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A PERIODIC WAVELET METHOD FOR THE SECOND KIND OF THE LOGARITHMIC INTEGRAL EQUATION

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A periodic wavelet Galerkin method is presented in this paper to solve a weakly singular integral equations with emphasis on the second kind of Fredholm integral equations. The kernel function, which includes of a smooth part and a log weakly singular part, is discretised by the periodic Daubechies wavelets. The wavelet compression strategy and the hyperbolic cross approximation technique are used to approximate the weakly singular and smooth kernel functions. Meanwhile, the sparse matrix of systems can be correspondingly obtained. The bi-conjugate gradient iterative method is used to solve the resulting algebraic equation systems. Especially, the analytical computational formulae are presented for the log weakly singular kernel. The computational error for the representative matrix is also evaluated. The convergence rate of this algorithm is $O(N^{-p}\log(N))$, where p is the vanishing moment of the periodic Daubechies wavelets. Numerical experiments are provided to illustrate the correctness of the theory presented here.

1. INTRODUCTION

Exterior boundary value problems for the two-dimensional Helmholtz equation are usually solved by boundary integral methods, which lead to a second kind of the Fredholm integral equation

$$\frac{1}{2\pi}(I-T)f = g, \tag{1.1}$$

where I is the identity operator and T is an integral operator as

$$Tf = \int_0^{2\pi} \big(k(x,y) + k_2(x,y)\big)f(y)\mathrm{d}y \tag{1.2}$$

in which $k(x,y)$ has a logarithm-like singularity along the diagonal $x = y$ but is continuous elsewhere on the unit square, $k_2(x,y)$ is a 2π-periodic smooth function in $L^2([0,2\pi]$

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$\otimes[0, 2\pi])$, $g(x)$ is a 2π-periodic function in $L^2[0, 2\pi]$, and $f(x)$ is the 2π-periodic unknown function in $L^2[0, 2\pi]$. There have been many theoretical studies on the second kind of the Fredholm integral equations, as well as numerical methods to solve them ([**2, 9, 10, 11, 13, 18**]). The traditional discretisation of integral equations suffers from the difficulty of computing and inverting dense matrices. If iterative methods are used, a single direct multiplication of matrix and vector requires $O(N^2)$ operations and the number of iterations is proportional to the condition number of the matrices. Therefore, some more efficient methods have to be found before the boundary integral method can be applied to practical, large scale, numerical computations.

The wavelet Galerkin method has recently been investigated by many authors; see [**1, 3, 4, 7, 12, 15, 16, 17**] for details. From the point of view of applications, although the three-dimensional boundary value problems are of higher interest, the development and practical realisation of two-dimensional wavelet Galerkin methods is of importance on its own. Two dimensional or axial symmetric boundary value problems play an important role in practical applications. The purpose of the paper is to provide a new method to reduce the computational complexity for the integral equation including weakly singular and smooth parts. We formulate and analyse the periodic Daubechies wavelet method. It combines the wavelet compression strategy and the hyperbolic cross approximation (sparse grid approximation) for the kernel function.

We restrict the wavelet basis to $[0, 2\pi]$ for the integral equation. After the wavelet Galerkin discretisation, the integral equation is transformed into an algebraic system with dense matrix. The multiplication complexity of the representative matrices plays the vital roles in the corresponding iterative method. Applying the compression truncation strategy and the hyperbolic cross approximation method to the resulting matrices, respectively, it is proved that the computational complexity is reduced from $O(N^2)$ to $O(N\log(N))$ with the convergence order $O(N^{-p}\log N)$, where p is the vanishing moment of the Daubechies wavelet. The advantage of the orthogonal wavelet basis is to make the representation of the identity operator I as an identity matrix. Furthermore, we give the computational formulae for the weakly singular part $k(x, y)$ which can be computed by the filter coefficients series of the periodic Daubechies wavelets. The computational error is also evaluated as $O(N_1^{-p}\log N_1)$ where N_1 is the selected truncation parameter. Note that when $N_1 \sim N$, the computational error does not affect the wavelet discretisation error.

The outline of the paper is as follows. In Section 2, we review some symbols and useful properties of the periodic Daubechies wavelets. Section 3 is devoted to our theoretical analysis. Applying the wavelet compression and hyperbolic cross approximation for the whole kernel function, we analyse the resulting representation matrix. In Section 4, we compute the entries of the representative matrix for the weakly singular part. The corresponding computational truncation error is proved. The convergence and complexity of this wavelet method are analysed. Numerical experiments are given in Section 5. Finally, we conclude our work in Section 6.

2. Daubechies Periodic Wavelets Basis in $L^2[0, 2\pi]$

Due to the weakly singular integral operator $Tf = \int T(x, y) f(y) \mathrm{d}y$ where $f(x) \in L^2[0, 2\pi]$ and $k(x, y) \in L^2([0, 2\pi] \otimes [0, 2\pi])$, in the section we use the periodic Daubechies wavelets with 2π period as in [**6, 14**]. The space $L^2[0, 2\pi]$ is equipped with the inner product $\langle f, g \rangle = (1/2\pi) \int f(x)\overline{g(x)} \mathrm{d}x$ and the norm $\|f\|_{L^2[0,2\pi]} = \langle f, f \rangle^{1/2}$.

We respectively write the periodic scaling functions and the wavelet functions as

$$\phi_{j,k}^{\text{per}}(x) = \sum_{l \in Z} \phi_{j,k}(x + 2\pi l) \quad \text{and} \quad \psi_{j,k}^{\text{per}}(x) = \sum_{l \in Z} \psi_{j,k}(x + 2\pi l). \tag{2.1}$$

It is obvious that

$$\phi_{j,k}^{\text{per}}(x + 2\pi) = \sum_{l \in Z} \phi_{j,k}\big(x + 2\pi(l+1)\big) = \sum_{l \in Z} \phi_{j,k}(x + 2\pi l) = \phi_{j,k}^{\text{per}}(x). \tag{2.2}$$

Thus, the following formulae are obtained,

$$\begin{aligned} &2\pi \langle \psi_{j,k}^{\text{per}}(x), \phi_{j,k'}^{\text{per}}(x) \rangle = 0; \qquad 2\pi \langle \psi_{j,k}^{\text{per}}(x), \psi_{j',k'}^{\text{per}}(x) \rangle = \delta_{jj'}\delta_{kk'}; \\ &\psi_{j,k}^{\text{per}}(x) = \psi_{j,k+2^{j+1}\pi l}(x); \qquad \phi_{j,k}^{\text{per}}(x) = \phi_{j,k+2^{j+1}\pi l}(x); \\ &\int x^k \phi^{\text{per}}(x) \mathrm{d}x = 0, 0 \leqslant k < p, \end{aligned} \tag{2.3}$$

where p is the wavelet vanishing moment. Thus, the basis set $\{\phi_{0,k}^{\text{per}}, \psi_{0,k}^{\text{per}}, \dots, \psi_{J-1,k}^{\text{per}}\}$ forms the base of the space $L^2[0, 2\pi]$ in [**6**]. Let $\phi_{0,k} = \psi_{-1,k}$. For any $f(x) \in L^2[0, 2\pi]$, we have $f(x) = \sum_{j=-1}^{J-1} \sum_{k=0}^{2^j-1} \widehat{f}_{j,k} \psi_{j,k}^{\text{per}}(x)$ where $\widehat{f}_{j,k} = \langle f, \psi_{j,k}^{\text{per}} \rangle$. Equally, let $n = 2^j + k$, we have $f(x) = \sum_{n=1}^{N} \widehat{f}_n \psi_n(x)$. For the smooth function in $L^2[0, 2\pi]$, using the periodic Daubechies wavelets approximation, we have Lemma 2.1.

Lemma 2.1. *Let $f(x) \in L^2[0, 2\pi]$ and the vanishing moment of the periodic Daubechies wavelets be p, then we have*

$$\sum_{j=-1}^{J-1} \sum_{k=0}^{2^j-1} \widehat{f}_{j,k} \psi_{j,k}^{\text{per}}(x) \to f(x), \quad J \to \infty \tag{2.4}$$

and

$$\left\| f(x) - \sum_{j=-1}^{J-1} \sum_{k=0}^{2^j-1} \widehat{f}_{j,k} \psi_{j,k}^{\text{per}}(x) \right\|_{L^2[0,2\pi]} \sim O(2^{-Jp}). \tag{2.5}$$

Next, we embark on discretising the integral equation (1.1).

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3. Discretisation Matrices Analysis

To derive the system of equations from (1.1), we introduce the approximations $\widetilde{f}$ and $\widetilde{g}$ defined by the following wavelet series:

$$\widetilde{f}(x) = \sum_{m=-1}^{J-1} \sum_{q=0}^{2^m-1} \widehat{f}_{m,q}\psi_{m,q}^{\text{per}}(x), \quad \text{and} \quad \widetilde{g}(x) = \sum_{j=-1}^{J-1} \sum_{k=0}^{2^j-1} \widehat{g}_{j,k}\psi_{j,k}^{\text{per}}(x). \tag{3.1}$$

The bases $\{\psi_{j,k}\} = \{\psi_n\}_{n=1}^N$, $n = 2^j + k$, $N = 2^J$, consist of all the scaling functions and the wavelets used in equation (3.1). The vectors $\widehat{f}_{m,q}$ and $\widehat{g}_{j,k}$ are the expansion coefficients.

In equation (1.1), the replacement of f and g by $\widetilde{f}$ and $\widetilde{g}$ yields the following residual r:

$$r(x) = \frac{1}{2\pi}\widetilde{f}(x) - \frac{1}{2\pi}\int_0^{2\pi} \big(k(x,y) + k_2(x,y)\big)\widetilde{f}(y)\mathrm{d}y - \widetilde{g}(x). \tag{3.2}$$

We now apply the Galerkin method to Equation (3.2). We wish to satisfy

$$\int_0^{2\pi} r(x)\psi_n(x)\mathrm{d}x = 0. \tag{3.3}$$

Then, the equation (1.1) is discretised by the wavelet Galerkin method as the algebraic system,

$$\widehat{f}_{j,k} - \sum_{j,m=-1}^{J-1} \sum_{k,q=0}^{2^j-1,2^m-1} w_{k,q}^{j,m}\widehat{f}_{j,k} - \sum_{j,m=-1}^{J-1} \sum_{k,q=0}^{2^j-1,2^m-1} s_{k,q}^{j,m}\widehat{f}_{j,k} = 2\pi\widehat{g}_{j,k}, \tag{3.4}$$

where

$$\begin{aligned}
M &= \{w_{k,q}^{j,m}\}_{2^J\times 2^J} = \{w_{n,n'}\}_{n,n'=1}^N,\\
S &= \{s_{k,q}^{j,m}\}_{2^J\times 2^J} = \{s_{n,n'}\}_{n,n'=1}^N,\\
\widehat{f} &= \{\widehat{f}_{j,k}\}_{2^J\times 1} = \{\widehat{f}_n\}_{n=1}^N,\\
\widehat{g} &= (2\pi\widehat{g}_{j,k})_{2^J\times 1} = \{2\pi\widehat{g}_n\}_{n=1}^N,\\
N &= 2^J.
\end{aligned}$$

This can be expressed in matrix form;

$$(I - M - S)\widehat{f} = \widehat{g}; \tag{3.5}$$

where I is the identity matrix.

In the next section, we give the concrete analysis of the resulting matrices M and S. The density and construction of these two matrices affect the complexity of this method. Throughout the paper we denote by C a positive constant which may take different values in different formulae.

3.1. Weakly singular kernel analysis. Firstly, we define $D^\alpha D^\beta$ as follows

$$D_x^\alpha D_y^\beta f(x,y) = \frac{\partial^{\alpha+\beta} f(x,y)}{\partial^\alpha x \partial^\beta y}. \tag{3.6}$$

The weakly singular function $k(x,y) = \log\left(4\sin^2(x-y)/2\right)$ has some critical properties for sparse representation. We have

$$\left|D_x^\alpha D_y^\beta k(x,y)\right| \leqslant \frac{1}{|x-y|^{\alpha+\beta}}. \tag{3.7}$$

We can easily get the evaluation of the elements $w_{k,q}^{j,m} = \langle k(x,y), \psi_{j,k}^{\text{per}}(x)\psi_{m,q}^{\text{per}}(y)\rangle$ of M by [**1, 5**].

Lemma 3.1. *Let p be the vanishing moment of the periodic Daubechies wavelets. $w_{k,q}^{j,m}$ is the entries of the matrix M, then we have*

$$w_{k,q}^{j,m} \sim O\left(2^{-(j+m)(p+1/2)}\frac{1}{|2^{-j}k - 2^{-m}q|^{2p}}\right). \tag{3.8}$$

By Lemma 3.1, we know that if $|2^{-j}k - 2^{-m}q| > \delta$ when δ is the selected hard threshold, that is, the elemets are far from the diagonal line, their values are very small. In other words, most of their entries are so small that they can be neglected without producing the larger error. The corresponding truncation strategy is

$$\widehat{w}_{k,q}^{j,m} = \begin{cases} w_{k,q}^{j,m}, & \text{if} |2^{-j}k - 2^{-m}q| < \delta; \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

So, we get the compressed matrix $M^{\text{comp}} = \{\widehat{w}_{k,q}^{j,m}\} = \{\widehat{w}_{n,n'}\}_{1\leqslant n,n'\leqslant N}$.

Note that the use of the bi-orthogonal wavelet gives optimal complexity. However the bi-orthogonal wavelet can not be extended to the wavelet packet method due to instability. Here we only develop the orthogonal wavelet method which facilitate the orthogonal wavelet packet method. Let P_J^{comp} be the compression operator of the operator P_J. By [**1**], we have

$$\left\|(K - P_J^{\text{comp}}K)f_J\right\| \leqslant CJ2^{-Jp}. \tag{3.10}$$

The non-zero entries of M^{comp} are $O(N\log N)$ with the convergence rate $O\left(N^{-p}\log(N)\right)$.

3.2. Smooth kernel analysis. In this part, we study the properties of the matrix S by the hyperbolic cross approximation. If the vanishing moment is p, then the support interval of $\psi_{j,k}^{\text{per}}(x)$ is $I_{j,k} = \left(2^{-j}(1-p+k) - 2\pi l,\ 2^{-j}(p+k) - 2\pi l\right)$. According to the smoothness of the function $k_2(x,y)$, we can obtain the following element value estimate.

Theorem 3.1 *Denote by p the vanishing moment of the periodic Daubechies wavelets. Let $k_2(x,y) \in H^{r',r}\left[(0,2\pi)\otimes(0,2\pi)\right]$, $r, r' \in (0,p]$, then we have*

$$|s_{k,q}^{j,m}| \leqslant C\frac{1}{(r-1)!}\frac{1}{(r'-1)!}\frac{(2m-1)^{r'+r+3}}{rr'\sqrt{(2r'+3)(2r+3)}}(2^{-j(r'+3/2)})(2^{-m(r+3/2)}). \tag{3.11}$$

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PROOF: Let x_0 and y_0 be the midpoints of the interval $I_{j,k}$ and $I_{m,q}$, respectively. By the Taylor expansion, for $(x,y) \in I_{j,k} \times I_{m,q}$ we have

$$k_2(x,y) = \sum_{\mu=0}^{r-1} \frac{k_2^{(0,\mu)}(x,y_0)}{\mu!}(y-y_0)^\mu + \frac{1}{(r-1)!}\int_{y_0}^{y} k_2^{(0,r)}\big(x, y_0+\theta(y-y_0)\big)(y-y_0)^r(1-\theta)^{r-1}d\theta,$$

where

$$\begin{aligned} &k_2^{(0,r)}\big(x, y_0+\theta(y-y_0)\big) \\ &\quad= \sum_{\mu'=0}^{r'-1}\frac{1}{\mu'!}k_2^{(\mu',r)}\big(x_0, y_0+\theta(y-y_0)\big)(x-x_0)^{\mu'} \\ &\qquad+ \frac{1}{(r'-1)!}\int_{x_0}^{x} k_2^{(r',r)}(x_0+\theta'(x-x_0), y_0+\theta(y-y_0))(x-x_0)^{r'}(1-\theta')^{r'-1}d\theta'. \end{aligned}$$

Because

$$\int_{I_{j,k}}(x-x_0)^{\mu'}\psi_{j,k}^{\text{per}}(x)dx = 0, \quad \mu' \in (0,r'], \qquad \int_{I_{m,q}}(y-y_0)^{\mu'}\psi_{m,q}^{\text{per}}(y)dy = 0, \quad \mu \in (0,r],$$

$$\left|k_2^{(r',r)}\big(x_0+\theta'(x-x_0), y_0+\theta(y-y_0)\big)\right| \leqslant C, \qquad \left\|\psi_{j,k}^{\text{per}}(x)\right\|_{L^2} = 1,$$

we have

$$\begin{aligned} s_{k,q}^{j,m} &= < k_2(x,y), \psi_{j,k}^{\text{per}}(x)\psi_{m,q}^{\text{per}}(y) > \\ &= \int_0^{2\pi}\int_0^{2\pi} k_2(x,y)\psi_{j,k}^{\text{per}}(x)\psi_{m,q}^{\text{per}}(y)dxdy \\ &= \int_{I_{j,k}}\int_{I_{m,q}} k_2(x,y)\psi_{j,k}^{\text{per}}(x)\psi_{m,q}^{\text{per}}(y)dxdy \\ &\leqslant \left(\frac{C}{(r'-1)!}\int_{I_{j,k}}\int_{x_0}^{x}(x-x_0)^{r'}(1-\theta')^{r'-1}d\theta'\left|\psi_{j,k}^{\text{per}}(x)\right|dx\right) \\ &\qquad \left(\frac{1}{(r-1)!}\int_{I_{m,q}}\int_{y_0}^{y}(y-y_0)^{r}(1-\theta)^{r-1}d\theta\left|\psi_{m,q}^{\text{per}}(y)\right|dy\right). \end{aligned}$$

Especially, it follows that

$$\begin{aligned} &\int_{I_{j,k}}\left(\int_{x_0}^{x}(1-\theta')^{r'-1}d\theta'\right)(x-x_0)^{r'}\left|\psi_{j,k}^{\text{per}}(x)\right|dx \\ &\qquad= \frac{1}{r}\int_{I_{j,k}}(x-x_0)^{r'+1}\left|\psi_{j,k}^{\text{per}}(x)\right|dx \\ &\qquad\leqslant \frac{1}{r'}\left(\int_{I_{j,k}}\left((x-x_0)^{r'+1}\right)^2dx\right)^{1/2}\left(\int_{I_{j,k}}\left|\psi_{j,k}^{\text{per}}(x)\right|^2dx\right)^{1/2} \\ &\qquad= \frac{1}{r'}\left(\frac{1}{2r'+3}(x-x_0)^{2r'+3}\big|_{I_{j,k}}\right)^{1/2} \end{aligned}$$

$$= \frac{(2m-1)^{r'+3/2}}{r'\sqrt{2r'+3}}(2^{-j(r'+3/2)}).$$

Finally, we obtain

$$|s_{k,q}^{j,m}| \leqslant C\frac{1}{(r-1)!}\frac{1}{(r'-1)!}\frac{(2m-1)^{r'+r+3}}{rr'\sqrt{(2r'+3)(2r+3)}}(2^{-j(r'+3/2)})(2^{-m(r+3/2)}).$$

□

Because of the good smoothness property of $k_2(x,y)$, we introduce the hyperbolic cross approximation to reduce the computational complexity without affecting the convergence rate. We give the definition of the n-dimensional Sobolev space $H_{\text{mix}}^{t,l}(\Omega^n)$ where Ω denotes the spatial domain such that, for $j = \{j_1, j_2, \dots, j_n\}$, $|j|_1 = \sum_{i=1}^{n} j_i$; $|j|_\infty = \max_{1\leqslant i\leqslant n} j_i$,

$$\|u\|_{H_{\text{mix}}^{t,l}}^2 \approx \sum_j 2^{2t|j|_1+2l|j|_\infty}\|z_j\|_{L^2}^2, \quad u = \sum_j z_j \in H_{\text{mix}}^{t,l}.$$

By using the vanishing moment p, the hyperbolic cross approximation rate can be obtained. Denote by $P_{\Lambda_J}^{\text{spar}}$ the operator that maps functions of interest to the space V_J^{per} whose base satisfies the index set Λ_J.

LEMMA 3.2. *Let $k_2(x,y) \in H_{\text{mix}}^{p,0}([0,2\pi]\times[0,2\pi])$, $f(y) \in H_{\text{mix}}^{p,0}([0,2\pi])$, then*

$$\left\| \int P_{\Lambda_J}^{\text{spar}} k_2(x,y) f_{\Lambda_J}(y)dy - k_2(x,y)f(y)dy \right\|_{L^2} \leqslant \max_{(j,m)\notin\Lambda_J} 2^{-(j+2m)p}\|k_2\|_{H_{\text{mix}}^{p,0}}\|f\|_{H_{\text{mix}}^{p,0}}. \tag{3.12}$$

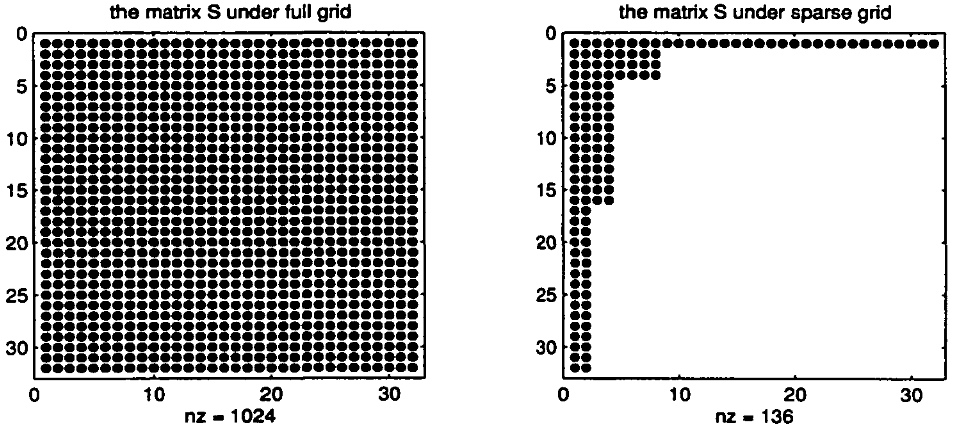
PROOF: The proof process can be deduced from [8] similarly. □

If $\Lambda = \{\max(j,m) \leqslant J\}$, we call it the full grid approximation with the approximation order $O(2^{-Jp})$. Using the sparse grid approximation, for $k_2(x,y)$, we can select the small index $\Lambda_J = \{-(j+2m)p \geqslant -pJ\}$. This sparse grid approximation also can preserve the approximation order of the full grid approximation. So we can obtain the standard index set is $\Lambda_J = \{(j,k)(m,q) : j+2m \leqslant J\}$. And we know that when the level J is odd, the non-zero elements number is $5\times 2^J - 3\times 2^{(J+1)/2}$. When the level J is even, the number is $5\times 2^J - 4\times 2^{J/2}$. The number of non-zero entries of S by sparse grid approximation is $O(2^J) \sim O(N)$.

The Figure shows such a representation matrices for the smooth kernel by using the Daubechies-2 wavelets with the level $J = 5$. It also displays the location of the nonzero entries of S under the hyperbolic cross approximation. The hyperbolic cross approximation produces a sparse matrix with the non-zero ratio 13.28%.

4. COMPUTATION OF M AND ERROR ANALYSIS FOR THIS METHOD

4.1. COMPUTATION OF REPRESENTATIVE MATRIX M. We compute the values $w_{k,q}^{j,m}$ of the matrix M by using the filter coefficients of the Daubechies wavelets. Let the Fourier



Compared maps of smooth kernel $k_2(x,y)$ matrix $S_{32\times 32}$.

expansion of the periodic Daubechies wavelet functions be $\psi_{j,k}^{\mathrm{per}} = \sum_{t'\in Z} d_{t'}^{j,k} e^{it'x}$ and the Fourier expansion of the periodic Daubechies scaling functions be $\phi_{j,k}^{\mathrm{per}} = \sum_{t'\in Z} b_{t'}^{j,k} e^{it'x}$.

Based on the Yu's way in [19], we have

$$-\log\left(\sin\frac{\omega}{2}\right) = \frac{1}{2}\sum_{t\neq 0, t=-\infty}^{+\infty}\frac{1}{|t|}e^{it\omega}. \tag{4.1}$$

Further, we can get the computational formula of the entries $w_{k,q}^{j,m}$ of the representative matrix M.

For $w_{k,q}^{j,m}$, $0\leqslant j,m\leqslant J-1, 0\leqslant k\leqslant 2^j-1, 0\leqslant q\leqslant 2^m-1$, we have

$$\begin{aligned} w_{k,q}^{j,m} &= \int_0^{2\pi}\int_0^{2\pi}\log(4\sin^2(x-y)/2)\psi_{j,k}^{\mathrm{per}}(x)\psi_{m,q}^{\mathrm{per}}(y)\mathrm{d}x\mathrm{d}y \\ &= -\int_0^{2\pi}\int_0^{2\pi}\sum_{t\neq 0,t=-\infty}^{+\infty}\frac{1}{|t|}e^{it(x-y)}\sum_{t'\in Z}d_{t'}^{j,k}e^{it'x}\sum_{t''\in Z}d_{t''}^{m,q}e^{it''y}\mathrm{d}x\mathrm{d}y \\ &= -\sum_{t\neq 0,t=-\infty}^{+\infty}\frac{1}{|t|}\sum_{t'\in Z}d_{t'}^{j,k}\sum_{t''\in Z}d_{t''}^{m,q}\int_0^{2\pi}\int_0^{2\pi}e^{it(x-y)}e^{it'x}e^{it''y}\mathrm{d}x\mathrm{d}y \\ &= -\sum_{t\neq 0,t=-\infty}^{+\infty}\frac{1}{|t|}d_{-t}^{j,k}d_t^{m,q}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} d_t^{j,k} &= \frac{1}{2\pi}\int_0^{2\pi}\psi_{j,k}^{\mathrm{per}}(x)e^{-itx}\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\sum_{l\in Z}2^{j/2}\psi(2^jx+2^{j+1}\pi l-k)e^{-itx}\mathrm{d}x \\ &= \frac{2^{j/2}}{2\pi}\sum_{l\in Z}\int_0^{2\pi}\psi(2^jx+2^{j+1}\pi l-k)e^{-itx}\mathrm{d}x \end{aligned}$$

$$= \frac{2^{j/2}}{2\pi}\sum_{l\in Z}\int_{2^{j+1}\pi l-k}^{2^{j+1}\pi(l+1)-k}\psi(y)e^{-it(2^{-j}y+2^{-j}k-2\pi l)}2^{-j}\mathrm{d}y$$
$$= \frac{2^{-j/2}}{2\pi}\int_{-\infty}^{+\infty}\psi(y)e^{-it2^{-j}y}e^{-it2^{-j}k}e^{it2\pi l}\mathrm{d}y$$
$$= \frac{2^{-j/2}}{2\pi}e^{-it2^{-j}k}\int_{-\infty}^{+\infty}\psi(y)e^{-it2^{-j}y}\mathrm{d}y$$
$$= \frac{2^{-j/2}}{2\pi}e^{-it2^{-j}k}\widehat{\psi}\Big(\frac{t}{2^j}\Big)$$

in which the mark $\wedge$ denotes the Fourier transform of function. Similarly, the values of $b_t^{j,k} = (2^{-j/2})/(2\pi)e^{-it2^{-j}k}\widehat{\phi}(t/2^j)$ can be obtained. Also, we have

$$w_{0,0}^{-1,-1} = -\sum_{t=-\infty,t\neq 0}^{+\infty}\frac{1}{|t|}b_{-t}^{-1,0}b_t^{-1,0}.$$

For $j=-1, m=0,\dots,J-1$, we know

$$w_{0,q}^{-1,m} = -\sum_{t=-\infty,t\neq 0}^{+\infty}\frac{1}{|t|}b_{-t}^{-1,0}d_t^{m,q}.$$

Denote by h_l the filter coefficients of the Daubechies wavelets. Thus, we have

$$\widehat{\phi}(\omega) = \prod_{n=1}^{+\infty} m_0(\omega/2^n), \quad \widehat{\psi}(\omega) = -e^{-i\omega/2}\overline{m_0(\omega/2+\pi)}\widehat{\phi}(\omega/2),$$
$$m_0(\xi) = \frac{1}{2}\sum_l h_l e^{-i\xi l}. \tag{4.3}$$

The values of $b_n^{j,k}$ and $d_n^{j,k}$ can be expressed in terms of the filter coefficients h_l. Thus, the entries of M are calculated by some discrete series without the analysis of the concrete expression form on the wavelets.

The function $k(x,y) = \log\big(4\sin^2(x-y)/2\big)$ is represented by

$$M = \big\{\{w_{k,q}^{j,m}\}_{-1\leqslant j,m\leqslant J-1}\big\}_{0\leqslant q\leqslant 2^m-1}^{0\leqslant k\leqslant 2^j-1}.$$

Let $\widetilde{w}_{k,q}^{j,m} = -\sum_{t\neq 0,t=-N_1}^{N_1}(1/|t|)d_{-t}^{j,k}d_t^{m,q}$. Denote the computed matrix with truncation series by $M^{num} = \{\widetilde{w}_{k,q}^{j,m}\}$. Now we give the computational error evaluation of $|w_{k,q}^{j,m} - \widetilde{w}_{k,q}^{j,m}|$.

THEOREM 4.1. *Assume the vanishing moment of the Daubechies wavelets functions be p and N_1 be the truncation parameter. Based on the definitions of the matrices M and M^{num}, we have*

$$|w_{k,q}^{j,m} - \widetilde{w}_{k,q}^{j,m}| \sim O(\log N_1 N_1{}^{-p}). \tag{4.4}$$

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PROOF: Due to the definitions of the matrices M and M^{num}, we have

$$\begin{aligned}
\left|w_{k,q}^{j,m} - \widetilde{w}_{k,q}^{j,m}\right| &= \left|\int_0^{2\pi}\int_0^{2\pi} \log\left(4\sin^2(x-y)/2\right)\psi_{j,k}^{\mathrm{per}}(x)\psi_{m,q}^{\mathrm{per}}(y)\,\mathrm{d}x\mathrm{d}y - \widetilde{w}_{k,q}^{j,m}\right| \\
&= \left|\int_0^{2\pi}\int_0^{2\pi} \log(4\sin^2(x-y)/2)\psi_{j,k}^{\mathrm{per}}(x)\psi_{m,q}^{\mathrm{per}}(y)\,\mathrm{d}x\mathrm{d}y\right. \\
&\qquad \left. - \int_0^{2\pi}\int_0^{2\pi} \sum_{t\neq 0, t=-N_1}^{N_1} \frac{1}{|t|} e^{it(x-y)} \sum_{t'=-N_1}^{N_1} d_{t'}^{j,k} e^{it'x} \sum_{t''=-N_1}^{N_1} d_{t''}^{m,q} e^{it''y}\right| \\
&\leqslant \int_0^{2\pi}\int_0^{2\pi} \left|\log\left(4\sin^2(x-y)/2\right) - \sum_{t\neq 0, t=-N_1}^{N_1} \frac{1}{|t|} e^{it(x-y)}\right| \left|\psi_{j,k}^{\mathrm{per}}\right| \times \left|\psi_{m,q}^{\mathrm{per}}\right| \mathrm{d}x\mathrm{d}y \\
&\qquad + \int_0^{2\pi}\int_0^{2\pi} \left|\sum_{t\neq 0, t=-N_1}^{N_1} \frac{1}{|t|} e^{it(x-y)}\right| \times \left|\psi_{j,k}^{\mathrm{per}} - \sum_{t'=-N_1}^{N_1} d_{t'}^{j,k} e^{it'x}\right| \\
&\qquad \times \left|\psi_{m,q}^{\mathrm{per}} - \sum_{t''=-N_1}^{N_1} d_{t''}^{m,q} e^{it''y}\right| \mathrm{d}x\mathrm{d}y \\
&\leqslant \int_0^{2\pi}\int_0^{2\pi} \left|\left(\sum_{t=-\infty}^{-N_1-1} + \sum_{t=N_1+1}^{+\infty}\right) \frac{1}{|t|} e^{it(x-y)}\right| |\psi_{j,k}^{\mathrm{per}}| \times |\psi_{m,q}^{\mathrm{per}}| \mathrm{d}x\mathrm{d}y \\
&\qquad + \int_0^{2\pi}\int_0^{2\pi} \left|\sum_{t\neq 0, t=-N_1}^{N_1} \frac{1}{|t|} e^{it(x-y)}\right| \times \left|\left(\sum_{t'=-\infty}^{-N_1-1} + \sum_{t'=N_1+1}^{+\infty}\right) d_{t'}^{j,k} e^{it'x}\right| \\
&\qquad \times \left|\left(\sum_{t''=-\infty}^{-N_1-1} + \sum_{t''=N_1+1}^{+\infty}\right) d_{t''}^{m,q} e^{it''y}\right| \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Now we show the evaluation of the corresponding items, respectively.

Firstly, we have

$$\left\|\left(\sum_{t=-\infty}^{-N_1-1} + \sum_{t=N_1+1}^{+\infty}\right) \frac{1}{|t|} e^{it(x-y)}\right\|_{L^2[0,2\pi]} \leqslant \left|\left(\sum_{t=-\infty}^{-N_1-1} + \sum_{t=N_1+1}^{+\infty}\right) \frac{1}{|t|}\right| \leqslant 2 \sum_{t=N_1+1}^{+\infty} \frac{1}{|t|} \leqslant \frac{2}{N_1}$$

and

$$\left\|\sum_{t\neq 0, t=-N_1}^{N_1} \frac{1}{|t|} e^{it(x-y)}\right\|_{L^2[0,2\pi]} \leqslant 2\left|\sum_{t=1}^{N_1} \frac{1}{|t|}\right| \leqslant \left(\log N_1 + O\left(\frac{1}{N_1}\right)\right).$$

Secondly, because p is the vanishing moment of the functions $\psi_{j,k}^{\mathrm{per}}(x)$ and $\phi_{j,k}^{\mathrm{per}}(x)$, these scaling and wavelet functions satisfy

$$\left|\psi_{j,k}^{\mathrm{per}}(x)\right| < \left(1+|x|\right)^{-1-p/2} \leqslant C, \quad \left|\phi_{j,k}^{\mathrm{per}}(x)\right| < \left(1+|x|\right)^{-1-p/2} \leqslant C.$$

By the analysis of the Fourier series, we have

$$\left\|\left(\sum_{t'=-\infty}^{-N_1-1} + \sum_{t'=N_1+1}^{+\infty}\right) d_{t'}^{j,k} e^{it'x}\right\|_{L^2[0,2\pi]} = \sum_{|t'|>N_1} |d_{t'}^{j,k}| \leqslant C N_1^{-p/2}.$$

Similarly,

$$\Big\|\Big(\sum_{t''=-\infty}^{-N_1-1}+\sum_{t''=N_1+1}^{+\infty}\Big)d_{t''}^{m,q}e^{it''y}\Big\|_{L^2[0,2\pi]}=\sum_{|t''|>N_1}|d_{t''}^{m,q}|\leqslant CN_1{}^{-p/2}.$$

Finally, we have

$$\begin{aligned}|w_{k,q}^{j,m}-\widetilde{w}_{k,q}^{j,m}|&\leqslant 4\pi^2\Big\{\frac{1}{N_1}C+C\log N_1N_1{}^{-p}\Big\}\\&\sim O(\log N_1N_1{}^{-p}).\end{aligned}$$

□

Based on the approximation and computation error, we analyse the error and complexity of the whole periodic Daubechies wavelet method.

4.2. Error analysis and complexity of this method. We can prove the convergence rate of the periodic Daubechies wavelet method.

Theorem 4.2. *Let J be the wavelet decomposition level. The resultant matrix dimension is N^2, where $N=2^J$. Denote $f_J=P_Jf$, $g_J=P_Jg$, the operator $T=K+K_2$, where $Kf(x)=\int_0^{2\pi}k(x,y)f(y)dy$ and $K_2f(x)=\int_0^{2\pi}k_2(x,y)f(y)dy$. Then we have*

$$\|f-f_J\|\leqslant CJ2^{-Jp}. \tag{4.5}$$

Proof: Since $(I-K-K_2)f=g$ and $(I-P_J^{\mathrm{comp}}K+P_J^{\mathrm{spar}}K_2)f_J=g_J$, we have

$$f-f_J=(I-T)^{-1}\big[(g_J-g)+(K-P_J^{\mathrm{comp}}K)f_J+(K_2-P_J^{\mathrm{spar}}K_2)f_J\big],$$

where

$$\begin{aligned}\big\|(K-P_J^{\mathrm{comp}}K)f_J+(K_2-P_J^{\mathrm{spar}}K_2)f_J\big\|&=\|Kf_J-P_J^{\mathrm{comp}}Kf_J\|+\|K_2f_J-P_J^{\mathrm{spar}}K_2f_J\|\\&\leqslant\big\|(K-P_J^{\mathrm{comp}}K)f_J\big\|+\big\|(K_2-P_J^{\mathrm{spar}}K_2)f_J\big\|.\end{aligned}$$

The error relies, therefore, on the fidelity of the operators $P_J^{\mathrm{comp}}K$, $P_J^{\mathrm{spar}}K_2$ and g_J. It holds that $\big\|(g_J-P_Jg)\big\|\leqslant C2^{-Jp}$. From Theorem 3.2, the formula $\big\|(K_2-P_J^{spar}K_2)f_J\big\|\leqslant C2^{-Jp}$ holds. On the other hand, we have $\big\|(K-P_J^{\mathrm{comp}}K)f_J\big\|\leqslant CJ2^{-Jp}$. Let $\big\|(I-T)\big\|^{-1}\leqslant B$, where B is the bounded number, we can estimate the total error of the algorithm as

$$\|f-f_J\|\leqslant CJ2^{-Jp}.$$

□

For the resultant matrix $B=I-M-S$, the number of nonzero elements affect the complexity of the algorithm. The numbers of nonzero elements of M and S are respectively $O\big(N\log(N)\big)$ and $O(N)$. So, the complexity of the algorithm is $O\big(N\log(N)\big)$.

The computational error of the matrix S is easy to keep smaller than the discretisation error. By Theorem 4.1, note that the computational error of the matrix M is $O\big(N_1^{-p}\log(N_1)\big)$. To preserve the computational error in accord with the discretisation error, the truncation parameter N_1 should be selected to satisfy $N_1\sim N$, where $N=2^J$.

5. Numerical Experiments

The numerical example we consider here has the form of (1.1) where

$$k(x,y)=\log\Big(4\sin^2\Big(\frac{x-y}{2}\Big)\Big),$$
$$k_2(x,y)=\log\big(a^2+b^2-(a^2-b^2)\cos(x+y)\big)+1,$$
$$g(x)=\exp(\cos(t))\Big(\cos(t)\cos(\sin(t))-\sin(t)\sin(\sin(t))\Big)-2+\exp(\cos(t))\cos(\sin(t))$$
$$+\exp(c\cos(t))\cos(c\sin(t))$$

in which $a=1$, $b=0.5$, and $c=(a-b)/(a+b)$.

We solve this example by using our wavelet method and the traditional Nyström method, respectively. For the log kernel, we firstly use the common Nyström method [10] to discretise the test example.

$$\frac{1}{2\pi}\int_0^{2\pi}\log\Big(4\sin^2\Big(\frac{x-y}{2}\Big)\Big)f(y)\mathrm{d}y\approx\sum_{j=0}^{2n-1}R_j^{(n)}(t)f(t_j),$$

where the selected equidistant nodes are $t_j=\pi j/n$ for $j=0,1,\dots,2n-1$,

$$R_j^{(n)}(t)f(t_j)=-\frac{1}{n}\Big\{\sum_{m=1}^{n-1}\frac{1}{m}\cos\big(m(t-t_j)\big)+\frac{1}{2n}\cos\big(n(t-t_j)\big)\Big\},j=0,1,\dots,2n-1.$$

The smooth part is discretised by the formula

$$\frac{1}{2\pi}\int_0^{2\pi}f(y)\mathrm{d}y\approx\frac{1}{2n}\sum_{j=0}^{2n-1}f(t_j).$$

Thus, the discretisation formula of the test example is

$$f_i^{(n)}-\sum_{j=0}^{2n-1}\Big\{R_{|i-j|}^{(n)}+\frac{\pi}{n}k_2(t_i,t_j)\Big\}f_j^{(n)}=g(t_i),i,j=0,1,\dots,2n-1.$$

Both of the system matrices of the log kernel M and the smooth kernel S are dense with $O(N^2)$ nonzero elements by the Nyström method.

Now, we turn our attention to the wavelet method. By the periodic Daubechies wavelet, we discretise the test example to get a system matrix. After obtaining the system matrix, the bi-conjugate gradient iterative method can be applied to solve the unknown. By the periodic wavelet transform, the approximation $f_J(x)$ will be obtained. We use the bi-conjugate gradient method to solve the resultant equation $B\widehat{f}=\widehat{g}$, where $B=I-M-S$.

We show the compression of the matrix M^{comp} in Table 1. Denote the threshold by ε. The symbol "*nnz*" stands for "(the number of non-zero entries of the matrix)/(all the entries of the matrix)". And, "*cond*" is the condition number of M^{comp}.

Table 1: Properties of the compressed matrix M

level J	size $N = 2^J$	threshold ε	number of nonzeros nnz	condition number cond
1	2	$1.0e-14$	0.5	4.0741
2	4	$1.0e-14$	0.5000	4.1342
3	8	$2.2000e-014$	0.4375	5.7956
4	16	$6.6000e-015$	0.3203	572.5716

Table 2: Numerical results of the system matrix $B = I - M - S$

level J	size $N = 2^J$	threshold ε	number of nonzeros nnz	condition number cond
2	4	$1.0000e-014$	1	1.35545980764243
3	8	$2.7900e-014$	0.70312500000000	1.35546030752614
4	16	$6.6000e-015$	0.50781250000000	1.35545975008947

In Table 2, for $B = I - M - S$, we give the property of the system matrix B. With increasing the numbers of wavelet basis, the system matrix has a stable condition number which is about 1.35546.

Table 3 compares the iterative error of the bi-conjugate gradient iterative method $e_n = \|r_n\| = \|\widehat{g} - B\widehat{f}_n\|$ between our wavelet method and the Nyström method. When the iterations number n increases, the wavelet method converges very rapidly. When $N = 8$, the condition number of B by the wavelet method is 1.35546030752614 which is compared with 19.81745055354439 by the traditional Nyström method.

A main advantage of wavelets is the reduction of the nonzero entries of matrices. It significantly reduces the time required by the iterative method. For the bi-conjugate gradient iterative method, each iteration requires two matrix multiplications, which take a time proportional to the number of nonzero elements of the matrix. The other major source of computational effort is to construct and store the matrices. For the resulting sparse linear systems by the wavelet method, it only takes $5\% - 10\%$ of the total computational time.

Table 4 displays the experimental accuracies. The errors are obtained by using the wavelet method with the true solution at the different level J. The computed result corroborates the theoretical analysis on the convergence rate $O\big(N^{-p}\log(N)\big)$ where $N = 2^J$ for our wavelet method.

6. Conclusions

We have shown that it is possible to use the periodic Daubechies wavelets to solve the weakly singular integral equations with log and smooth kernels. The algorithm leads

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Table 3: Iterative error of the bi-conjugate gradient method

number of iterations n	error of the wavelet method ε_n	error of the Nyström method ε_n
1	1.41981583488315	5.83502366943766
2	0.02688971889605	1.13970208509284
3	$1.282844345898094e-006$	0.08607546684821
4	$1.699156985238018e-011$	0.00311810828786
5	$1.972132533753210e-017$	$8.815650367406799e-004$
6	$1.355252715606881e-019$	$2.366416351984084e-014$
7	$8.131516293641283e-020$	$4.215924841687874e-015$
8	$2.710505431213761e-020$	$1.395528702594244e-015$
9	$2.710505431213761e-020$	$1.087791964408415e-015$
10	$2.710505431213761e-020$	$1.185393811112968e-015$
11	$2.710505431213761e-020$	$4.710277376051325e-016$
16	$2.710505431213761e-020$	$1.047382306668854e-015$
18	$2.710505431213761e-020$	$1.047382306668854e-015$

Table 4: Computed error of solutions

level J	size N	error of the wavelet method $\|f - f_J\|_{L_2}$
1	2	2.60887262610224
2	4	$1.67636915514826e-01$
3	8	$6.38325521955033e-02$
4	16	$1.69717428982700e-02$

to sparse matrices which can be efficiently realised by the bi-conjugate gradient iterative method. Application of the compression strategy and the hyperbolic cross approximation are shown to be necessary to achieve less complexity. Particularly, we study the properties of the resulting matrices M and S for the weakly singular and smooth parts. The traditional Nyström method typically produce dense matrices that require a large amount of storage cost and time to invert.

The method of the log integral kernel by the periodic Daubechies wavelets can be applied to calculate other integral kernels. The application of the hyperbolic cross approximation for the smooth kernel can help reducing the nonzero elements of the system matrix and accelerating the convergence of the method.

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