D}} = (a^{\mathrm{D}})^n$ [**4**, Theorem 5.4]. In view of (1.3), the spectral idempotent of a^n is also equal to p:
$$e - (a^n)^{\mathrm{D}} a^n = e - (a^{\mathrm{D}}a)^n = e - a^{\mathrm{D}}a = p.$$
Applying Theorem 2.2 to a^n in place of a and using equation $pa^m = 0$, we get
$$\int_0^{\infty} \exp(ta^n)a^m\,\mathrm{d}t = \int_0^{\infty} \exp(ta^n)(e-p)a^m\,\mathrm{d}t = -(a^n)^{\mathrm{D}}a^m. \tag{2.3}$$
By (1.3) again,
$$(a^{\mathrm{D}})^n a^m = (a+p)^{-n}(e-p)(a+p)^m = (a+p)^{m-n}(e-p),$$
from which (2.2) follows when we observe that $x^r = a^r(e-p)$ for any $r > 0$. □

Specializing the preceding theorem, we get a new integral representation for the Drazin inverse.

Theorem 2.4. *Let $a \in \mathcal{A}$ be a Drazin invertible element of a finite Drazin index $k \geqslant 1$ such that the non-zero spectrum of a^{m+1} lies in the open left half of the complex plane for some $m \geqslant k$. Then*
$$a^{\mathrm{D}} = -\int_0^{\infty} \exp(ta^{m+1})a^m\,\mathrm{d}t. \tag{2.4}$$

The condition that the non-zero spectrum of a^{m+1} lies in the open left half of the complex plane is equivalent to the condition that the non-zero spectrum of a lies in the union of $m+1$ angular regions
$$\frac{4j+1}{2(m+1)}\pi < \theta < \frac{4j+3}{2(m+1)}\pi, \quad j = 0, 1, \dots, m.$$
(Divide the unit 'pie' into $2(m+1)$ equal slices starting at $\theta = \pi/(2(m+1))$ and keep every second slice starting with $\pi/(2(m+1)) < \theta < 3\pi/(2(m+1))$.)

There is also a 'right half-plane' version of Theorem 2.4.

Corollary 2.5. *Let $a \in \mathcal{A}$ be a Drazin invertible element of a finite Drazin index $k \geqslant 1$ such that the non-zero spectrum of a^{m+1} lies in the open right half of the complex plane for some $m \geqslant k$. Then*

$$a^{\mathrm{D}} = \int_0^\infty \exp(-ta^{m+1})a^m \,\mathrm{d}t. \tag{2.5}$$

3. Application to Moore–Penrose inverse

Let $\mathcal{A}$ be a unital C^*-algebra. According to [**5**, Theorem 2.5], $a \in \mathcal{A}$ is *Moore–Penrose invertible* if and only if a^*a (respectively, aa^*) is Drazin invertible with the Drazin index not exceeding 1. We observe that

$$ap = 0 = pa, \tag{3.1}$$

where p is the (self-adjoint) spectral idempotent of a^*a (and also of aa^*):

$$\|ap\|^2 = \|(ap)^*ap\| = \|pa^*ap\| = 0, \qquad \|pa\|^2 = \|pa(pa)^*\| = \|paa^*p\| = 0.$$

The *Moore–Penrose inverse* of a can be then defined by

$$a^\dagger = (a^*a)^{\mathrm{D}}a^* = a^*(aa^*)^{\mathrm{D}}. \tag{3.2}$$

Since the non-zero spectrum of a^*a always lies in the open right half of the complex plane and the Drazin index of a^*a does not exceed 1, Proposition 2.1 and Corollary 2.5 apply to give the following representation of the Moore–Penrose inverse.

Theorem 3.1. *Let $a \in \mathcal{A}$ be a Moore–Penrose invertible element of a C^*-algebra $\mathcal{A}$. Then, for each $m \geqslant 0$,*

$$a^\dagger = \int_0^\infty \exp(-t(a^*a)^{m+1})(a^*a)^m a^* \,\mathrm{d}t = \int_0^\infty a^* \exp(-t(aa^*)^{m+1})(aa^*)^m \,\mathrm{d}t. \tag{3.3}$$

Proof. Equation (3.3) with $m = 0$ is obtained when we apply Proposition 2.1 to the formula (3.2) for the Moore–Penrose inverse, taking into account that $a^*p = pa^* = 0$ in view of (3.1). We have thus obtained Showalter's representation [**9**] by a different method.

The case $m > 0$ follows from Corollary 2.5. □

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