SOME RESULTS INVOLVING WEAK COMPACTNESS IN $C(X)$, $C(\upsilon X)$ AND $C(X)'$

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1. Introduction

The main aim of the present note is to compare $C(X)$ and $C(\upsilon X)$, the spaces of real-valued continuous functions on a completely regular space X and its real 1-1 compactification υX, with regard to weak compactness and weak countable compactness. In a sense to be made precise below, it is shown that $C(X)$ and $C(\upsilon X)$ have the same absolutely convex weakly countably compact sets. In certain circumstances countable compactness may be replaced by compactness, in which case one obtains a nice representation of the Mackey completion of the dual space of $C(X)$ (Theorems 5, 6, 7).

These results all depend to some extent on Theorem 1, where it is shown that the same absolutely convex subsets of $C(X)$ are compact for the weak topology and the topology of pointwise convergence. This fact is used to give a proof of a recent result of H. Buchwalter (**1**, Théorème 2.6) which asserts that $C(X)$ and $C(\upsilon X)$ have the same *disques bornés complétants* for their topologies of compact convergence. (A disque borné complétant in a locally convex space is a bounded absolutely convex set whose span is a Banach space with respect to the associated norm. Such a set will be referred to here as a *dbc-set*.) Various convergence criteria are also considered in Section 3.

Section 4 is concerned with conditions under which $C(X)$ is barrelled or σ-barrelled in its topology of compact convergence. In (**2**, Théorème 4.1), H. Buchwalter and J. Schmets have given various conditions equivalent to σ-barrelledness for $C(X)$. Theorem 9 contains some further results in this direction, while Theorem 10 is an analogue for the barrelled case of one of their results.

I am very grateful to Professor Schmets and Professor Buchwalter for providing me with preprint copies of (**2**), and to the referee for simplifying the proof of Theorem 5 (ii).

2. Notation

It will be assumed *without specific mention* that all the underlying topological spaces are non-empty and completely regular.

For $f \in C(X)$, f^{υ} denotes its unique extension to an element of $C(\upsilon X)$ and if $A \subseteq C(X)$, $A^{\upsilon} = \{f^{\upsilon}\colon f \in A\}$. The statement that $C(X)$ and $C(\upsilon X)$ have the same sets satisfying a property (P) means that A satisfies property (P) in $C(X)$ if and only if A^{υ} satisfies this property in $C(\upsilon X)$.

$C_c(X)$ denotes $C(X)$ with the topology of compact convergence and $C(X)'$ is the dual of $C_c(X)$. $C_\sigma(X)$, $C_s(X)$, $[C(X)']_\sigma$, $[C(X)']_\tau$ denote respectively $C(X)$ with the weak topology $\sigma(C(X), C(X)')$, $C(X)$ with the topology s_X of pointwise convergence, $C(X)'$ with the weak topology $\sigma(C(X)', C(X))$ and $C(X)'$ with the Mackey topology $\tau(C(X)', C(X))$.

For each $x \in X$, δ_x denotes the element of $C(X)'$ defined by $\delta_x(f) = f(x)$.

3. Weak compactness and convergence: completion of the dual space

Let $\mathcal{K}$ be the family of all non-empty compact subsets of X. For each $K \in \mathcal{K}$ let $H(K)$ be the subset $\{f \in C(X): |f(x)| \leqq 1, x \in K\}^0$ of $C(X)'$. Then:

(i) each $H(K)$ is an absolutely convex compact subset of $[C(X)']_\sigma$;
(ii) $\bigcup\{H(K): K \in \mathcal{K}\}$ spans $C(X)'$;
(iii) the set of extremal points of $H(K)$ is $\{\pm\delta_x: x \in K\}$.

The first and second observations follow since $H(K)$ is the polar of a neighbourhood in $C_c(X)$ and each element of $C(X)'$ is bounded on such a neighbourhood. It is well known that when X is compact, the extremal points of the closed unit ball of $C(X)'$ are precisely the point measures $\pm\delta_x$ $(x \in K)$ (**7**, Section 25, 2 (2)). The third observation is easily deduced from this by using the fact that each $K \in \mathcal{K}$ is C-embedded in $C(X)$, (**4**, 3.11 (c)).

If F' is the vector subspace of $C(X)'$ spanned by the extremal points of the sets $H(K)$ $(K \in \mathcal{K})$, $(C(X), F')$ is a dual pair, and the topology $\sigma(C(X), F')$ is simply s_X. (For later reference, note that in the terminology of (**13**, Section 2), $\mathcal{S} = \{H(K): K \in \mathcal{K}\}$ is an S-family for $(C(X), C(X)')$ and $C(X)'(\mathcal{S}) = F'$.) The following is now an immediate consequence of (**12**, Theorem 2).

Theorem 1. *$C_\sigma(X)$ and $C_s(X)$ have the same absolutely convex compact sets.*

Corollary. *Let $f_n \to 0$ in $C_\sigma(X)$ and suppose that the closed absolutely convex envelope A of $\{f_n: n = 1, 2, \ldots\}$ in $C_\sigma(X)$ is compact. Then A^υ is compact in $C_\sigma(\upsilon X)$.*

Proof. The proof of (**7**, Section 20, 9 (6)) shows that

$$A = \left\{\sum_{n=1}^{\infty} \xi_n f_n: \sum_{n=1}^{\infty} |\xi_n| \leqq 1, \quad \xi_n \in \mathbb{R}, \quad n = 1, 2, \ldots\right\},$$

where each series $\Sigma\xi_n f_n$ converges in $C_\sigma(X)$. For any such series $\Sigma\xi_n f_n$ and any $y \in \upsilon X$, there exists $x \in X$ such that

$$\left(\sum_{n=1}^{\infty} \xi_n f_n\right)^\upsilon (y) = \left(\sum_{n=1}^{\infty} \xi_n f_n\right)(x)$$

and $f_n^\upsilon(y) = f_n(x)$ $(n = 1, 2, \ldots)$ (**4**, 7C). Then $f_n^\upsilon(y) \to 0$ and

$$\left(\sum_{n=1}^{\infty} \xi_n f_n\right)^\upsilon (y) = \sum_{n=1}^{\infty} \xi_n f_n(x) = \sum_{n=1}^{\infty} \xi_n f_n^\upsilon(y),$$

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or equivalently,

$$f_n^\upsilon \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \xi_n f_n^\upsilon = \left(\sum_{n=1}^{\infty} \xi_n f_n\right)^\upsilon$$

in $C_s(\upsilon X)$. Again by (7, Section 20, 9 (6)), A^υ is compact in $C_s(\upsilon X)$ and therefore also in $C_\sigma(\upsilon X)$ by the theorem.

Théorème 2.6 of (**1**) can now be obtained by an application of this corollary.

Theorem 2 (Buchwalter). *$C_c(X)$ and $C_c(\upsilon X)$ have the same dbc-sets.*

Proof. As in (**1**) it is enough to show that if D is a *dbc*-set in $C_c(X)$, D^υ is bounded in $C_c(\upsilon X)$. If this is not the case, there is an element $\mu \in C(\upsilon X)'$ and a sequence (f_n) in D such that

$$|\mu(f_n^\upsilon)| \geqq n^2 \quad (n = 1, 2, \ldots) \quad (*).$$

If E is the span of D in $C(X)$, $(1/n)f_n \to 0$ in the norm topology defined on E by D, and since E is complete in this topology, the closed absolutely convex envelope A of $\{(1/n)f_n: n = 1, 2, \ldots\}$ is a norm compact subset of E. Since $C_\sigma(X)$ induces a coarser topology on E, A is compact in $C_\sigma(X)$. By the Corollary to Theorem 1, A^υ is compact and therefore bounded in $C_\sigma(\upsilon X)$. Thus there is a constant $M \geq 0$ such that $|\mu((1/n)f_n^\upsilon)| \leq M$ $(n = 1, 2, \ldots)$, which contradicts (*).

Remark. As observed in (**2**, Remark 1 following 3.5), in fact $C_s(X)$ and $C_c(\upsilon X)$ have the same *dbc*-sets. This may be proved as above with an application of Theorem 1.

The next two results are interpretations in the present context of some general convergence theorems of (**12**) and (**13**). The first of these is well known.

Theorem 3. (i) *$f_n \to f$ in $C_\sigma(X)$ if and only if $\{f_n: n = 1, 2, \ldots\}$ is bounded in $C_\sigma(X)$ and $f_n(x) \to f(x)$ for each $x \in X$.*

(ii) *$g_n \to g$ in $C_\sigma(\upsilon X)$ if and only if $\{g_n: n = 1, 2, \ldots\}$ is bounded in $C_\sigma(\upsilon X)$ and $g_n(x) \to g(x)$ for each $x \in X$.*

Proof. (i) follows immediately from (**12**, Corollary 2). As in the proof of the Corollary to Theorem 1, if $g_n(x) \to g(x)$ for each $x \in X$, then $g_n \to g$ in $C_s(\upsilon X)$. (ii) then follows from (i).

A series Σx_n in a locally convex space is said to have property (O) if each subseries is weakly convergent (see (**14**)).

Theorem 4. *A series Σf_n has property (O) in $C_c(X)$ if and only if Σf_n^υ has property (O) in $C_c(\upsilon X)$. Further if Σf_n has property (O) in $C_c(X)$, Σf_n^υ converges unconditionally in $C_c(\upsilon X)$.*

Proof. The sufficiency of the condition in the first assertion is immediate.

Conversely, if Σf_n has property (O) in $C_c(X)$, then Σf_n^υ is certainly subseries convergent under $s_{\upsilon X}$. The necessity of the condition and the second assertion then follow immediately from (**13**, Corollary to Theorem 7), when the interpretation of the topology of pointwise convergence given at the beginning of the section is applied to $s_{\upsilon X}$.

Theorem 5. (i) *$C_\sigma(X)$ and $C_\sigma(\upsilon X)$ have the same absolutely convex countably compact sets.* (ii) *If $C_c(\upsilon X)$ is complete, $C_\sigma(X)$ and $C_\sigma(\upsilon X)$ have the same absolutely convex compact sets.*

In both cases, if $C_c(X)$ is sequentially complete, the hypothesis of absolute convexity may be omitted.

Proof. (i) Let A be any non-empty absolutely convex countably compact subset of $C_\sigma(X)$. Let (f_n) be a sequence in A and let $f_0 \in A$ be a cluster point of (f_n) in $C_\sigma(X)$. Given $y_1, y_2, \ldots, y_r \in \upsilon X$, there exist $x_1, x_2, \ldots, x_r \in X$ such that

$$f_n^\upsilon(y_s) = f_n(x_s) \quad (n = 0, 1, \ldots;\ s = 1, 2, \ldots, r).$$

Then, if $\varepsilon > 0$,

$$\{n : |f_n^\upsilon(y_s) - f_0^\upsilon(y_s)| < \varepsilon,\ s = 1, 2, \ldots, r\}$$
$$= \{n : |f_n(x_s) - f_0(x_s)| < \varepsilon,\ s = 1, 2, \ldots, r\},$$

from which it follows that f_0^υ is a cluster point of (f_n^υ) in $C_s(\upsilon X)$.

Let $\mu_1, \mu_2, \ldots, \mu_t \in C(\upsilon X)'$. By (**9**, Theorem 2.1) there is a subsequence $(f_{n(k)}^\upsilon)$ such that $f_{n(k)}^\upsilon(x) \to f_0^\upsilon(x)$ for all $x \in \bigcup_{r=1}^{t} \operatorname{supp} \mu_r$. Now A is a *dbc*-set in $C_c(X)$ since it is sequentially complete in $C_\sigma(X)$ (the proof of **10**, Chapter V, Lemma 2 is easily adapted to this case) and so by Theorem 2, $(f_{n(k)}^\upsilon)$ is a bounded sequence in $C_c(\upsilon X)$. Applying Theorem 3 (i) in $C_\sigma\left(\bigcup_{r=1}^{t} \operatorname{supp} \mu_r\right)$ then shows that

$$\mu_r(f_{n(k)}^\upsilon) \to \mu_r(f_0^\upsilon \quad \text{as} \quad k \to \infty \quad (r = 1, 2, \ldots, t),$$

from which it follows that for an infinite number of suffices n,

$$f_n^\upsilon - f_0^\upsilon \in \{g \in C(\upsilon X) : |\mu_r(g)| \leqq 1, \quad r = 1, 2, \ldots, t\}.$$

f_0^υ is therefore a cluster point of (f_n^υ) in $C_\sigma(\upsilon X)$ and A^υ is a countably compact subset of $C_\sigma(\upsilon X)$.

The converse is immediate since the mapping θ: $C_\sigma(\upsilon X) \to C_\sigma(X)$ defined by $\theta(g) = g|_X$ is continuous.

(ii) Let B be any absolutely convex compact subset of $C_\sigma(X)$. By (i) and Eberlein's theorem it follows that if $C_c(\upsilon X)$ is complete, then B^υ is relatively compact in $C_\sigma(\upsilon X)$. Since the mapping θ defined in (i) is a continuous bijection and since B is closed in $C_\sigma(X)$, $B^\upsilon = \theta^{-1}(B)$ is closed and therefore compact in $C_\sigma(\upsilon X)$.

The converse follows as in (i).

Finally, if $C_c(X)$ is sequentially complete, $C_c(X)$ and $C_c(\upsilon X)$ have the same bounded sets (**1**, 3.15). The assumption of absolute convexity may therefore be dropped in (i) and consequently also in (ii).

Remark. If $C_\sigma(X)$ and $C_\sigma(\upsilon X)$ have the same compact sets, one can show by an argument similar to that used in the proof of Theorem 2 that they must have the same bounded sets. However H. Buchwalter has shown that in general the families of bounded sets are not the same (**1**, Section 3).

If there exists a discrete space Z with measurable cardinal (**4**, Chapter 12), then $C_\sigma(Z)$ and $C_\sigma(\upsilon Z)$ do not have the same absolutely convex compact sets. Since $C_c(Z)$ is a product of copies of $\mathbb{R}$, $[C(Z)']_\tau$ is a direct sum of copies of $\mathbb{R}$ and so is complete. Since Z is not realcompact (**4**, 12.2), there is an element $y \in \upsilon Z \backslash Z$ and the mapping $f \to f^\upsilon(y)$ is a linear form on $C(Z)$ which must be discontinuous on some absolutely convex compact subset A of $C_\sigma(Z)$ (**10**, Chapter VI, Theorem 3). A^υ cannot then be compact in $C_\sigma(\upsilon Z)$ (see Theorem 7 below).

Theorem 6. *If there is a sequence (X_n) of bounded subsets of X such that $X = \text{cl} \bigcup_{n=1}^{\infty} X_n$, then $C_\sigma(X)$ and $C_\sigma(\upsilon X)$ have the same absolutely convex compact sets.*

Proof. Let A be an absolutely convex compact subset of $C_\sigma(X)$. By Theorem 5 (i), A^υ is countably compact in $C_\sigma(\upsilon X)$. Now each X_n is also a bounded subset of υX and so its closure in υX is compact (**4**, 8E). Thus by hypothesis υX has a σ-compact dense subset and so by (**9**, Theorem 2.5) A^υ is compact in $C_s(\upsilon X)$. It now follows from Theorem 1 that A^υ is compact in $C_\sigma(\upsilon X)$. As before, the converse is immediate.

While the problem of comparing the absolutely convex compact subsets of $C_\sigma(X)$ and $C_\sigma(\upsilon X)$ would seem to be of interest for its own sake, the next result gives it an added importance.

Theorem 7. *If $C_\sigma(X)$ and $C_\sigma(\upsilon X)$ have the same absolutely convex compact sets, then $[C(\upsilon X)']_\tau$ is the completion of $[C(X)']_\tau$.*

Proof. The mapping θ: $C_\sigma(\upsilon X) \to C_\sigma(X)$ defined by $\theta(g) = g\,|_X$ is bijective and continuous. Hence its transpose θ' maps $C(X)'$ injectively onto a vector subspace G' of $C(\upsilon X)'$ which is dense in $[C(\upsilon X)']_\tau$. By hypothesis, if G' is provided with the topology induced by $\tau(C(\upsilon X)', C(\upsilon X))$, θ' defines a topological isomorphism of $[C(X)']_\tau$ onto G'.

The result will follow if it is shown that $[C(\upsilon X)']_\tau$ is complete. M. de Wilde and J. Schmets have shown that $C_c(\upsilon X)$ is ultrabornological (**3**, Théorème). Consequently $C(\upsilon X)'$ may be represented as a closed vector subspace of a product $\prod_{\gamma \in \Gamma} E'_\gamma$, where each E'_γ is the dual space of a Banach space E_γ (**10**, Chapter V, Propositions 3, 25, 28). Since a product of complete spaces is complete and each E'_γ is complete under $\tau(E'_\gamma, E_\gamma)$ (**10**, Chapter VI, Corollary 2 of Proposition 1), $C(\upsilon X)'$ must be complete in the topology induced by the product of these Mackey topologies. This is a topology of the dual pair $(C(\upsilon X)', C(\upsilon X))$, so that $[C(\upsilon X)']_\tau$ is also complete (**10**, Chapter VI, Proposition 5).

Now suppose that $C_c(\upsilon X)$ is sequentially complete. For each series Σf_n in $C_c(X)$ having property (O), let $A(\Sigma f_n)$ denote the closed absolutely convex envelope of $\left\{\sum_{r=1}^{n} f_r:\quad n=1, 2, \ldots\right\}$ in $C_\sigma(X)$. By Theorem 4, $C_c(X)$ and $C_c(\upsilon X)$ have the same series with property (O). It follows from (**14**, Lemma 1), as in the corollary to Theorem 1, that for each such series Σf_n, the closed absolutely convex envelope of $\left\{\sum_{r=1}^{n} f_r^\upsilon:\quad n=1, 2, \ldots\right\}$ in $C_\sigma(\upsilon X)$ is compact, and so $A(\Sigma f_n)$ is compact in $C_\sigma(X)$. The topology ξ on $C(X)'$ of uniform convergence on all sets of the form $A(\Sigma f_n)$ is then a topology of the dual pair $(C(X)', C(X))$.

The celebrated results of L. Nachbin (**8**) and T. Shirota (**11**) show that $C_c(\upsilon X)$ is bornological, so that by hypothesis and the remark following Proposition 2 of (**14**), $C(\upsilon X)'$ is complete in the topology η of uniform convergence on the sets $(A(\Sigma f_n))^\upsilon$.

An argument similar to that in the first paragraph of the proof of Theorem 7 then establishes the following.

Theorem 8. *In the above notation, if $C_c(\upsilon X)$ is sequentially complete, the ξ-completion of $C(X)'$ is $C(\upsilon X)'$ with the topology η.*

Remark. It may seem more natural so assume in Theorem 5 (ii) that $C_c(\upsilon X)$ is quasicomplete and in Theorem 8 that $C_c(X)$ is sequentially complete. However, S. Warner has shown that quasicompleteness and completeness are equivalent for such function spaces (**15**, Theorem 1) and as pointed out by H. Buchwalter, if $C_c(X)$ is sequentially complete so also is $C_c(\upsilon X)$ (**1**, following proof of 4.4).

Some results related to the material of this section are mentioned in (**6**).

4. σ-barrelled and barrelled spaces

A separated locally convex space is said to be *σ-barrelled* if each weak* bounded sequence in its dual space forms an equicontinuous set. In (**2**, Théorème 4.1), H. Buchwalter and J. Schmets consider conditions equivalent to σ-barrelledness for $C_c(X)$; in particular they show that $C_c(X)$ is σ-barrelled if and only if $[C(X)']_\tau$ is sequentially complete. Theorem 10 is the analogue of this result for the barrelled case. Theorem 9 contains some further conditions equivalent to σ-barrelledness.

A *real* separated locally convex space E with dual E' is said to have property (J) if for each $x \in E$ and each non-empty $\sigma(E', E)$-closed bounded set B there exists $x' \in B$ such that

$$\langle x, x'\rangle = \sup\{\langle x, y'\rangle:\quad y' \in B\}.$$

The following characterisation of property (J) may be well known. A proof is included for completeness.

Lemma. *E has property (J) if and only if each $\sigma(E', E)$-bounded sequence has a $\sigma(E', E)$-cluster point.*

Proof. Let B be any non-empty $\sigma(E', E)$-closed bounded set and let $x \in E$. Choose a sequence (x'_n) in B such that

$$\langle x, x'_n \rangle \to \sup \{\langle x, y' \rangle : \quad y' \in B\}.$$

If (x'_n) has a $\sigma(E', E)$-cluster point x', then $x' \in B$ and

$$\langle x, x' \rangle = \sup \{\langle x, y' \rangle : \quad y' \in B\}.$$

This establishes the sufficiency of the condition.

Now suppose that (x'_n) is a $\sigma(E', E)$-bounded sequence in E' which has no $\sigma(E', E)$-cluster point. Let x' be a $\sigma(E^*, E)$-cluster point of (x'_n). (E^* denotes the algebraic dual of E.) Since $x' \neq 0$, there exists $x \in E$ such that $\langle x, x' \rangle > 0$. Choose a subsequence (y'_n) of (x'_n) such that $\langle x, y'_n \rangle \to \langle x, x' \rangle$ and construct a subset B of E' as follows:

(*a*) if there is a subsequence (z'_n) of (y'_n) such that

$$\langle x, z'_n \rangle < \langle x, x' \rangle \quad (n = 1, 2, \ldots),$$

put $B = \{z'_n : \; n = 1, 2, \ldots\}$;

(*b*) if not and there is a subsequence (w'_n) of (y'_n) such that

$$\langle x, w'_n \rangle > \langle x, x' \rangle \quad (n = 1, 2, \ldots),$$

put $B = \{-w'_n : \; n = 1, 2, \ldots\}$;

(*c*) if neither of these conditions is satisfied, there is a subsequence (u'_n) of (y'_n) such that

$$\langle x, u'_n \rangle = \langle x, x' \rangle \quad (n = 1, 2, \ldots).$$

Put $B = \{(1 - 1/n)u'_n : \; n = 1, 2, \ldots\}$.

In each case B is $\sigma(E', E)$-closed and bounded and x does not attain its supremum on B. This establishes the necessity of the condition.

A separated locally convex space is said to be *sequentially barrelled* if each weak* null sequence forms an equicontinuous set (**16**, Section 4). Although there are sequentially barrelled spaces which are not σ-barrelled, the two ideas coincide for $C_c(X)$.

Theorem 9. *The following are equivalent:*

(i) *$C_c(X)$ is σ-barrelled;*

(ii) *$C_c(X)$ is sequentially barrelled;*

(iii) *$C_c(X)$ has property (J);*

(iv) *each absolutely convex bounded metrisable subset of $[C(X)']_\sigma$ is equicontinuous.*

Proof. It is clear that (i)⇒(ii). The proof that (ii)⇒(i) is a slight modification of the proof of (**2**, Théorème 4.1, (*e*)⇒(*a*)). Let (μ_n) be a bounded sequence in $[C(X)']_\sigma$. Then $(1/n)\mu_n \to 0$ in $[C(X)']_\sigma$. Thus if $C_c(X)$ is sequentially barrelled,

$$\text{supp}\{\mu_n : \; n = 1, 2, \ldots\} = \text{supp}\{(1/n)\mu_n : \; n = 1, 2, \ldots\}$$

is compact, and the rest follows as in (**2**).

(i)$\Rightarrow$(iii). This follows immediately from the Lemma.

(iii)$\Rightarrow$(i). If $C_c(X)$ has property (J), $[C(X)']_\sigma$ must be sequentially complete by the Lemma. The result now follows by (**2**, Théorème 4.1, (b)$\Leftrightarrow$(a)).

(i)$\Rightarrow$(iv). Let B be any absolutely convex bounded metrisable subset of $[C(X)']_\sigma$. By (**5**, Proposition 1.3 and Theorem 1.4), the closure $\bar{B}$ of B in $[C(X)']_\sigma$ is also metrisable, and by hypothesis it is countably compact. $\bar{B}$ is therefore compact in $[C(X)']_\sigma$, and since a compact metric space is separable, it follows that $\bar{B}$, and therefore also B, is equicontinuous.

(iv)$\Rightarrow$(i). Let (μ_n) be any Cauchy sequence in $[C(X)']_\sigma$. By (**5**, Corollary to Theorem 1.4), the closed absolutely convex envelope of $\{\mu_n: n = 1, 2, \ldots\}$ in $[C(X)']_\sigma$ is metrisable and therefore equicontinuous. It now follows that $[C(X)']_\sigma$ is sequentially complete and the result follows as in (iii)$\Rightarrow$(i).

Theorem 10. *$C_c(X)$ is barrelled if and only if $[C(X)']_\tau$ is quasicomplete.*

Proof. If $C_c(X)$ is barrelled, $[C(X)']_\sigma$ is quasicomplete. This implies that $[C(X)']_\tau$ is also quasicomplete.

To establish the reverse implication, it is enough to show that each non-empty closed bounded subset of X is compact (**8**, Theorem 1; **11**, Theorem 1). Let Y be such a subset and let $\mathscr{F}$ be an ultrafilter in Y. Then as in the proof of (**11**, Theorem 1):

(i) $\mathscr{F}$ converges to an element x of υX;

(ii) $\mathscr{G} = \{\{\delta_x: x \in G\}: G \in \mathscr{F}\}$ is an ultrafilter in $B = \{\delta_x: x \in Y\}$ and for each $f \in C(X)$, $\lim \langle f, \mathscr{G}\rangle = f^\upsilon(x)$.

Since $[C(X)']_\tau$ is quasicomplete it is also sequentially complete, and so $C_c(X)$ is σ-barrelled (**2**, Théorème 4.1). B is bounded in $[C(X)']_\sigma$ and therefore each sequence in B has a cluster point in $[C(X)']_\sigma$. Since $[C(X)']_\tau$ is quasi-complete, it follows from Eberlein's theorem that B is relatively compact in $[C(X)']_\sigma$. The linear form $f \to f^\upsilon(x)$ must therefore be an element of $C(X)'$. Consequently $x \in X$. Since Y is closed in X, $x \in Y$ and Y is therefore compact.

Added in Proof: After submission of this note, papers (17) and (18) appeared. Corollary 7.4 of (17) contains a generalisation of Theorem 1 while Theorem 10.6 of (18) is a version of this result. In both cases the methods used are quite different from these employed here.

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