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A NOTE ON THE PERIODICITY OF ENTIRE FUNCTIONS

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Abstract

We give some sufficient conditions for the periodicity of entire functions based on a conjecture of C. C. Yang, using the concepts of value sharing, unique polynomial of entire functions and Picard exceptional value.

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1. Introduction and main results

We assume that the reader is familiar with elementary Nevanlinna theory [3, 12]. Given a meromorphic function f, the family of all meromorphic functions w safisfying $T(r, w) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, is denoted by $S(f)$. For convenience, we also include all constant functions in $S(f)$.

It is well known that any periodic entire function $f(z)$ with period c can be written as an everywhere convergent series

$$f(z) = \sum_{-\infty}^{+\infty} a_n e^{2i\pi nz/c}.$$

Baker [1] proved that if $f(z)$ is a nonconstant entire function and $p(z)$ is a polynomial with $\deg(p(z)) \geq 3$, then $f(p)$ is not a periodic function. Recently, it has been shown that value sharing can provide sufficient conditions for the periodicity of meromorphic functions. Heittokangas *et al.* [4, Theorem 2] showed that if a finite order transcendental meromorphic function $f(z)$ shares a_1, a_2 CM and a_3 IM with $f(z + c)$, then $f(z)$ is a periodic function, where $a_1, a_2, a_3 \in S(f) \cup \infty$. Obviously, if $f(z)$ is a transcendental entire function with finite order and $f(z)$ and $f(z + c)$ share a_1 CM and a_2 IM, then $f(z)$ is a periodic function, where $a_1, a_2 \in S(f)$. Very recently,

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related to the periodicity of entire functions, Wang and Hu [8] mentioned the following conjecture given by C. C. Yang.

Yang's Conjecture. Let f be a transcendental entire function and k be a positive integer. If $ff^{(k)}$ is a periodic function, then f is also a periodic function.

We give the following result related to Yang's conjecture.

Theorem 1.1. *Let f be a transcendental entire function with a nonzero Picard exceptional value and let k be a positive integer. If $ff^{(k)}$ is a periodic function, then f is also a periodic function.*

Wang and Hu [8] also considered Yang's conjecture and obtained the following result.

Theorem A [8, Theorem 1]. *Let f be a transcendental entire function and k a positive integer. If $(f^2)^{(k)}$ is a periodic function, then f is also a periodic function.*

Remark 1.2. (i) Theorem A shows that Yang's conjecture is true when $k = 1$. However, f^2 cannot be replaced by f in Theorem A. For example $f(z) = e^z + z$ is not a periodic function, but $f^{(k)}$ is a periodic function.

(ii) The function f^2 can be replaced by f^n in Theorem A provided that $n \geq 3$. In fact, if $(f^n)^{(k)}$ is a periodic function with period c, then

$$f(z+c)^n - f(z)^n = p(z), \tag{1.1}$$

where $p(z)$ is a polynomial with $\deg(p(z)) \leq k-1$. Equation (1.1) has no nonconstant entire solutions if $p(z) \not\equiv 0$, which is a direct corollary of Yang's result [9, Theorem 1] related to Fermat functional equations: if m, n are positive integers satisfying $m^{-1} + n^{-1} < 1$, then there are no nonconstant entire solutions $f(z)$ and $g(z)$ that satisfy $a(z)f(z)^n + b(z)g(z)^m = 1$, where $a(z), b(z) \in S(f)$. Using Yang's result, we get $p(z) \equiv 0$. Hence $f(z+c) = tf(z)$ and $t^n = 1$, thus $f(z)$ is a periodic function with period nc.

We summarise Theorem A and Remark 1.2(ii) in the following corollary.

Corollary 1.3. *Let f be a transcendental entire function, k and n positive integers and $n \geq 2$. If $(f^n)^{(k)}$ is a periodic function, then f is also a periodic function.*

From Theorem A and Remark 1.2(i), we raise the following question.

Question 1.4. Let f be a transcendental entire function and k a positive integer. Let $a_1, a_2, \ldots, a_n$ be constants and $a_n \neq 0$. If $(a_nf^n + \cdots + a_1f)^{(k)}$ $(n \geq 2)$ is a periodic function, is it true that f is also a periodic function?

The following result gives a partial answer to this question.

Theorem 1.5. *Let f be a transcendental entire function and k a positive integer. Let a_1 and a_2 be constants and $a_2 \neq 0$. If $(a_2f^2 + a_1f)^{(k)}$ is a periodic function, then f is also a periodic function.*

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REMARK 1.6. Suppose we replace $a_2 f^2 + a_1 f$ by $a_n f^n + \cdots + a_1 f$ in Theorem 1.5, where $a_n \neq 0$ and at least one $a_i \neq 0$ $(i = 1, \ldots, n-1)$. We can attempt a similar approach to that in the proof of Theorem 1.5 below, but it seems difficult to obtain expressions for $f(z)$ and $f(z+c)$.

THEOREM 1.7. *Let k and n be positive integers with $n \geq 2$. Let f be a transcendental entire function with $\rho_2(f) < 1$ and $N(r, 1/f) = S(r, f)$. If $(a_n f^n + \cdots + a_1 f)^{(k)}$ is a periodic function and $a_n \neq 0$, then f is also a periodic function.*

The unique polynomial of entire functions (UPE) can also be applied to the periodicity of entire functions. We first recall the definition of UPE [11]. A polynomial $P(z)$ is called UPE if, whenever $P(f) = P(g)$ for two nonconstant entire functions f and g, then $f = g$. Li and Yang [6] gave the following results for meromorphic functions. From the proofs in [6], if f and g are entire functions, then the condition $n > 2m + 2$ $(m \geq 2)$ is enough to give the conclusion.

THEOREM B [6, Theorem 1]. *Let a_i, b_i $(i = 1, 2)$ and c be nonzero meromorphic functions. Suppose m and n with $n > 2m + 2$ $(m \geq 2)$ are relatively prime to each other. Then the functional equation*

$$f^n + a_1 f^{n-m} + b_1 = c(g^n + a_2 g^{n-m} + b_2)$$

has a pair of admissible entire solutions (f, g), if and only if $c = b_1/b_2$ and $f = hg$, where h is a meromoprhic function satisfying $h^n = c$ and $h^m = a_1/a_2$.

THEOREM C [6, Corollary 1]. *Given integers m and n with $n > 2m + 2$ $(m \geq 2)$, relatively prime to each other, and rational functions a_1, a_2, a_3, $a_4 \neq 0$, the functional equation*

$$f^n + a_1 f^{n-m} + a_2 g^n + a_3 g^{n-m} + a_4 = 0$$

has no transcendental entire solutions f and g.

Combining Theorem B with Theorem C gives the following corollary.

COROLLARY 1.8. *Let f be a transcendental entire function, k a positive integer, m and n relatively prime integers with $n > 2m + 2$ $(m \geq 2)$ and let a_n, a_1 be nonzero rational functions. If $(a_n f^n + a_1 f^{n-m})^{(k)}$ is a periodic function, then f is also a periodic function.*

In addition, Wang and Hu [8, Theorem 2] used the method in [10] to obtain the following theorem as a partial result towards Yang's conjecture.

THEOREM D. *Let n, m, k, p, q be positive integers and f be a transcendental entire function with finite order. If $f^n(f^{(k)})^m$ is a periodic function with period c, f and $f^{(k)}$ have zeros with multiplicities p, q and the multiplicity of the zeros of $f^n(f^{(k)})^m$ is more than np, then f is a periodic function with period $(m + n)c$.*

REMARK 1.9. The condition that the multiplicity of the zeros of $f^n(f^{(k)})^m$ is more than np in Theorem D means that f and $f^{(k)}$ have common zeros. We give some further observations on Theorem D. Let $f(z)$ be a transcendental entire function with finite order.

(1) Assume that f and $f^{(k)}$ have no zeros. Then $f^{(k)} = Af$, where A is a nonzero constant. Thus

$$f(z)^n(f^{(k)}(z))^m = f(z+c)^n(f^{(k)}(z+c))^m.$$

We have

$$f(z)^{n+m} = f(z+c)^{n+m},$$

that is $f(z+c) = tf(z)$ and $t^{n+m} = 1$. So $f(z)$ is a periodic function with period $(n+m)c$.

(2) Assume that $f(z)$ has no zeros but $f^{(k)}(z)$ has zeros. Thus $f(z) = e^{p(z)}$, where $p(z)$ is a nonconstant polynomial, so $f^{(k)}(z) = H(z, p(z))e^{p(z)}$, where $H(z, p(z))$ is a differential polynomial in $p(z)$ and $m(r, H) = S(r, f)$. Thus,

$$H(z, p(z))^m e^{(m+n)p(z)} = H(z, p(z+c))^m e^{(m+n)p(z+c)},$$

and hence $p(z)$ must be a linear polynomial, that is $p(z) = Az + B$, where $A \neq 0, B$ are constants. Hence $f(z) = e^{Az+B}$ and so $f(z)$ is a periodic function with period $2k\pi i/A$.

(3) Assume that $f(z)$ has zeros but $f^{(k)}(z)$ has no zeros. In this case, $f(z)$ and $f(z+c)$ share 0 CM. Using the result given by Li and Gao [7, Theorem 1.6], if $f(z)$ is not a periodic function, then $\rho(f) > 1$ and $f(z)$ satisfies one of the following three cases:

(a) $\rho(f) \leq \lambda(f) + 1$ and $f(z) = P(z)e^{Q(z)}$, where $P(z)$ is an entire function such that $\rho(P) < \lambda(f)$ and $Q(z)$ is a polynomial with $\deg(Q) \leq \rho(P) + 1$.

(b) $\rho(f) > \lambda(f) + 1$ and $f(z) = P(z)e^{Q(z)}$, where $P(z)$ is an entire function such that $\rho(P) < \lambda(f)$ and $Q(z)$ is a polynomial with $\deg(Q) = \rho(P) \geq 3$.

(c) $\rho(f) > \lambda(f) + 1$ and $f(z) = e^{Q(z)}$, where $Q(z)$ is a polynomial with $\deg(Q) = 2$.

(4) Assume that $f(z)$ and $f^{(k)}$ have zeros. It is not easy to consider this case from the point of view of value sharing.

Obviously, case (c) in (3) does not occur for the case that $f(z)$ has zeros. However, cases (a) and (b) in (3) and case (4) remain open.

2. Proofs of the theorems

To prove Theorems 1.1 and 1.5, we need the following lemma.

LEMMA 2.1 [12, Theorem 1.56]. *Let $f_j(z)$ $(j = 1, 2, 3)$ be meromorphic functions and $f_1(z)$ nonconstant. If*

$$\sum_{j=1}^{3} f_j = 1$$

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and

$$\sum_{j=1}^{3} N\Big(r, \frac{1}{f_j}\Big) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r),$$

where $\lambda < 1$ *and* $T(r) = \max_{j=1,2,3}\{T(r, f_j)\}$, *then* $f_2(z) \equiv 1$ *or* $f_3(z) \equiv 1$.

Proof of Theorem 1.1. Assume that d is the nonzero Picard exceptional value. Then $f(z) = e^{p(z)} + d$, where $p(z)$ is an entire function and $T(r, p(z)) = S(r, f)$. Since $ff^{(k)}$ is a periodic function with period c, we assume that

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c).$$

Hence,

$$H(z, p(z))e^{2p(z)} + dH(z, p(z))e^{p(z)} = H(z, p(z+c))e^{2p(z+c)} + dH(z, p(z+c))e^{p(z+c)},$$

where $H(z, p(z))$ is a differential polynomial in $p(z)$. Thus,

$$\frac{H(z, p(z))}{dH(z, p(z+c))}e^{2p(z)-p(z+c)} + \frac{H(z, p(z))}{H(z, p(z+c))}e^{p(z)-p(z+c)} - \frac{1}{d}e^{p(z+c)} = 1.$$

Obviously $-e^{p(z+c)}/d$ is not a constant. From Lemma 2.1, we have two cases.

Case 1: The first possibility is

$$\frac{H(z, p(z))}{dH(z, p(z+c))}e^{2p(z)-p(z+c)} = 1 \quad \text{and} \quad \frac{H(z, p(z))}{H(z, p(z+c))}e^{p(z)-p(z+c)} - \frac{1}{d}e^{p(z+c)} = 0.$$

From this, $e^{p(z)+p(z+c)} = d^2$, therefore, $p(z) = B - p(z+c)$ where B is a constant. Thus, $(H(z, p(z))/dH(z, p(z+c)))e^{3p(z)-B} = 1$, so $T(r, e^{p(z)}) = S(r, e^{p(z)})$, which is impossible.

Case 2: The second possibility is

$$\frac{H(z, p(z))}{H(z, p(z+c))}e^{p(z)-p(z+c)} = 1 \quad \text{and} \quad \frac{H(z, p(z))}{dH(z, p(z+c))}e^{2p(z)-p(z+c)} - \frac{1}{d}e^{p(z+c)} = 0.$$

This gives $e^{p(z)} = e^{p(z+c)}$ and so $f(z) = f(z+c)$. □

Proof of Theorem 1.5. Assume that $(a_2 f^2 + a_1 f)^{(k)}$ is a periodic function with period c. Thus,

$$(a_2 f(z+c)^2 + a_1 f(z+c))^{(k)} = (a_2 f(z)^2 + a_1 f(z))^{(k)}.$$

Furthermore,

$$a_2 f(z+c)^2 + a_1 f(z+c) = a_2 f(z)^2 + a_1 f(z) + P(z),$$

where $P(z)$ is a polynomial with $\deg(P(z)) \le k-1$. Thus,

$$[f(z+c) - f(z)][a_2 f(z+c) + a_2 f(z) + a_1] = P(z). \tag{2.1}$$

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Case 1: $P(z) \equiv 0$. Either $f(z+c) = f(z)$ which implies that $f(z)$ is a periodic function with period c, or $a_2 f(z+c) + a_2 f(z) + a_1 = 0$, which implies that $f(z)$ is a periodic function with period $2c$.

Case 2: $P(z) \not\equiv 0$. Since $f(z)$ is an entire function, by (2.1),

$$\begin{cases} f(z+c) - f(z) = P_1(z)e^{Q(z)}, \\ a_2 f(z+c) + a_2 f(z) + a_1 = P_2(z)e^{-Q(z)}, \end{cases}$$

where $P_1(z)P_2(z) = P(z)$ and $Q(z)$ is an entire function. A computation gives

$$f(z+c) = \frac{P_2(z)e^{-Q(z)} - a_1 + a_2 P_1(z)e^{Q(z)}}{2a_2}, \quad f(z) = \frac{P_2(z)e^{-Q(z)} - a_1 - a_2 P_1(z)e^{Q(z)}}{2a_2}.$$

Hence,

$$\frac{P_2(z+c)}{P_2(z)}e^{Q(z)-Q(z+c)} - \frac{a_2 P_1(z+c)}{P_2(z)}e^{Q(z+c)+Q(z)} - \frac{a_2 P_1(z)}{P_2(z)}e^{2Q(z)} = 1.$$

Obviously, $-(a_2 P_1(z)/P_2(z))e^{2Q(z)} \not\equiv 1$, since $f(z)$ is a transcendental entire function. From Lemma 2.1, we have to consider two cases.

Case 1: The first possibility is

$$\frac{P_2(z+c)}{P_2(z)}e^{Q(z)-Q(z+c)} \equiv 1 \quad \text{and} \quad -\frac{a_2 P_1(z+c)}{P_2(z)}e^{Q(z+c)+Q(z)} - \frac{P_1(z)}{P_2(z)}e^{2Q(z)} \equiv 0.$$

In this case, $P_1(z+c)P_2(z+c) = -P_1(z)P_2(z)$, which implies that $P(z+c) = -P(z)$, a contradiction since $P(z)$ is a polynomial.

Case 2: The second possibility is

$$-\frac{a_2 p_1(z+c)}{p_2(z)}e^{Q(z+c)+Q(z)} = 1 \quad \text{and} \quad \frac{p_2(z+c)}{p_2(z)}e^{Q(z)-Q(z+c)} - \frac{p_1(z)}{p_2(z)}e^{2Q(z)} = 0.$$

Here, we also get $P(z+c) = -P(z)$, a contradiction. □

Proof of Theorem 1.7. If $(a_n f^n + \cdots + a_1 f)^{(k)}$ is a periodic function with period c, then

$$a_n f(z+c)^n + \cdots + a_1 f(z+c) = a_n f(z)^n + \cdots + a_1 f + P(z), \tag{2.2}$$

where $P(z)$ is a polynomial with $\deg(P(z)) \le k-1$.

Let $f(z+c) = f(z)h(z)$, where $h(z)$ is a meromorphic function. From the difference analogue of the logarithmic derivative lemma for meromorphic functions with hyper-order less than one [2], $m(r,h) = S(r,f)$. Since $N(r,1/f) = S(r,f)$, it follows that $N(r,h) = S(r,f)$ and $T(r,h) = S(r,f)$. From (2.2),

$$a_n[h(z)^n - 1]f(z)^n + a_{n-1}[h(z)^{n-1} - 1]f(z)^{n-1} + \cdots + a_1[h(z) - 1]f(z) = P(z).$$

We claim that $h(z)$ is a constant and $h^n = 1$. Otherwise, from

$$m\Big(r, \frac{h^j - 1}{h^n - 1}\Big) = S(r,f)$$

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follows $T(r,h) = S(r,f)$ for $j = 1, 2, \ldots, n-1$. By the Clunie lemma [5, Lemma 2.4.2], $m(r,f) = S(r,f)$, which is a contradiction. Thus, $h^n = 1$. Assume the next nonzero coefficient is $a_k \neq 0$. Then $h^k = 1$ and $h^{n-k} = 1$. From this, $f(z+c) = h^{n-k}f(z)$, that is, $f(z)$ is a periodic function with period $(n-k)c$. This proves Theorem 1.7. □

Proof of Corollary 1.8. If $(a_n f^n + a_1 f^{n-m})^{(k)}$ is a periodic function with period c, then

$$a_n f^n + a_1 f^{n-m} = a_n f(z+c)^n + a_1 f(z+c)^{n-m} + P(z),$$

where $P(z)$ is a polynomial with $\deg(P(z)) \leq k-1$.

If $P(z) \neq 0$, then Theorem C implies that there is no entire solution f that satisfies this equation. If $P(z) \equiv 0$, then Theorem B implies that $f(z) = hf(z+c)$, where $h^m = 1$. Thus, $f(z)$ is a periodic function with period mc. □

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