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## A proof that the middle points of parallel chords of a conic lie on a fixed straight line.

By Professor JACK.

FIGURES 12, 13, 14.

Let S be the focus of the conic, FX the directrix, $e$ the eccentricity.

Let V be the middle point of PP′.

Draw VK perpendicular to the directrix, and with centre V describe a circle, radius equal to $e$.VK.

Join SP, SP′ and draw radii V$p'$, V$p$, parallel to SP′, SP, and let PP′ meet the directrix in L. Then $p$, $p'$ are on the line SL.

Now $$\mathrm{PV} = \mathrm{VP'} \quad \text{(by hypothesis)}$$

$$\therefore \quad \frac{\mathrm{PV}}{\mathrm{VL}} = \frac{\mathrm{P'V}}{\mathrm{VL}},$$

$$\text{and} \quad \frac{\mathrm{PV}}{\mathrm{VL}} = \frac{\mathrm{S}p}{p\mathrm{L}} \quad \text{and} \quad \frac{\mathrm{P'V}}{\mathrm{VL}} = \frac{\mathrm{S}p'}{p'\mathrm{L}};$$

$$\therefore \quad \frac{\mathrm{S}p}{p\mathrm{L}} = \frac{\mathrm{S}p'}{p'\mathrm{L}};$$

$\therefore$ L$p'$S$p$ is a harmonic range,

and S is on the polar of L with regard to the circle.

Draw this polar, meeting PP′ in Y, the directrix in F and VK in H.

Now in the quadrilateral LYHK,

$$\angle \mathrm{LKH} = \angle \mathrm{LYH} = \frac{\pi}{2};$$

$\therefore$ the figure is cyclic.

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$$\therefore \quad VH.VK = VY.VL = (e.VK)^2.$$

$$\therefore \quad VH = e^2VK.$$

Now produce VF to meet SX (which is perpendicular to the directrix) in C;

$$\therefore \quad \frac{CS}{CX} = \frac{VH}{VK} = e^2;$$

∴ C is a fixed point; and so is F; and ∴ V is always on the fixed line CF.

This proof applies to central conics (Fig. 12 has been drawn for the ellipse, Figs. 13 and 14 for the hyperbola).

The case of the parabola may be got by observing that C moves off to infinity and therefore all diameters are parallel to the axis; or it may be investigated as follows.

FIGURE 15.

Describe a circle with centre V and radius VK ($e$ being 1).

Join SP, SP′ and draw V$p$, V$p'$ parallel to them. Then as before $p$, $p'$ are on LS and L$p$S$p'$ is a harmonic range.

∴ S is on the polar of L with regard to the circle; (and this polar always goes through K)

∴ KV is perpendicular to the directrix, *i.e.*, parallel to the axis.

*The Converse* (for Central Conics).

With almost the same construction, we get

$$\frac{VH}{VK} = \frac{CS}{CX} = e^2, \quad \therefore \ VH = e^2.VK.$$

$$\therefore \quad VH.VK = (e.VK)^2;$$

$$\therefore \quad VY.VL = (\text{the radius})^2;$$

∴ SF is polar of L;

∴ L$p$S$p'$ is a harmonic range;

∴ VP = VP′.

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