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SUBGROUP COVERINGS OF SOME LINEAR GROUPS

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Dedicated to Bernhard Neumann on his 90th birthday.

A *cover* for a group is a collection of proper subgroups whose union is the whole group. A cover is *minimal* if no other cover contains fewer members. We term *minimised* a minimal cover with the property that substituting for a member of the cover by a proper subgroup of that member produces a collection which is no longer a cover. We here describe the minimised covers for the groups $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$.

1. INTRODUCTION

Let G be a group. A *cover* of G is a collection $\mathcal{A} = \{A_i : 1 \leqslant i \leqslant n\}$ of proper subgroups of G whose union is G. The cover $\mathcal{A}$ is *irredundant* if no proper sub-collection is also a cover; and *minimal* if no cover of G has fewer than n members. In this minimal case we write $\sigma(G) = n$.

Covers of groups have been studied by many authors. For example Neumann [5] shows that the intersection of the members of an irredundant cover with n members has index bounded by a function of n. Tomkinson [8] improved this bound. Minimal covers seem to have been introduced by Cohn [1]; and Tomkinson [9] showed that, for a finite soluble group G, $\sigma(G)$ is $p^\alpha + 1$ where p^α is the size of the smallest chief factor of G with multiple complements. He confirmed a conjecture of Cohn [1] that $\sigma(G) = 7$ for no group G. His proof suggests that investigating minimal covers of insoluble groups might be of interest. Here we make a small beginning by looking at the groups $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ (Theorem 3.5). We find σ for these groups and, more to the point, we give a description of all the minimal covers which are minimised in a sense to be described below (Theorem 4.4).

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2. Quotations and notation

In what follows G will always denote one of the groups $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ where $q = p^\alpha$ is a prime power, always at least 4 except in Section 5. In contexts where the symbol G is unquantified it may be interpreted as any one of the four groups. Of course, when q is even, the four groups comprise just two isomorphism classes represented by $\mathrm{GL}_2(q)$ and $\mathrm{SL}_2(q)$. We associate a parameter ε with G as follows: $\varepsilon = 1$ except in the single case when q is odd and G is $\mathrm{PSL}_2(q)$ when $\varepsilon = 1/2$. Let V be the space on which $\mathrm{GL}_2(q)$ acts so that all four groups have a natural action on the projective line $\mathcal{P}(1,q)$ thought of as the set of one-dimensional subspaces of V. It will be convenient to assign a labelling $\{1, 2, \ldots, q+1\}$ to the points of $\mathcal{P}(1,q)$; this assignment is arbitrary, but then fixed. The stabiliser in G of point i we denote by P_i and its unipotent radical by N_i; note that $N_i = O_p(P_i)$, and that N_i is a Sylow p-subgroup of G. It is well-known that P_i is maximal in G.

The structure of a point stabiliser P_i is well-known. We may write $P_i = F_i Z$ where Z is the centre of G, and $F_i Z/Z$ is a Frobenius group of order $\varepsilon q(q-1)$ with Frobenius kernel $N_i Z/Z$. Of course, except for $\mathrm{GL}_2(q)$, Z in the product $F_i Z$ is redundant; and, in the case of $\mathrm{GL}_2(q)$, the product $F_i Z$ is direct. F_i is not unique when $G = \mathrm{GL}_2(q)$, but each choice for it is a supplement for $\mathrm{SL}_2(q)$ in G.

Let us write $K_{ij} := F_i \cap P_j$ $(i \neq j)$. Then

$$|K_{ij}| = \varepsilon(q-1), \tag{1}$$

and

$$|K_{ij} \cap Z| = \begin{cases} 2, & G = \mathrm{SL}_2(q)\ (q \text{ odd}) \\ 1, & \text{otherwise}\ . \end{cases} \tag{2}$$

Moreover

$$K_{ij}Z = K_{ji}Z = P_i \cap P_j\ (i \neq j).$$

K_{ij} is an example of an *eccentric* cycle in $P_i \cap P_j$, namely a cyclic supplement for Z in $P_i \cap P_j$. Every eccentric cycle arises as $F_i \cap P_j$ for some choice of F_i. In $\mathrm{GL}_2(q)$ eccentric cycles are complements for Z in $P_i \cap P_j$, and distinct eccentric cycles generate $P_i \cap P_j$. $GL_2(q) = \mathrm{SL}_2(q)K_{ij}$ for every eccentric cycle K_{ij}.

A Singer cycle of $\mathrm{GL}_2(q)$ is a cyclic subgroup of order $q^2 - 1$. Every Singer cycle acts irreducibly on V. The Singer cycles therefore form a single conjugacy class of $\mathrm{GL}_2(q)$, and Singer cycles are self-centralising; in particular Z is in every Singer cycle. If S is a Singer cycle of $\mathrm{GL}_2(q)$ we shall term $S \cap \mathrm{SL}_2(q)$, $\big(S \cap \mathrm{SL}_2(q)\big)/\big(Z \cap \mathrm{SL}_2(q)\big)$ and S/Z Singer cycles of $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ respectively. By Huppert [**3**, 7.3] $\big|S \cap \mathrm{SL}_2(q)\big| = q+1$ and $\mathrm{GL}_2(q) = \mathrm{SL}_2(q)S$. Also, using this same result from Huppert, whenever D is a Singer normaliser in $\mathrm{GL}_2(q)$, then $D \cap \mathrm{SL}_2(q)$, $\big(D \cap \mathrm{SL}_2(q)\big)/\big(Z \cap \mathrm{SL}_2(q)\big)$ and D/Z are Singer normalisers in $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ respectively. The same result shows that the Singer cycles of $SL_2(q)$ are a single conjugacy class of $SL_2(q)$; that the order of the normaliser of a Singer cycle S is $2|S|$; and that the number of Singer cycles is $s := q(q-1)/2$ whichever of the four groups G might be.

Now it is easy to see that, whenever S is a Singer cycle and $1 \leqslant i \leqslant q+1$,

$$G = N_G(S)P_i \tag{3}$$

and

$$\left|P_i \cap N_G(S)\right| = \begin{cases} 2(q-1), & G = \mathrm{GL}_2(q) \\ 2, & G = \mathrm{SL}_2(q) \\ 2, & G = \mathrm{PGL}_2(q) \\ 1, & G = \mathrm{PSL}_2(q)\ (q \text{ odd}) . \end{cases} \tag{4}$$

We denote the set of Singer subgroups of G by Σ and the set of their normalisers in G by $\overline{\Sigma}$. In the same spirit if $\Sigma_0 \subseteq \Sigma$ then $\overline{\Sigma}_0$ denotes the set of the normalisers of the members of Σ_0.

We denote by Π the set of all point stabilisers in G. It will be convenient to write Π_i for $\Pi \setminus \{P_i\}$ and Π_{ij} for $\Pi \setminus \{P_i, P_j\}$ $(i \neq j)$.

It will be useful to make the following definition in the case that q is odd:

$$M := \begin{cases} \mathrm{SL}_2(q)Z, & G = \mathrm{GL}_2(q)\ (q \text{ odd}) \\ \mathrm{SL}_2(q)Z/Z, & G = \mathrm{PGL}_2(q)\ (q \text{ odd}) . \end{cases}$$

Note that M is a maximal subgroup of G.

We rely throughout this article on Dickson's list of subgroups of $\mathrm{PSL}_2(q)$ as given in Huppert (8.27 in [3]) and, to a lesser extent where convenient, on the Atlas [2]. Observe that in Huppert's 8.27 (7) there is a misprint: the orders of the subgroups involved are divisors of $\varepsilon q(q-1)$.

3. Preliminary results

Here we build up the results we need for the description of the minimised covers in the next section. One of the results we obtain here is the value of $\sigma(G)$ for all but a few small cases. These missing cases are treated in Section 5.

Lemma 3.1. *The union $\Sigma \cup \Pi$ is a cover for G. When q is even $\overline{\Sigma} \cup \Pi_i$ is a cover of G for each $i \in \{1, 2, \ldots, q+1\}$.*

Proof: An element of $\mathrm{GL}_2(q)$ acting without a fixed point in $\mathcal{P}(1,q)$ is contained in a Singer cycle. Hence $\Sigma \cup \Pi$ is a cover for G.

The second statement comes from the fact that

$$P_i = N_i Z \cup \bigcup_{j \neq i} (P_i \cap P_j)$$

and that, by (4), for q even, each element of $N_i Z$ is contained in some Singer normaliser. $\square$

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Corollary 3.2.

$$\sigma(G) \leqslant \begin{cases} \frac{1}{2}q(q+1), & q \text{ even} \\ \frac{1}{2}q(q+1)+1, & q \text{ odd} . \end{cases}$$

Lemma 3.3. *For $q \geqslant 4$ the normaliser of a Singer cycle S is the unique maximal subgroup of G containing S except when $G = (P)SL_2(q)$ and $q = 5, 7, 9$.*

Proof: When $G = (P)\,\mathrm{GL}_2\,(q)$ this follows from a special case of Kantor [4]. When $G = (P)\mathrm{SL}_2(q)$ the result follows by reference to Dickson's list. A maximal subgroup U of $\mathrm{SL}_2(q)$ containing a Singer cycle S satisfies $(q+1) \bigm| |U|$; and $Z \subseteq U$ because $\mathrm{SL}_2(q)$ is perfect. Then $\varepsilon(q+1) \bigm| |U/Z|$ and it is straight-forward to check that, unless q is odd and $q \leqslant 9$, U/Z is the normaliser of Z. ∎

Lemma 3.4.

1. *Let $G = \mathrm{GL}_2\,(q)$. Then*
   (a) $\langle N_i, N\rangle = SL_2(q) \quad (1 \neq N \subseteq N_j,\ i \neq j,\ q \geqslant 4)$
   (b) $\langle K_{ij}, N_k\rangle = G \quad (i \neq k \neq j \neq i,\ q \geqslant 4)$
   (c) $\langle K_{ij}, K_{k\ell}\rangle = G \quad \big(\{i,j\} \cap \{k,\ell\} = \emptyset,\ q \geqslant 7\big)$;
2. *When $G = SL_2(q)$ 1(b) holds for $q \geqslant 4$; and 1(c) holds for $q \geqslant 4$ but $q \neq 5, 7, 9, 11$.*

Proof: Let G be either $\mathrm{GL}_2\,(q)$ or $\mathrm{SL}_2(q)$. Now K_{ij} acts on $\mathcal{P}(1,q)$ with orbit lengths 1, 1 and $q-1$; and N_i acts with orbit lengths 1 and q. Hence, under the conditions imposed, $\langle N_i, N\rangle$, $\langle K_{ij}, N_k\rangle$ and $\langle K_{ij}, K_{k\ell}\rangle$ act transitively on $\mathcal{P}(1,q)$. Therefore $q+1$ divides the order of each.

1(a) Now $H := \langle N_i, N\rangle \subseteq \mathrm{SL}_2(q)$; and $q(q+1) \bigm| |H|$. If $H \neq \mathrm{SL}_2(q)$ then HZ/Z is a proper subgroup of $\mathrm{PSL}_2\,(q)$ and $\varepsilon q(q+1) \bigm| |HZ/Z|$. From Dickson's list [3, 8.27] the only proper subgroups of $\mathrm{PSL}_2\,(q)$ whose orders are divisible by q are contained in a Frobenius group of order $\varepsilon q(q-1)$, which is not divisible by $\varepsilon q(q+1)$, or are in subgroups isomorphic to A_4, S_4 or A_5; an easy calculation shows that, for $q \geqslant 4$, none of these is possible. This contradiction means that $H\big(Z \cap \mathrm{SL}_2(q)\big) = \mathrm{SL}_2(q)$ and so $H = \mathrm{SL}_2(q)$ since $\mathrm{SL}_2(q)$ is perfect.

1(b) Let $H := \langle K_{ij}, N_k\rangle$ and $H_0 := H \cap \mathrm{SL}_2(q)$ so that $H = K_{ij}H_0$. Also let $Z_0 := Z \cap \mathrm{SL}_2(q)$. Suppose $H_0 \neq \mathrm{SL}_2(q)$ and let U_0 be a maximal subgroup of $\mathrm{SL}_2(q)$ containing H_0; since $\mathrm{SL}_2(q)$ is perfect $Z_0 \subseteq U_0$. Now $q(q^2-1) \bigm| |H|$ or $q(q^2-1)/2$ according as q is even or odd, so either $\varepsilon q(q+1) \bigm| |U_0/Z_0|$ $\big(q \equiv 1 \pmod 4\big)$ or $\varepsilon^2 q(q+1) \bigm| |U_0/Z_0|$ $\big(q \equiv 3 \pmod 4\big)$. Much as in the last paragraph we conclude that, for $q \geqslant 4$, $H_0/Z_0 = \mathrm{PSL}_2\,(q)$ whence $H_0 = \mathrm{SL}_2(q)$, and so $H = G$, a contradiction.

1(c) Let $H = \langle K_{ij}, K_{k\ell}\rangle$. Define $H_0 := HZ \cap \mathrm{SL}_2(q)$ so that $HZ = K_{ij}H_0$. Write $Z_0 := Z \cap \mathrm{SL}_2(q)$. $K_{ij} \times Z \subseteq HZ$ so $(q^2-1)(q-1) \,\big|\, |HZ|$ or $(q^2-1)(q-1)/4$ according as q is even or odd, whence $\varepsilon^3(q^2-1) \,\big|\, |H_0/Z_0|$.

Suppose that $HZ \neq G$ so that $H_0 \neq \mathrm{SL}_2(q)$. We produce a contradiction to this assumption using Dickson's list of subgroup of $\mathrm{PSL}_2\,(q)$.

We show, first of all, that H_0/Z_0 is not isomorphic to one of A_4, S_4, A_5. Since $q \geqslant 7$, H_0/Z_0 does not in these cases have an automorphism of order $q-1$ (note that $H_0/Z_0 \cong A_5$ does not arise when $q = 7$). Hence K_{ij} does not act faithfully on H_0/Z_0, so there is a non-identity (non-central) element $y \in K_{ij}$ satisfying $[H_0, y] \subseteq Z_0$. Now the function $H_0 \to Z_0$ defined by $x \mapsto [x, y]$ is a homomorphism, and so $\left[O^2(H_0), y\right] = 1$ since $|Z_0| = 2$. Since $O^2(H_0)$ is an irreducible subgroup of G its centraliser in G is cyclic. However $\langle y\rangle Z$ is not cyclic, a contradiction in each of the three cases.

For $q \geqslant 7$, the other possibilities for H_0/Z_0 are: cyclic of order dividing $\varepsilon(q \pm 1)$, dihedral of order dividing $2\varepsilon(q \pm 1)$, subgroups of Frobenius groups of order $\varepsilon q(q-1)$, $\mathrm{PSL}_2\,(p^m)$ $(m \mid \alpha)$ and $\mathrm{PGL}_2\,(p^m)$ $(2m \mid \alpha)$. By checking these for divisibility of their orders by $\varepsilon^3(q^2-1)$ we eliminate all but the cases $q = 7$ and $H_0/Z_0 \cong S_3$, and $q = 9$ and $H_0/Z_0 \cong D_{10}$. The second case is eliminated exactly as in the last paragraph. In the first case H_0 is a non-Abelian semi-direct product of a cyclic group of order 3 by a cyclic group of order 4. Now V is irreducible for H_0, in fact absolutely irreducible since its dimension is prime. Therefore $C_G(H_0) = Z$. Also the automorphism group of H_0 has unique Sylow subgroup of order 3. Let y_1 and y_2 be elements of order 3 in K_{ij} and $K_{k\ell}$ respectively. It follows that $y_1 y_2^{\pm 1} \in C_G(H_0) = Z$. However this contradicts the fact that y_1, y_2 fix different points.

We have shown, therefore, that $HZ = G$. Hence $\mathrm{SL}_2(q) = G' \subseteq H$ so $G = \mathrm{SL}_2(q)K_{ij} \subseteq H$ whence $H = G$.

This concludes the proof of 1(c).

(2) If $H := \langle K_{ij}, N_k\rangle \neq G$ we see that $\varepsilon^2 q(q^2-1) \,\big|\, |H/Z|$. This is dismissed, for $q \geqslant 4$, much as in the proof of 1(a) above.

If $H := \langle K_{ij}, K_{k\ell}\rangle \neq \mathrm{SL}_2(q) = G$ we deduce that $\varepsilon^2(q^2-1) \,\big|\, |H/Z|$. Checking through Dickson's list yields that q is odd and $q \leqslant 11$. ∎

It is not difficult to see that the proscription $q \neq 5, 7, 9, 11$ is necessary in the second part of (2) of this lemma.

Before stating the next theorem we introduce some convenient notation. A poset may be defined on the set of collections of subgroups of a group H as follows. If $\mathcal{A} = \{A_i : 1 \leqslant i \leqslant m\}$ and $\mathcal{B} = \{B_i : 1 \leqslant i \leqslant n\}$ are collections of subgroups of H we say $\mathcal{A} \preceq \mathcal{B}$ if for some one-to-one function $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ we have $A_i \subseteq B_{f(i)}$ $(1 \leqslant i \leqslant m)$. The poset so defined we denote by Δ_H.

THEOREM 3.5. *Let $q \geqslant 4$. Then*

$$\sigma(G) = \begin{cases} \frac{1}{2}q(q+1), & q \text{ even} \\ \frac{1}{2}q(q+1)+1, & q \text{ odd} \end{cases}$$

provided either $G = (P)\,\mathrm{GL}_2(q)$, or $G = (P)SL_2(q)$ *and* $q \neq 5, 7, 9$.

PROOF: Let $\mathcal{A}$ be a minimal cover for G. If each member of $\mathcal{A}$ is replaced by a maximal subgroup of G containing it the resulting set is still a (minimal) cover for G. Hence we may suppose that $\mathcal{A}$ consists of maximal subgroups A_i $\big(1 \leqslant i \leqslant \sigma(G)\big)$ of G.

Now each Singer cycle S is in some member of $\mathcal{A}$. Since, by Lemma 3.3, the unique maximal subgroup containing a Singer cycle is its normaliser, which has order $2|S|$, it follows that different Singer cycles are in different members of $\mathcal{A}$, let us say in A_k $\big(1 \leqslant k \leqslant s = q(q-1)/2\big)$.

Using Lemma 3.4 1(a) we see that distinct F_i generate G, so do not belong to the same member of $\mathcal{A}$. Hence, if q is odd and $\{F_i\} \preceq \mathcal{A}$ $(1 \leqslant i \leqslant q+1)$, or if q is even and $\{F_i\} \preceq \mathcal{A}$ for all but one i, we are done by Corollary 3.2. Suppose, for some i, that

$$\{F_i\} \not\preceq \mathcal{A}.$$

None of the subgroups K_{ij} $(j \neq i)$ is in A_k $(1 \leqslant k \leqslant s)$; for, under our conditions, a Singer normaliser contains no eccentric cycle of order $\varepsilon(q-1)$. Moreover no two are in the same member of $\mathcal{A}$ since two together generate F_i. Hence

$$K_{ij} \subseteq A_{s+j} \ (j \neq i),$$

say. If q is even then $n \geqslant q(q-1)/2 + q = q(q+1)/2$ and we are done, by Corollary 3.2. If q is odd then

$$N_i \cap A_k = 1 \ (k \neq s+i). \tag{5}$$

This is because q is co-prime to the order of the Singer normalisers on the one hand and, if $N_i \cap A_{s+k} \neq 1$ for some $k \neq i$, then $F_i = \langle K_{ik}, N_i \cap A_{s+k}\rangle \subseteq A_{s+k}$, a contradiction, on the other. Therefore N_i intersects another member of $\mathcal{A}$ non-trivially. Hence,

$$n \geqslant \frac{1}{2}q(q-1) + q + 1$$

so, by Corollary 3.2 again, we are done. ∎

4. MINIMISED COVERS

The subposet of Δ_H consisting of minimal covers for H we denote Γ_H. Our aim now is to describe the minimal elements of Γ_G: these are the *minimised* covers of the heading of this section. The relative conditions imposed on q and G in the lemma we now state are designed to ensure that the hypotheses of Lemmas 3.3, 3.4 hold.

LEMMA 4.1. *Let $q \geqslant 7$ in the case that $G = (P)\mathrm{GL}_2(q)$, and $q \geqslant 4$ but $q \neq 5, 7, 9, 11$ when $G = (P)SL_2(q)$. Also let $\mathcal{A}$ be a minimal cover of G. Then, for some integers i, j satisfying $1 \leqslant i < j \leqslant q+1$,*

$$\Sigma \cup \Pi_{ij} \preceq \mathcal{A}.$$

PROOF: As before write $\mathcal{A} := \{A_i : 1 \leqslant i \leqslant n\}$ where $n = \sigma(G)$. It is clear that $\Sigma \preceq \mathcal{A}$, say the subgroups A_i $(1 \leqslant i \leqslant s)$ contain the Singer cycles.

Using Lemma 3.4 1(a) no two F_is are in the one member of $\mathcal{A}$ since, under our hypotheses, they generate G. Suppose that two F_is do not belong to members of $\mathcal{A}$; say, for notational simplicity, that $\{F_1\} \not\preceq \mathcal{A}$ and $\{F_2\} \not\preceq \mathcal{A}$. Now the q cyclic subgroups K_{1k} $(1 < k \leqslant q+1)$ do not belong to any A_ℓ $(1 \leqslant \ell \leqslant s)$, and no two belong to the same member of $\mathcal{A}$ since together they generate F_1. Let us say that

$$K_{1k} \subseteq A_{s+k} \ (1 < k \leqslant q+1).$$

When q is odd, moreover,

$$N_1 \subseteq A_{s+1}$$

because the argument following (5) shows that N_1 intersects every other member of $\mathcal{A}$ trivially.

The cyclic subgroups K_{2k} do not belong to A_i $(1 \leqslant i \leqslant s)$; and, from Lemma 3.4 1(c), 2, $K_{2k} \not\subseteq A_{s+\ell}$ $(\ell \neq k)$. Hence $K_{2k} \subseteq A_{s+k}$. Therefore,

$$F_k = \langle K_{1k}, K_{2k} \rangle \subseteq A_{s+k} \ (1 \neq k \neq 2).$$

Indeed this remains true whatever eccentric cycle in $K_{2k}Z$ we use in place of K_{2k}, so $Z \subseteq A_{s+k}$ whence $P_k = F_kZ \subseteq A_{s+k}$. This confirms the claim of the Lemma in the case that two F_is are not contained in members of $\mathcal{A}$.

Suppose just one F_i, say F_1, is not in a member of $\mathcal{A}$. Then we may suppose that

$$F_j \subseteq A_{s+j} \ (1 < j \leqslant q+1).$$

If q is odd then, as above, $N_1 \subseteq A_{s+1}$. In any case, as above, every eccentric cycle in $K_{1\ell}Z$ is in $A_{s+\ell}$ $(1 < \ell \leqslant q+1)$. It follows that $P_\ell \subseteq A_{s+\ell}$ $(1 < \ell \leqslant q+1)$. This concludes the proof in this case.

Suppose finally that every F_i is contained in some member of $\mathcal{A}$ (so that q is odd), say

$$F_i \subseteq A_{s+i} \ (1 \leqslant i \leqslant q+1).$$

Now

$$N_iZ \cap A_j \subseteq Z \ (j \neq s+i)$$

on account of Lemma 3.4 1(a) and the fact that Singer normalisers have order co-prime to q (recall that q is odd here). Hence $N_iZ \subseteq A_{s+i}$ and so $P_i = F_iZ \subseteq A_{s+i}$. This again confirms the conclusion of the Lemma. ∎

We now begin a description of the minimised covers, always under the hypotheses of Lemma 4.1 so that Lemmas 3.3, 3.4 are in force. To this end we first of all derive necessary conditions on a cover $\mathcal{A}$ of G in order that it be minimal.

At most two point stabilisers are not in $\mathcal{A}$, by Lemma 4.1. Suppose that P_i and P_j are not in $\mathcal{A}$. First consider the case when q is odd. Now, by Lemma 3.4 1(a), and the fact that p does not divide the order of a Singer normaliser, N_i and N_j each intersect trivially the members of $\mathcal{A}$ containing the subgroups in $\Sigma \cup \Pi_{ij}$. By Theorem 3.5 there are two other members of $\mathcal{A}$, say X, Y, and

$$N_iZ \cap A = N_jZ \cap A = Z \ (A \in \mathcal{A}, X \neq A \neq Y).$$

It follows that

$$N_iZ = Z \cup (N_iZ \cap X) \cup (N_iZ \cap Y).$$

This union is redundant. If not, as is well-known (it is an old result of Scorza [**6**]),

$$\begin{aligned} 4 = \left|N_iZ : Z \cap (N_iZ \cap X) \cap (N_iZ \cap Y)\right| &= |N_iZ : Z \cap X \cap Y| \\ &= |N_iZ : Z||Z : Z \cap X \cap Y|, \end{aligned}$$

a contradiction since $|N_iZ : Z| = |N_i|$ is odd. Hence N_iZ is contained in X or Y. Similarly N_jZ is in X or Y. Therefore, changing names if necessary, either

(a) $M = \langle N_i, N_j\rangle Z = X$, and in this case $G = (P)\,\mathrm{GL}_2(q)$; or

(b) $N_iZ \subseteq X$ and $N_jZ \subseteq Y$.

Now K_{ij} must be in X or Y, using Lemma 3.4 and the fact that, under our hypotheses, no Singer normaliser contains an eccentric cycle. Case (b) therefore does not arise since P_i, P_j are not in $\mathcal{A}$. In case (a) it must be that every eccentric cycle from $P_i \cap P_j$ is in Y.

It follows that, when q is odd, and $\mathcal{A}$ lacks two point stabilisers P_i and P_j, $G = (P)\,\mathrm{GL}_2(q)$, $X = M$ and $K_{ij}Z \subseteq Y$.

When q is even, and $\mathcal{A}$ lacks two point stabilisers, P_i and P_j say, then there is a unique member X of $\mathcal{A}$ other than those containing the members of $\Sigma \cup \Pi_{ij}$, by Lemma 3.5. By Lemma 3.4, $K_{ij}Z \subseteq X$. Now $X \cap N_i = X \cap N_j = 1$, and N_i and N_j intersect trivially the point stabilisers P_k $(i \neq k \neq j)$. Hence $N_i \cap W \neq 1$ for some $W \in \mathcal{A}$ containing a Singer cycle. Therefore $W \in \overline{\Sigma}$. However no two involutions in a Singer normaliser commute, so $|W \cap N_i| = 2$. It follows that, for each non-trivial $v \in N_i$, there is a Singer normaliser $\overline{S}_v \in \mathcal{A}$ such that $\overline{S}_v \cap N_i = \langle v\rangle$. Similarly for each non-trivial $v \in N_j$ there exits $T_v \in \Sigma$ such that $\overline{T_v} \in \mathcal{A}$ and $\overline{T_v} \cap N_j = \langle v\rangle$.

Let us write, for $1 \leqslant k \leqslant q+1$ and for $1 \neq v \in N_k$,

$$\Sigma_{k,v} := \left\{S \in \Sigma : \overline{S} \cap N_k = \langle v\rangle\right\}.$$

Observe that, by (4), $N_G(S) \cap N_i \neq 1$ $(S \in \Sigma,\ 1 \leqslant i \leqslant q+1)$. Then, for each $k \in \{1, 2, \ldots, q+1\}$, $\Sigma = \bigcup_{1 \neq v \in N_k} \Sigma_{k,v}$; conjugation by N_k permutes $\Sigma_{k,v}$ transitively with kernel $\langle v \rangle$; and conjugation action by P_k permutes the set $\{\Sigma_{k,v} : 1 \neq v \in N_k\}$ transitively. Let us call a subset Σ_k of Σ a *k-transversal* if $|\Sigma_k \cap \Sigma_{k,v}| = 1$ $(1 \neq v \in N_k)$.

What we showed in the penultimate paragraph was that, when q is even, and when $\mathcal{A}$ does not contain P_i and P_j, then $\mathcal{A}$ contains $\overline{\Sigma}_i$ and $\overline{\Sigma}_j$ where Σ_i and Σ_j are respectively an i- and a j-transversal of Σ. Notice that Σ_i and Σ_j may overlap.

To sum up so far:

LEMMA 4.2. *Under the hypotheses of Lemma 4.1, if $\mathcal{A}$ is a minimal cover of G, and if two point stabilisers, say P_i and P_j, are not in $\mathcal{A}$, then:*

(a) *if q is odd, $G = (P)\,\mathrm{GL}_2(q)$ and*

$$\Sigma \cup \Pi_{ij} \cup \{M\} \cup \{P_i \cap P_j\} \preceq \mathcal{A};$$

(b) *and if q is even,*

$$\overline{\Sigma}_{ij} \cup (\Sigma \setminus \Sigma_{ij}) \cup \Pi_{ij} \cup \{P_i \cap P_j\} \preceq \mathcal{A}$$

where Σ_{ij} is the union of an i-transversal and a j-transversal of Σ.

Analogously, when just one point stabiliser is absent from a minimal cover, we have the following lemma.

LEMMA 4.3. *Under the hypotheses of Lemma 4.1, if $\mathcal{A}$ is a minimal cover of G lacking just one point stabiliser P_i then:*

(a) *if q is odd, $\Sigma \cup \Pi_i \cup \{N_i Z\} \preceq \mathcal{A}$;*

(b) *and if q is even, $\overline{\Sigma}_i \cup (\Sigma \setminus \Sigma_i) \cup \Pi_i \preceq \mathcal{A}$ where Σ_i is an i-transversl of Σ.*

PROOF: We have $\Sigma \cup \Pi_i \preceq \mathcal{A}$. When q is odd $N_i Z$ intersects trivially every Singer normaliser and every P_j $(j \neq i)$, using a familiar argument. This proves the first statement. The second statement uses the argument given in the preamble to Lemma 4.2. ∎

We are now in a position to describe the minimised covers of G.

THEOREM 4.4. *Let $q \geqslant 7$ and $G = (\mathrm{P})\,\mathrm{GL}_2(q)$, or $q \geqslant 4$ but $q \neq 5, 7, 9, 11$ and $G = (P)SL_2(q)$. The minimal members of Γ_G are precisely:*

1. *q odd*

   (a) $\Sigma \cup \Pi_{ij} \cup \{M\} \cup \{P_i \cap P_j\}$ $(1 \leqslant i < j \leqslant q+1)$ ($G = (\mathrm{P})\mathrm{GL}_2(q)$ only);

   (b) $\Sigma \cup \Pi_i \cup \{N_i Z\}$ $(1 \leqslant i \leqslant q+1)$.

2. *q even*

   (a) $\overline{\Sigma}_{ij} \cup (\Sigma \setminus \Sigma_{ij}) \cup \Pi_{ij} \cup \{P_i \cap P_j\}$ $(1 \leqslant i < j \leqslant q+1)$ *where Σ_{ij} is the union of an i-transversal and a j-transversal of Σ.*

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(b) $\overline{\Sigma}_i \cup (\Sigma \setminus \Sigma_i) \cup \Pi_i$ *where* Σ_i *is an* i*-transversal of* Σ *which, for no* $j \neq i$, *is a* j*-transversal.*

PROOF: Observe first of all that the unions displayed in the theorem are covers of G. By Lemma 3.1 each of 1(a) and 2(a) is a cover if the elements of $(P_i \setminus P_j) \cup (P_j \setminus P_i)$ are accounted for. These are either in $\{N_i Z\} \cup \{N_j Z\}$, or have order dividing $q-1$ and do not fix both i and j. The first type are in M (q odd) or in $\overline{\Sigma}_{ij}$ (q even); and, by Maschke's Theorem, the elements of the second type fix a point other than i, j and so are picked up by the members of Π_{ij}. A similar argument shows that the unions 1(b) and 2(b) are covers for G.

Let $\mathcal{A}$ be an arbitrary minimised cover of G. Note that $\mathcal{A}$ lacks at least one point stabiliser. When q is even this follows from Theorem 3.5. When q is odd $\mathcal{A}$ would be $\Sigma \cup \Pi$ if it contained every point stabiliser; but then $\Sigma \cup \Pi_1 \cup \{N_1 Z\}$, for example, would be a cover properly below $\mathcal{A}$ in Γ_G, contradicting the assumption that $\mathcal{A}$ is minimised.

By Lemma 4.1 at most two point stabilisers do not appear in $\mathcal{A}$. If there are two, say P_i and P_j then, by Lemma 4.2, the unions displayed in 1(a) and 2(a) are below $\mathcal{A}$ in Γ_G when q is odd or even respectively. If just one point stabiliser P_i is not in $\mathcal{A}$ then, by Lemma 4.3, the unions in 1(b) and 2(b) are below $\mathcal{A}$ in Γ_G, according as q is odd or even. Since these unions are all covers for G each is $\mathcal{A}$. Notice that in the case 2(b) Σ_i is not a j-transversal for $j \neq i$ since otherwise we could replace P_j in $\mathcal{A}$ by $\mathcal{P}_i \cap P_j$ obtaining a cover of G strictly below $\mathcal{A}$.

Finally the unions $\mathcal{B}$ displayed in the theorem *are* minimised covers. In the cases 1(a) and 2(a) this follows at once from Lemma 4.2 since a minimised cover dominated by $\mathcal{B}$ in Γ_G lacks two point stabilisers. In the cases 1(b) and 2(b) if $\mathcal{B}$ is not minimised then it dominates a minimal member $\mathcal{C}$ of Γ_G. By Lemma 4.3, $\mathcal{C}$ lacks a point stabiliser other than P_i, sayP_j. By Lemma 4.2 this means that $M \in \mathcal{C} \preceq \mathcal{B}$ in case 1(b), a contradiction, and in case 2(b) that $\overline{\Sigma}_j \subseteq \mathcal{C} \preceq \mathcal{B}$, whence $\overline{\Sigma}_j = \overline{\Sigma}_i$, also a contradiction. ∎

COROLLARY 4.5. *If* $\mathcal{A}$ *is a minimal cover of* G *then* Z *is in every member of* $\mathcal{A}$.

5. THE MISSING VALUES OF $\sigma(G)$

Here we calculate σ for the groups $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ missing from Theorem 3.5.

CASE $q = 2$. In this case $Z = 1$, all four groups are isomorphic to S_3, and plainly

$$\sigma(G) = 4. \tag{6}$$

CASE $q = 3$. Observe that $\mathrm{PGL}_2(3) \cong S_4$ and that A_4 and the three Sylow 2-subgroups constitute a minimal cover of S_4. On the other hand $\mathrm{PSL}_2(3) \cong A_4$. It is easily seen that a minimal cover for A_4 consists of its five Sylow subgroups, so

$$\sigma\big((\mathrm{P})\,\mathrm{GL}_2(3)\big) = 4; \quad \sigma\big((\mathrm{P})\mathrm{SL}_2(3)\big) = 5. \tag{7}$$

Case $q = 5$. Only (P)$SL_2(q)$ escape Theorem 3.5. Since $PSL_2(5) \cong PSL_2(4)$,

$$\sigma\big((P)SL_2(5)\big) = 10. \tag{8}$$

Case $q = 7$. Again only (P)$SL_2(q)$ are missing from Theorem 3.5. Since $G = PSL_2(7) \cong GL_3(2)$ we can regard G as acting on the projective space $\mathcal{P}(2, 2)$. In this action every element of G either fixes a point, or lies in a Singer cycle. There are eight Singer cycles, and seven point stabilisers, so $\sigma(G) \leqslant 15$.

By Kantor [4] the normalisers of these eight Singer subgroups, Frobenius groups of order 21, are the unique maximal subgroups of G containing a Singer cycle. Hence the eight Singer normalisers all occur in a minimal cover $\mathcal{A}$ of maximal subgroups of G.

The only other maximal subgroups of G are the point and line stabilisers, all isomorphic to S_4: see the Atlas [2]. Hence no proper subgroup of G contains more than three cyclic subgroups of order 4. There are 21 such subgroups, so there are at least 7 subgroups in $\mathcal{A}$ different from the Singer normalisers. Thus $\sigma(G) \geqslant 15$ whence

$$\sigma\big((P)SL_2(7)\big) = 15. \tag{9}$$

Case $q = 9$. Let $G = PSL_2(q)$, of order 360. The maximal subgroups of G are either isomorphic to A_5, are point stabilisers which are Frobenius of order 36, or isomorphic to S_4: see [2]. The Singer cycles here have order 5, their normalisers have order 10, and so there are 36 in all. No more than six 5-cycles lie in a proper subgroup of G.

Note that $G \cong A_6$, so every Singer cycle is in one of the six copies of A_5 in a conjugacy class. As before G is the union of Singer cycles and point stabilisers, and therefore the union of six copies of A_5 and the ten point stabilisers. Hence $\sigma(G) \leqslant 16$.

On the other hand if $\mathcal{B} := \big\{B_i : 1 \leqslant i \leqslant \sigma(G)\big\}$ is a minimal cover of G of maximal subgroups then at least $36/6 = 6$ members of $\mathcal{B}$ contain a 5-cycle, and these members are all isomorphic to A_5. None of these contains a 4-cycle. Let us suppose that $\sigma(G) < 16$. Then there are at most nine point stabilisers in $\mathcal{B}$, say P is a point stabiliser not in $\mathcal{B}$. The nine 4-cycles in P are then in different members of $\mathcal{B}$ none of which are isomorphic to A_5. Hence at most six members of $\mathcal{B}$ are copies of A_5, and therefore exactly six. It follows that $\sigma(G) = 15$. Let us say that $B_i \cong A_5$ $(1 \leqslant i \leqslant 6)$ and that, for $7 \leqslant i \leqslant 15$, each B_i contains a 4-cycle from P (but, of course, contains no 5-cycle).

From now on we shall regard G as A_6, and expressions such as 'point stabiliser' and 'fixed point' will refer to the natural action of A_6 on $\{1, 2, 3, 4, 5, 6\}$. We denote by Q_i the stabiliser of i in this action $(1 \leqslant i \leqslant 6)$. Each Q_i is isomorphic to A_5 and the set $\{Q_i : 1 \leqslant i \leqslant 6\}$ is a conjugacy class. It is well-known that there is just one other conjugacy class $\{R_i : 1 \leqslant i \leqslant 6\}$ of copies of A_5 in G (whose actions are transitive). These two classes are distinguished by the action of their 3-cycles: those in the Q_i fix a point, those in the R_i fix no point.

In P there are four 3-cycles of which just two fix a point. Let us say c_1 fixes a point but c_2 fixes no point. Moreover $c_i \notin B_j$ $(1 \leqslant i \leqslant 2,\ 7 \leqslant j \leqslant 15)$ since P is generated by every pair consisting of a 3-cycle and a 4-cycle. Hence each of c_1, c_2 is in some B_i $(1 \leqslant i \leqslant 6)$. It follows that at least one point stabiliser, say Q_1, is not in $\mathcal{B}$. Now each of the six 5-cycles in Q_1 is in some B_i $(1 \leqslant i \leqslant 6)$, and no two are in the same one since two together generate Q_1. Since 1 is the unique point fixed by these 5-cycles from Q_1 it follows that *no* Q_i is in $\mathcal{B}$. This contradicts that c_1 lies in some B_i $(1 \leqslant i \leqslant 6)$.

Hence the assumption $\sigma(G) < 16$ is false. Therefore

$$\sigma\Big((\mathrm{P})\mathrm{SL}_2(9)\Big) = 16. \tag{10}$$

The values of σ for the groups $\mathrm{PSL}_2(7)$ and A_6 were obtained by Shieh in [**7**].

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