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Nagoya Math. J.
Vol. 132 (1993), 175–197

# NEW INVARIANTS AND CLASS NUMBER PROBLEM IN REAL QUADRATIC FIELDS

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In recent papers [10, 11, 12, 13, 14], we defined some new $p$-invariants for any rational prime $p$ congruent to $1 \bmod 4$ and $D$-invariants for any positive square-free integer $D$ such that the fundamental unit $\varepsilon_D$ of real quadratic field $\mathbf{Q}(\sqrt{D})$ satisfies $N\varepsilon_D = -1$, and studied relationships among these new invariants and already known invariants.

One of our main purposes in this paper is to generalize these $D$-invariants to invariants valid for all square-free positive integers containing $D$ with $N\varepsilon_D = 1$. Another is to provide an improvement of the theorem in [14] related closely to class number one problem of real quadratic fields. Namely, we provide, in a sense, a most appreciable estimation of the fundamental unit to be able to apply, as usual (cf. [3, 4, 5, 9, 12, 13]), Tatuzawa's lower bound of $L(1, \chi_D)$ (cf. [7]) for estimating the class number of $\mathbf{Q}(\sqrt{D})$ from below by using Dirichlet's classical class number formula.

In §1, we shall define and consider properties of new $D$-invariant $m_D$ valid for all square-free positive integers $D$, and in §2 we shall deal with new $D$-invariant $n_D$, which is more valuable when $m_D = 0$. At that time, we shall partly alter notations of some new $D$-invariants related to $n_D$ defined in [13] for avoiding any confusion of notations.

In §3, we shall consider real quadratic fields of R-D type, for which fields explicit forms of the fundamental units are well-known, and as an application of results in §1-2, we shall characterize each case of R-D type by using $D$-invariants studied there.

Finally, in §3 we shall provide an improvement of the theorem in [14] and Theorem 2 in [12].

Throughout this paper, we denote by $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ the set of all non-negative rational integers and by $[x]$ the greatest integer less than or equal to $x$. Moreover, we set

Received January 26, 1993.

$$\mathbf{D} = \{D > 0 : \text{square-free positive integer}\},$$

and for any $D$ in $\mathbf{D}$, we denote by $h_D$ and by

$$\varepsilon_D = (t_D + u_D\sqrt{D})/2 \quad (> 1)$$

the class number and the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ respectively.

Furthermore, we set

$$\mathbf{D}_+ = \{D \in \mathbf{D} : N\varepsilon_D = 1\},$$

and

$$\mathbf{D}_- = \{D \in \mathbf{D} : N\varepsilon_D = -1\},$$

where $N$ means the norm mapping from $\mathbf{Q}(\sqrt{D})$ to the rational number field $\mathbf{Q}$.

## §1

We first prove the following:

THEOREM 1.1. *For any $D$ in $\mathbf{D}$ greater than 13 (or $t_D \geqq 5$), it holds*

$$[u_D^2/t_D] = [t_D/D] = [\varepsilon_D/D] = [u_D/\sqrt{D}].$$

*Proof.* First we show that $D > 13$ implies $t_D \geqq 5$. It follows from $t_D^2 - Du_D^2 = \pm 4$ that $D \leqq Du_D^2 = t_D^2 \pm 4$. Hence,

$$D \geqq 29 \text{ implies } t_D^2 \pm 4 \geqq 29, \text{ and so } t_D \geqq 5.$$

On the other hand, for any $D$ satisfying $13 < D < 29$ we can confirm $t_D \geqq 5$ by practical calculations.

Next, we put $m_D = [u_D^2/t_D]$. Then, for any $D$ in $\mathbf{D}_+$ satisfying $t_D \geqq 5$ we can prove the following inequality:

$$Dm_D \leqq t_D - 1 < u_D\sqrt{D} < \varepsilon_D < t_D < D(m_D + 1), \tag{1}$$

and for any $D$ in $\mathbf{D}_-$ satisfying $t_D \geqq 5$ we can prove the following inequality:

$$Dm_D < t_D < \varepsilon_D < u_D\sqrt{D} < t_D + 1 \leqq D(m_D + 1). \tag{2}$$

In the case $D$ in $\mathbf{D}_+$ from $u_D^2 D = t_D^2 - 4$ and $t_D \geqq 5$ we know

$$t_D > u_D^2 D/t_D = t_D - 4/t_D > t_D - 1.$$

https://doi.org/10.1017/S0027763000004700 Published online by Cambridge University Press

Moreover we get

$$Dm_D \leqq u_D^2 D/t_D < D(m_D + 1).$$

Hence we get

$$Dm_D \leqq t_D - 1 \quad \text{and} \quad t_D \leqq D(m_D + 1).$$

Furthermore, we know

$$u_D\sqrt{D} < \varepsilon_D < t_D$$

from Lemma 2 in [9], and

$$t_D - 1 < \sqrt{t_D^2 - 4} = u_D\sqrt{D}$$

from $t_D \geqq 5$.

In the case $D$ in $\mathbf{D}_-$ from $u_D^2 D = t_D^2 + 4$ we know

$$t_D < u_D^2 D/t_D = t_D + 4/t_D < t_D + 1.$$

Hence we get similarly

$$Dm_D \leqq t_D \quad \text{and} \quad t_D + 1 \leqq D(m_D + 1).$$

Moreover, we know

$$t_D < \varepsilon_D < u_D\sqrt{D}$$

from Lemma 2 in [9], and

$$u_D\sqrt{D} = \sqrt{t_D^2 + 4} < t_D + 1$$

from $t_D \geqq 5$.

Here, if we assume $t_D = Dm_D$ or $D(m_D + 1)$, then $t_D \equiv 0 \pmod{D}$. Hence, it follows

$$\pm 4 = t_D^2 - Du_D^2 \equiv 0 \pmod{D},$$

which conflicts with $D > 13$.

Our theorem is immediately follows from (1) and (2).

For any $D$ in $\mathbf{D}_+$, we put

$$A_D = \{a : 0 < a < D,\ a^2 \equiv 4 \pmod{D}\},$$

and

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$$(A, B)_D = \{(a, b) : a \in A_D,\ a^2 - 4 = bD\}.$$

For any $D$ in $\mathbf{D}_-$, we put

$$A_D = \{a : 0 \leqq a < D,\ a^2 \equiv -4 \pmod{D}\},$$

and

$$(A, B)_D = \{(a, b) : a \in A_D,\ a^2 + 4 = bD\}.$$

Then we can prove the following:

THEOREM 1.2. *For any $D$ in $\mathbf{D}$, there are uniquely determined $m_D$ in $\mathbf{N}_0$ and $(a_D, b_D)$ in $(A, B)_D$ such that*

$$\begin{cases} t_D = D \cdot m_D + a_D, \\ u_D^2 = D \cdot m_D^2 + 2a_D \cdot m_D + b_D. \end{cases}$$

*Additionally, if $D > 5$, then*

$$m_D = [t_D/D] \quad \textit{and} \quad 0 \leqq b_D < a_D < D,$$

*and moreover*

$$b_D = 0 \quad \textit{if and only if} \quad a_D = 2.$$

*Proof.* In the case $D$ in $\mathbf{D}_+$, we put

$$[t_D/D] = m_D \quad \text{and} \quad t_D = D \cdot m_D + a_D.$$

Then, $m_D$ and $a_D$ are uniquely determined, and we know

$$m_D \in \mathbf{N}_0, \quad 0 \leqq a_D < D.$$

Moreover, since

$$Du_D^2 = t_D^2 - 4 = D(Dm_D^2 + 2a_D m_D) + (a_D^2 - 4),$$

we obtain

$$a_D^2 - 4 \equiv 0 \pmod{D}.$$

Therefore, we put

$$a_D^2 - 4 = D \cdot b_D.$$

Then, $b_D$ is also uniquely determined and we obtain

$$u_D^2 = D \cdot m_D^2 + 2a_D \cdot m_D + b_D.$$

Here, if we assume $a_D = 0$, then we get $D = 2$.
However, from $\varepsilon_2 = 1 + \sqrt{2}$ we get $N\varepsilon_2 = -1$, which contradicts with $D$ in $\mathbf{D}_+$.
Hence, $a_D \neq 0$ holds, and so $(a_D, b_D)$ is in $(A, B)_D$.

Next, it is clear that $b_D = 0$ if and only if $a_D = 2$.
Furthermore,

$$b_D < 0 \quad \text{if and only if} \quad a_D = 1,$$

which is equivalent to $D = 3 \quad (b_D = -1)$.
Hence, $D > 3$ implies $b_D \geqq 0$.

If we assume $b_D \geqq D$, then from $a_D^2 - 4 = Db_D \geqq D^2$ we get at once

$$-4 \geqq D^2 - a_D^2 = (D - a_D)(D + a_D) > D,$$

which is a contradiction. Hence, we get $b_D < D$.

Finally, we put $f(x) = -x^2 + Dx + 4$.
Then

$$f(0) = f(D) = 4 > 0,$$

and

$$0 < f(b_D) = (a_D + b_D)(a_D - b_D)$$

holds.
Hence, we know $a_D > b_D$ from $a_D + b_D > 0$.

In the case $D$ in $\mathbf{D}_-$, we proved already in [13].

## §2

In the case $m_D = 0$, i.e. $t_D < D$, we know by Theorem 1.1

$$u_D^2 < t_D \quad \text{provided } D > 13 \text{ or } t_D \geqq 5.$$

Therefore, in this case the invariant $n_D = [t_D / u_D^2]$ is more useful than the invariant $m_D$.

For any $D$ in $\mathbf{D}_+$, we put

$$V_D = \{v : 0 \leqq v < u_D^2,\ v^2 \equiv 4 \pmod{u_D^2}\},$$

and

$$(V, W)_D = \{(v, w) : v \in V_D,\ v^2 - 4 = wu_D^2\}.$$

For any $D$ in $\mathbf{D}_-$, we put

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$$V_D = \{v : 0 \leqq v < u_D^2,\ v^2 \equiv -4 \pmod{u_D^2}\},$$

and

$$(V, W)_D = \{(v, w) : v \in V_D,\ v^2 + 4 = wu_D^2\}.$$

Then we can prove the following:

THEOREM 2.1. *For any $D$ in $\mathbf{D}$, there are uniquely determined $n_D$ in $\mathbf{N}_0$ and $(v_D, w_D)$ in $(V, W)_D$ such that*

$$\begin{cases} t_D = u_D^2 \cdot n_D + v_D \\ D = u_D^2 \cdot n_D^2 + 2v_D \cdot n_D + w_D. \end{cases}$$

*Additionally, if $u_D > 2$, then*

$$0 \leqq w_D < v_D < u_D^2 \quad \textit{and} \quad n_D = [t_D / u_D^2] = [D / t_D].$$

*Proof.* In the case $D$ in $\mathbf{D}_+$, we put

$$[t_D / u_D^2] = n_D \quad \text{and} \quad t_D = u_D^2 \cdot n_D + v_D.$$

Then, $n_D$ and $v_D$ are uniquely determined, and

$$n_D \in \mathbf{N}_0, \quad 0 \leqq v_D < u_D^2$$

holds.

Since

$$Du_D^2 = t_D^2 - 4 = u_D^2(u_D^2 n_D^2 + 2v_D n_D) + (v_D^2 - 4),$$

we obtain

$$v_D^2 - 4 \equiv 0 \pmod{u_D^2},$$

and so $v_D$ is in $V_D$.

Moreover, we put

$$v_D^2 - 4 = u_D^2 \cdot w_D.$$

Then, $w_D$ is also uniquely determined and we obtain

$$D = u_D^2 \cdot n_D^2 + 2v_D \cdot n_D + w_D,$$

and so $(v_D, w_D)$ is in $(V, W)_D$.

Especially, since $w_D u_D^2 = v_D^2 - 4 < u_D^4 - 4$, if we assume $u_D > 2$, then we get

$$w_D < u_D^2 - (4 / u_D^2) < u_D^2.$$

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Furthermore, it is clear that

$$w_D < 0 \quad \text{if and only if} \quad v_D = 0 \text{ or } 1.$$

On the other hand,

$$v_D = 0 \quad \text{implies} \quad w_D u_D^2 = -4, \quad \text{and so} \quad u_D = 1 \text{ or } 2,$$

and

$$v_D = 1 \quad \text{implies} \quad w_D u_D^2 = -3, \quad \text{and so} \quad u_D = 1.$$

Hence, $u_D > 2$ implies $0 \leqq w_D < u_D^2$.

Next, we put

$$g(x) = -x^2 + u_D^2 x + 4.$$

Then we get

$$g(0) = g(u_D^2) = 4 > 0$$

and

$$0 < g(w_D) = (v_D + w_D)(v_D - w_D).$$

Hence, if $u_D > 2$, then we know $v_D > w_D$ from $v_D + w_D \geqq 0$.

Finally, in $D = n_D t_D + (n_D v_D + w_D)$,

$$u_D > 2 \quad \text{implies} \quad n_D v_D + w_D \geqq 0,$$

and

$$t_D - (n_D v_D + w_D) = (u_D^2 - v_D) \cdot n_D + (v_D - w_D) > 0.$$

Hence, we get

$$[t_D / u_D^2] = n_D = [D / t_D].$$

In the case $D$ in $\mathbf{D}_-$, we proved already in [13].

## §3

In the unique expression

$$D = k^2 + r \quad (-k < r \leqq k)$$

for any $D$ in $\mathbf{D}$, if

$$4k \equiv 0 \pmod r$$

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holds, then the real quadratic field $\mathbf{Q}(\sqrt{D})$ is called of Richaud-Degert type (simply R-D type).

In this section, we consider new invariants of real quadratic fields of R-D type. For real quadratic fields of R-D type, the explicit form of the fundamental unit is well-known as follows (cf. [1, 2, 6, 8]):

THEOREM 3.1 (Richaud-Degert). *Let* $\mathbf{Q}(\sqrt{D})$ $(D = k^2 + r,\ -k < r \leqq k)$ *be real quadratic fields of* R-D *type* $(4k \equiv 0 \bmod r)$.
*Then, the fundamental unit* $\varepsilon_D$ *of* $\mathbf{Q}(\sqrt{D})$ *is of the following form*:

$$\begin{cases} \varepsilon_D = k + \sqrt{D} & \text{with } N\varepsilon_D = -\operatorname{sgn} r \quad \text{for } |r| = 1 \\ \qquad (\text{except for } D = 5;\ k = 2,\ r = 1), & \\ \varepsilon_D = (k + \sqrt{D})/2 & \text{with } N\varepsilon_D = -\operatorname{sgn} r \quad \text{for } |r| = 4, \\ \varepsilon_D = \{(2k^2 + r) + 2k\sqrt{D})\}/|r| & \text{with } N\varepsilon_D = 1 \qquad \text{for } |r| \neq 1,\ 4. \end{cases}$$

Using this theorem, we consider each case of R-D type as follows: In the case $r = 1$, we get $D = k^2 + 1 \quad (k \geqq 1)$ and $\varepsilon_D = k + \sqrt{D} \quad (N\varepsilon_D = -1)$, and so

$$t_D = 2k, \quad u_D = 2, \quad t_D/u_D^2 = k/2,$$

which implies

$$n_D = k/2 \quad (k : \text{even}) \quad \text{or} \quad (k-1)/2 \quad (k : \text{odd}).$$

Hence,

$$\text{if } k \text{ is even, then } v_D = 0 \text{ and } w_D = 1.$$

On the other hand,

$$\text{if } k \text{ is odd, then } v_D = 2 \text{ and } w_D = 2.$$

In the case $r = -1$, we get $D = k^2 - 1 \quad (1 < k : \text{even})$ and $\varepsilon_D = k + \sqrt{D}$ $(N\varepsilon_D = 1)$, and so

$$t_D = 2k, \quad u_D = 2, \quad t_D/u_D^2 = k/2,$$

which implies

$$n_D = k/2.$$

Hence, we obtain

$$v_D = 0 \quad \text{and} \quad w_D = -1.$$

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In the case $r = 2$, we get $D = k^2 + 2 \quad (k \geqq 2)$ and $\varepsilon_D = k^2 + 1 + k\sqrt{D}$ $(N\varepsilon_D = 1)$, and so

$$t_D = 2(k^2 + 1), \quad u_D = 2k, \quad u_D^2/t_D = 2k^2/(k^2 + 1),$$

which implies

$$m_D = 1$$

from Theorem 1.1, because of $1 < 2k^2/(k^2 + 1) < 2$.
Hence, we get

$$a_D = k^2, \quad b_D = k^2 - 2.$$

In the case $r = -2$, we get $D = k^2 - 2 \quad (k > 2)$ and $\varepsilon_D = k^2 - 1 + k\sqrt{D}$ $(N\varepsilon_D = 1)$, and so

$$t_D = 2(k^2 - 1), \quad u_D = 2k, \quad u_D^2/t_D = 2k^2/(k^2 - 1),$$

which implies

$$m_D = 2$$

from Theorem 1.1, because of $2 < 2k^2/(k^2 - 1) < 3$.
Hence, we get

$$a_D = 2, \quad b_D = 0.$$

In the case $r = 3$, we get $D = k^2 + 3$ $(3 \leqq k \equiv 0 \bmod 3)$ and $\varepsilon_D = \{(2k^2 + 3) + 2k\sqrt{D}\}/3 \quad (N\varepsilon_D = 1)$, and so

$$t_D = 2(2k^2 + 3)/3, \quad u_D = 4k/3, \quad u_D^2/t_D = 8k^2/(6k^2 + 9),$$

which implies

$$m_D = 1$$

from Theorem 1.1, because of $1 < 8k^2/(6k^2 + 9) < 2$.
Hence, we get

$$a_D = (k^2 - 3)/3, \quad b_D = (k^2 - 9)/9.$$

In the case $r = -3$, we get $D = k^2 - 3 \quad (3 < k \equiv 0 \bmod 3)$ and $\varepsilon_D = \{(2k^2 - 3) + 2k\sqrt{D}\}/3 \quad (N\varepsilon_D = 1)$, and so

$$t_D = 2(2k^2 - 3)/3, \quad u_D = 4k/3, \quad u_D^2/t_D = 8k^2/(6k^2 - 9),$$

which implies

$$m_D = 1$$

from Theorem 1.1, because of $1 < 8k^2/(6k^2 - 9) < 2$.
Hence, we get

$$a_D = (k^2 + 3)/3, \quad b_D = (k^2 + 9)/9.$$

In the case $|r| = 4$, we get $D = k^2 \pm 4$ $(k \geqq 4)$ and $\varepsilon_D = (k + \sqrt{D})/2$ $(N\varepsilon_D = -1)$, and so

$$t_D = k, \quad u_D = 1, \quad t_D/u_D^2 = k,$$

which implies $n_D = k$.
Hence, we obtain

$$v_D = 0 \quad \text{and} \quad w_D = \pm 4.$$

In the case $|r| \geqq 5$, we get $t_D = 2(2k^2 + r)/|r|$, $u_D = 4k/|r|$, which implies

$$t_D/u_D^2 = (|r|/4) + (\operatorname{sgn} r)(r^2/8k^2).$$

Here, since $0 < r^2/8k^2 \leqq 1/8$, we obtain

$$n_D = [|r|/4] - 1 \quad (\text{for } 0 > r \equiv 0 \bmod 4),$$

or

$$n_D = [|r|/4] \qquad (\text{for other cases}).$$

From these considerations, we obtain the following theorem and table:

THEOREM 3.2. *For real quadratic fields* $\mathbf{Q}(\sqrt{D})$ *of* R-D *type, if* $D > 5$, *then*

$$m_D = \begin{cases} 2 & \text{for } r = -2, \\ 1 & \text{for } r = 2, \pm 3, \\ 0 & \text{for } |r| \neq 2, 3. \end{cases}$$

| $D = k^2 + r$ | $r$ | $t_D$ | $u_D$ | $m_D$ | $a_D$ | $b_D$ |
|---|---|---|---|---|---|---|
| $D \geqq 6$ | 2 | $2k^2 + 2$ | $2k$ | 1 | $k^2$ | $k^2 - 2$ |
| $D \geqq 7$ | $-2$ | $2k^2 - 2$ | $2k$ | 2 | 2 | 0 |
| $D \geqq 39$ | 3 | $2(2k^2 + 3)/3$ | $4k/3$ | 1 | $(k^2 - 3)/3$ | $(k^2 - 9)/9$ |
| $D \geqq 33$ | $-3$ | $2(2k^2 - 3)/3$ | $4k/3$ | 1 | $(k^2 + 3)/3$ | $(k^2 + 9)/9$ |

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| $D=k^2+r$ | $r$ | $t_D$ | $u_D$ | $n_D$ | $v_D$ | $w_D$ | |
|---|---|---|---|---|---|---|---|
| $D \geqq 17$ | 1 | $2k$ | 2 | $k/2$ | 0 | 1 | $k$ : even |
| $D \geqq 10$ | 1 | $2k$ | 2 | $(k-1)/2$ | 2 | 2 | $k$ : odd |
| $D \geqq 15$ | $-1$ | $2k$ | 2 | $k/2$ | 0 | $-1$ | $k$ : even only |
| $D \geqq 29$ | 4 | $k$ | 1 | $k$ | 0 | 4 | |
| $D \geqq 21$ | $-4$ | $k$ | 1 | $k$ | 0 | $-4$ | |

Furthermore, we can prove the following three propositions, which characterize each case of R-D type:

PROPOSITION 3.1. (1) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $|r|=1$, *if and only if* $u_D=2$.
(2) $Q(\sqrt{D})$ *is of* R-D *type with* $|r|=4$, *if and only if* $u_D=1$.

*Proof.* From $t_D^2-Du_D^2=\pm 4$ we can obtain the following:
In the case $u_D=1$, we get directly $D=t_D^2\pm 4$.
In the case $u_D=2$, we know first that $t_D$ is even, and so we can put $t_D=2k$ with a positive integer $k$. Hence we get $D=k^2\pm 1$.

The converse is clear from the above table.

PROPOSITION 3.2. (1) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 1$ ($k$: *even*) *if and only if* $(v_D, w_D)=(0, \pm 1)$.
(2) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 2$, *if and only if* $u_D^2=4D\mp 8$.
(3) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 3$, *if and only if* $9u_D^2=16D\mp 48$.
(4) $\mathbf{Q}(\sqrt{D})$ *is of* R-D *type with* $r=\pm 4$, *if and only if* $(v_D, w_D)=(0, \pm 4)$.

*Proof.* (1), (4) In the case $v_D=0$, by Theorem 2.1 we obtain immediately $D=u_D^2n_D^2+w_D$. Hence, if additionally we assume $w_D=\pm 1$, or $\pm 4$ respectively, then $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=w_D$.

The converse is clear from the above table.

(2) If $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 2$, then $u_D=2k$, and hence

$$u_D^2=4k^2=4(k^2\pm 2)\mp 8=4D\mp 8.$$

Conversely, if $u_D^2=4D\mp 8$, then $u_D^2\equiv 0 \pmod 4$, and so $u_D\equiv 0 \pmod 2$. Hence, we can put $u_D=2k$ with a suitable natural number $k$, and obtain immediately $D=k^2\pm 2$, which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 2$.

(3) If $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r=\pm 3$, then $u_D=4k/3$, and hence

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$$9u_D^2 = 16k^2 = 16(k^2 \pm 3) \mp 48 = 16D \mp 48.$$

Conversely, if $9u_D^2 = 16D \mp 48$, then $9u_D^2 \equiv 0 \pmod{4^2}$, and so $u_D \equiv 0 \pmod 4$. Hence, similarly we can put $u_D = 4k$ with a suitable natural number $k$, and obtain $D = 9k^2 \pm 3$, which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = \pm 3$.

PROPOSITION 3.3. (1) *$\mathbf{Q}(\sqrt{D})$ is of R-D type with $r = 1$ ($k$: odd), if and only if $u_D = 2$ and $(v_D, w_D) = (2, 2)$.*

(2) *$\mathbf{Q}(\sqrt{D})$ is of R-D type with $r = -2$, if and only if $m_D = 2$ and $(a_D, b_D) = (2, 0)$.*

*Proof.* (1) If we assume $u_D = 2$ and $(v_D, w_D) = (2, 2)$, then from Theorem 2.1 we get immediately

$$D = 4n_D^2 + 4n_D + 2 = (2n_D + 1)^2 + 1,$$

which shows that $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = 1$.

The converse is clear from the above table.

(2) If we assume $m_D = 2$ and $(a_D, b_D) = (2, 0)$, then from Theorem 1.2, we get similarly $u_D^2 = 4(D + 2)$. Hence, there exists a natural number $k$ satisfying $D + 2 = k^2$ i.e. $D = k^2 - 2$. Therefore, $\mathbf{Q}(\sqrt{D})$ is of R-D type and $r = -2$.

The converse is clear from the above table.

## §4

In this section, in connection with class number problem, we consider finiteness properties and estimations from below for the class number of real quadratic fields.

We first prove the following theorem related to class number one problem for real quadratic fields:

THEOREM 4.1. *For arbitrarily chosen and fixed natural number $h_0$ and real number $c$ greater than 2, there exists only a finite number of real quadratic fields $\mathbf{Q}(\sqrt{D})$ $(D \in \mathbf{D})$ such that*

$$\varepsilon_D < D \cdot e^{D^{\frac{1}{c}}} \quad \textit{and} \quad h_D \leqq h_0.$$

*Proof.* We first define a symbol $\delta_D$ depend on $D$ in $\mathbf{D}$ by

$$\delta_D = 0 \quad \text{for} \quad D \equiv 1 \pmod 4,$$

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and

$$\delta_D = 1 \quad \text{for} \quad D \equiv 2,\ 3 \pmod 4).$$

Then, we get $d = 4^{\delta_D} \cdot D$ for the discriminant $d$ of real quadratic field $\mathbf{Q}(\sqrt{D})$.

Moreover, by applying Tatuzawa's result (cf. [7])

$$L(1, \chi_d) > 0.655/(sd^{1/s})$$

(for any $s \geqq 11.2$, $d \geqq e^s$ and with one possible exception of $d$) to Dirichlet's

classical class number formula

$$h_d = (2 \log \varepsilon_D)^{-1} \sqrt{d} \cdot L(1, \chi_d),$$

we obtain

$$h_D > 4^{\delta_D (s-2)/2s} \cdot 0.3275 s^{-1} \cdot D^{(s-2)/2s} / \log \varepsilon_D$$

for any $s \geqq 11{,}2$ and $D \geqq e^s$ in $D$.

Here, if we assume $\varepsilon_D < D \cdot e^{D^{\frac{1}{c}}}$, and put

$$\alpha = (s-2)/(2s), \quad \beta = 1/c,$$

then

$$\alpha > \beta \quad \text{if and only if} \quad s > 2c/(c-2).$$

On the other hand, if we put moreover

$$f_s(D) = D^{\alpha} / (D^{\frac{1}{c}} + \log D),$$

then we get for any $s > 2c/(c-2)$

$$h_D > 4^{\delta_D (s-2)/2s} \cdot 0.3275 s^{-1} \cdot f_s(D),$$

and under the assumption $\alpha > \beta > 0$, $f_s(D)$ tends to infinity as $D$ tends to infinity. Therefore, if we choose any $s$ satisfying

$$s > \max\{11.2,\ 2c/(c-2)\},$$

then there exists a positive number $D_0$ such that

$$h_D > h_0 \quad \text{holds for any } D \text{ in } \mathbf{D} \text{ with } D > D_0,$$

in other words,

$$h_D \leqq h_0 \quad \text{implies} \quad D \leqq D_0.$$

From this theorem, we can obtain immediately the following two corollaries:

COROLLARY 4.1. *For arbitrarily chosen and fixed real number $c$ greater than 2, there exists only a finite number of real quadratic fields $\mathbf{Q}(\sqrt{D})$ $(D \in \mathbf{D})$ such that*

$$\varepsilon_D < D \cdot e^{D^{\frac{1}{c}}} \quad \text{and} \quad h_D = 1.$$

COROLLARY 4.2. *There exist infinitely many real quadratic fields of class number one if and only if there exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{D})$ $(D \in \mathbf{D})$ satisfying*

$$\varepsilon_D > D \cdot e^{D^{\frac{1}{c}}} \quad \text{and} \quad h_D = 1$$

*for any fixed number $c$ greater than 2.*

Furthermore, we can provide the following lower bounds for $h_D$:

PROPOSITION 4.1. *For any $s \geqq 11.2$ and $D \geqq e^s$ in $D$,*
(1) *if $m_D \neq 0$, then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log (m_D + 1) D\}$$

*holds with one possible exception of $D$.*
(2) *if $m_D = 0$ (i.e. $n_D \neq 0$), then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \{\log (D/n_D) + 1)\}$$

*holds with one possible exception of $D$,*
(3) *if $\mathbf{Q}(\sqrt{D})$ is a real quadratic field of* R-D *type, then*

$$h_D > 0.3275 \cdot 4^{\delta_D (s-2)/2s} \cdot s^{-1} \cdot D^{(s-2)/2s} / \log 3D.$$

*holds with one possible exception of $D$.*

*Proof.* In case of $m_D \neq 0$, from Theorem 1.1 we know first

$$\varepsilon_D < D(m_D + 1).$$

In case of $m_D = 0$, we know $n_D \neq 0$ from Theorem 1.1, and so from Theorem 1.3 in [13] we get $\varepsilon_D < (D/n_D) + 1$ for $D$ in $\mathbf{D}_-$, and also get similarly $\varepsilon_D < D/n_D$ for $D$ in $\mathbf{D}_+$.

In case of real quadratic field of R-D type, from Theorem 3.2 we get $\varepsilon_D < 3D$.

https://doi.org/10.1017/S0027763000004700 Published online by Cambridge University Press

Hence, in each case, by applying these upper bounds for $\varepsilon_D$ to the formula

$$h_D > 4^{\delta_D^{(s-2)/2s}} \cdot 0.3275 s^{-1} \cdot D^{(s-2)/2s} / \log \varepsilon_D$$

obtained in the proof of Theorem 4.1, we can prove Proposition 4.1.

---

$$t_D = Dm_D + a_D \qquad t_D = u_D^2 n_D + v_D$$
$$u_D^2 = Dm_D^2 + 2a_D m_D + b_D \qquad D = u_D^2 n_D^2 + 2v_D n_D^2 + 2v_D n_D + w_D$$
$$a_D^2 \pm 4 = b_D D \qquad v_D^2 \pm 4 = w_D u_D^2$$
$$m_D = [t_D / D] \qquad n_D = [t_D / u_D^2]$$

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| # 2 | 1 | 2 | 2 | −1 | 1 | 0 | 2 | | | |
| # 3 | −1 | 4 | 2 | 1 | 1 | 1 | −1 | | | |
| # 5 | | 1 | 1 | −1 | | | | 1 | 0 | 4 |
| 6 | 2 | 10 | 4 | 1 | 1 | 4 | 2 | | | |
| # 7 | −2 | 16 | 6 | 1 | 2 | 2 | 0 | | | |
| 10 | 1 | 6 | 2 | −2 | | | | 1 | 2 | 2 |
| # 11 | 2 | 20 | 6 | 1 | 1 | 9 | 7 | | | |
| # 13 | | 3 | 1 | −1 | | | | 3 | 0 | 4 |
| 14 | −2 | 30 | 8 | 1 | 2 | 2 | 0 | | | |
| 15 | −1 | 8 | 2 | 2 | | | | 2 | 0 | −1 |
| # 17 | 1 | 8 | 2 | −1 | | | | 2 | 0 | 1 |
| # 19 | | 340 | 78 | 1 | 17 | 17 | 15 | | | |
| 21 | −4 | 5 | 1 | 1 | | | | 5 | 0 | −4 |
| 22 | | 394 | 84 | 1 | 17 | 20 | 18 | | | |
| # 23 | −2 | 48 | 10 | 1 | 2 | 2 | 0 | | | |
| 26 | 1 | 10 | 2 | −2 | | | | 2 | 2 | 2 |
| # 29 | 4 | 5 | 1 | −1 | | | | 5 | 0 | 4 |
| 30 | 5 | 22 | 4 | 2 | | | | 1 | 6 | 2 |
| # 31 | | 3040 | 546 | 1 | 98 | 2 | 0 | | | |
| 33 | −3 | 46 | 8 | 1 | 1 | 13 | 5 | | | |
| 34 | −2 | 70 | 12 | 2 | 2 | 2 | 0 | | | |
| 35 | −1 | 12 | 2 | 2 | | | | 3 | 0 | −1 |
| # 37 | 1 | 12 | 2 | −1 | | | | 3 | 0 | 1 |
| 38 | 2 | 74 | 12 | 1 | 1 | 36 | 34 | | | |
| 39 | 3 | 50 | 8 | 2 | 1 | 11 | 3 | | | |
| # 41 | | 64 | 10 | −1 | 1 | 23 | 13 | | | |
| 42 | 6 | 26 | 4 | 2 | | | | 1 | 10 | 6 |
| # 43 | | 6964 | 1062 | 1 | 161 | 41 | 39 | | | |
| 46 | | 48670 | 7176 | 1 | 1058 | 2 | 0 | | | |

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| # 47 | −2 | 96 | 14 | 1 | 2 | 2 | 0 | | | |
| 51 | 2 | 100 | 14 | 2 | 1 | 49 | 47 | | | |
| # 53 | 4 | 7 | 1 | −1 | | | | 7 | 0 | 4 |
| 55 | | 178 | 24 | 2 | 3 | 13 | 3 | | | |
| 57 | | 302 | 40 | 1 | 5 | 17 | 5 | | | |
| 58 | | 198 | 26 | −2 | 3 | 24 | 10 | | | |
| # 59 | | 1060 | 138 | 1 | 17 | 57 | 55 | | | |
| # 61 | | 39 | 5 | −1 | | | | 1 | 14 | 8 |
| 62 | −2 | 126 | 16 | 1 | 2 | 2 | 0 | | | |
| 65 | 1 | 16 | 2 | −2 | | | | 4 | 0 | 1 |
| 66 | 2 | 130 | 16 | 2 | 1 | 64 | 62 | | | |
| # 67 | | 97684 | 11934 | 1 | 1457 | 65 | 63 | | | |
| 69 | | 25 | 3 | 1 | | | | 2 | 7 | 5 |
| 70 | | 502 | 60 | 2 | 7 | 12 | 2 | | | |
| # 71 | | 6960 | 826 | 1 | 98 | 2 | 0 | | | |
| # 73 | | 2136 | 250 | −1 | 29 | 19 | 5 | | | |
| 74 | | 86 | 10 | −2 | 1 | 12 | 2 | | | |
| 77 | −4 | 9 | 1 | 1 | | | | 9 | 0 | −4 |
| 78 | −3 | 106 | 12 | 2 | 1 | 28 | 10 | | | |
| # 79 | −2 | 160 | 18 | 3 | 2 | 2 | 0 | | | |
| 82 | 1 | 18 | 2 | −4 | | | | 4 | 2 | 2 |
| # 83 | 2 | 164 | 18 | 1 | 1 | 81 | 79 | | | |
| 85 | 4 | 9 | 1 | −2 | | | | 9 | 0 | 4 |
| 86 | | 20810 | 2244 | 1 | 241 | 84 | 82 | | | |
| 87 | 6 | 56 | 6 | 2 | | | | 1 | 20 | 11 |
| # 89 | | 1000 | 106 | −1 | 11 | 21 | 5 | | | |
| 91 | | 3148 | 330 | 2 | 34 | 54 | 32 | | | |
| 93 | | 29 | 3 | 1 | | | | 3 | 2 | 0 |
| 94 | | 4286590 | 442128 | 1 | 45602 | 2 | 0 | | | |
| 95 | −5 | 78 | 8 | 2 | | | | 1 | 14 | 3 |
| # 97 | | 11208 | 1138 | −1 | 115 | 53 | 29 | | | |
| # 101 | 1 | 20 | 2 | −1 | | | | 5 | 0 | 1 |
| 102 | 2 | 202 | 20 | 2 | 1 | 100 | 98 | | | |
| # 103 | | 455056 | 44838 | 1 | 4418 | 2 | 0 | | | |
| 105 | 5 | 82 | 8 | 2 | | | | 1 | 18 | 5 |
| 106 | | 8010 | 778 | −2 | 75 | 60 | 34 | | | |
| # 107 | | 1924 | 186 | 1 | 17 | 105 | 103 | | | |
| # 109 | | 261 | 25 | −1 | 2 | 43 | 17 | | | |
| 110 | 10 | 42 | 4 | 2 | | | | 2 | 10 | 6 |
| 111 | | 590 | 56 | 2 | 5 | 35 | 11 | | | |
| # 113 | | 1552 | 146 | −1 | 13 | 83 | 61 | | | |

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 114 | | 2050 | 192 | 2 | 17 | 112 | 110 | | | |
| 115 | | 2252 | 210 | 2 | 19 | 67 | 39 | | | |
| 118 | | 613834 | 56508 | 1 | 5201 | 116 | 114 | | | |
| 119 | −2 | 240 | 22 | 2 | 2 | 2 | 0 | | | |
| 122 | 1 | 22 | 2 | −2 | | | | 5 | 2 | 2 |
| 123 | 2 | 244 | 22 | 2 | 1 | 121 | 119 | | | |
| ♯127 | | 9461248 | 839550 | 1 | 74498 | 2 | 0 | | | |
| 129 | | 33710 | 2968 | 1 | 261 | 41 | 13 | | | |
| 130 | | 114 | 10 | −4 | | | | 1 | 14 | 2 |
| ♯131 | | 21220 | 1854 | 1 | 161 | 129 | 127 | | | |
| 133 | | 173 | 15 | 1 | 1 | 40 | 12 | | | |
| 134 | | 291850 | 25212 | 1 | 2177 | 132 | 130 | | | |
| ♯137 | | 3488 | 298 | −1 | 25 | 63 | 29 | | | |
| 138 | −6 | 94 | 8 | 2 | | | | 1 | 30 | 14 |
| ♯139 | | 155126500 | 13157658 | 1 | 1116017 | 137 | 135 | | | |
| 141 | −3 | 190 | 16 | 1 | 1 | 49 | 17 | | | |
| 142 | −2 | 286 | 24 | 3 | 2 | 2 | 0 | | | |
| 143 | −1 | 24 | 2 | 2 | | | | 6 | 0 | −1 |
| 145 | 1 | 24 | 2 | −4 | | | | 6 | 0 | 1 |
| 146 | 2 | 290 | 24 | 2 | 1 | 144 | 142 | | | |
| ♯149 | | 61 | 5 | −1 | | | | 2 | 11 | 5 |
| ♯151 | | 3456296080 | 281269386 | 1 | 22889378 | 2 | 0 | | | |
| 154 | | 42590 | 3432 | 2 | 276 | 86 | 48 | | | |
| 155 | | 498 | 40 | 2 | 3 | 33 | 7 | | | |
| ♯157 | | 213 | 17 | −1 | 1 | 56 | 20 | | | |
| 158 | | 15486 | 1232 | 1 | 98 | 2 | 0 | | | |
| 159 | | 2648 | 210 | 2 | 16 | 104 | 68 | | | |
| 161 | | 23550 | 1856 | 1 | 146 | 44 | 12 | | | |
| ♯163 | | 128160052 | 10038270 | 1 | 786257 | 161 | 159 | | | |
| 165 | −4 | 13 | 1 | 2 | | | | 13 | 0 | −4 |
| 166 | −3 | * | | 1 | | | | | | |
| ♯167 | −2 | 336 | 26 | 1 | 2 | 2 | 0 | | | |
| 170 | 1 | 26 | 2 | −4 | | | | 6 | 2 | 2 |
| ♯173 | 4 | 13 | 1 | −1 | | | | 13 | 0 | 4 |
| 174 | | 2902 | 220 | 2 | 16 | 118 | 80 | | | |
| 177 | | 124846 | 9384 | 1 | 705 | 61 | 21 | | | |
| 178 | | 3202 | 240 | 2 | 17 | 176 | 174 | | | |
| ♯179 | | 8380420 | 626382 | 1 | 46817 | 177 | 175 | | | |
| ♯181 | | 1305 | 97 | −1 | 7 | 38 | 8 | | | |
| 182 | 13 | 54 | 4 | 2 | | | | 3 | 6 | 2 |
| 183 | | 974 | 72 | 2 | 5 | 59 | 19 | | | |

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 185 | | 136 | 10 | −2 | | | | 1 | 36 | 13 |
| 186 | | 15002 | 1100 | 2 | 80 | 122 | 80 | | | |
| 187 | | 3364 | 246 | 2 | 17 | 185 | 183 | | | |
| 190 | | 104042 | 7548 | 2 | 547 | 112 | 66 | | | |
| # 191 | | 17988000 | 1301566 | 1 | 94178 | 2 | 0 | | | |
| # 193 | | 3528264 | 253970 | −1 | 18281 | 31 | 5 | | | |
| 194 | −2 | 390 | 28 | 2 | 2 | 2 | 0 | | | |
| 195 | −1 | 28 | 2 | 4 | | | | 7 | 0 | −1 |
| # 197 | 1 | 28 | 2 | −1 | | | | 7 | 0 | 1 |
| # 199 | | 32532393040 | 2306160198 | 1 | * | | | | | |
| 201 | | 1030190 | 72664 | 1 | 5125 | 65 | 21 | | | |
| 202 | | 6282 | 442 | −2 | 31 | 20 | 2 | | | |
| 203 | 7 | 114 | 8 | 2 | | | | 1 | 50 | 39 |
| 205 | | 43 | 3 | 2 | | | | 4 | 7 | 5 |
| 206 | | 119070 | 8296 | 1 | 578 | 2 | 0 | | | |
| 209 | | 93102 | 6440 | 1 | 445 | 97 | 45 | | | |
| 210 | 14 | 58 | 4 | 4 | | | | 3 | 10 | 6 |
| # 211 | | 556708747300 | * | 1 | * | 209 | 207 | | | |
| 213 | −12 | 73 | 5 | 1 | | | | 2 | 23 | 21 |
| 214 | | * | | 1 | | | | | | |
| 215 | −10 | 88 | 6 | 2 | | | | 2 | 16 | 7 |
| 217 | | 7688126 | 521904 | 1 | 35429 | 33 | 5 | | | |
| 218 | | 502 | 34 | −2 | 2 | 66 | 20 | | | |
| 219 | −6 | 148 | 10 | 4 | | | | 1 | 48 | 23 |
| 221 | −4 | 15 | 1 | 2 | | | | 15 | 0 | −4 |
| 222 | −3 | 298 | 20 | 2 | 1 | 76 | 26 | | | |
| # 223 | −2 | 448 | 30 | 3 | 2 | 2 | 0 | | | |
| 226 | 1 | 30 | 2 | −8 | | | | 7 | 2 | 2 |
| # 227 | 2 | 452 | 30 | 1 | 1 | 225 | 223 | | | |
| # 229 | 4 | 15 | 1 | −3 | | | | 15 | 0 | 4 |
| 230 | 5 | 182 | 12 | 2 | | | | 1 | 38 | 10 |
| 231 | 6 | 152 | 10 | 4 | | | | 1 | 52 | 27 |
| # 233 | | 46312 | 3034 | −1 | 198 | 178 | 136 | | | |
| 235 | 10 | 92 | 6 | 6 | | | | 2 | 20 | 11 |
| 237 | 12 | 77 | 5 | 1 | | | | 3 | 2 | 0 |
| 238 | | 23326 | 1512 | 2 | 98 | 2 | 0 | | | |
| # 239 | | 12390240 | 801458 | 1 | 51842 | 2 | 0 | | | |
| # 241 | | 142022136 | 9148450 | −1 | 589303 | 113 | 53 | | | |
| 246 | | 177610 | 11324 | 2 | 721 | 244 | 242 | | | |
| 247 | | 170584 | 10854 | 2 | 690 | 154 | 96 | | | |
| # 251 | | 7349780 | 463914 | 2 | 29281 | 249 | 247 | | | |

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 253 | | 1861 | 117 | 1 | 7 | 90 | 32 | | | |
| 254 | −2 | 510 | 32 | 3 | 2 | 2 | 0 | | | |
| 255 | −1 | 32 | 2 | 4 | | | | 8 | 0 | −1 |
| # 257 | 1 | 32 | 2 | −3 | | | | 8 | 0 | 1 |
| 258 | 2 | 514 | 32 | 2 | 1 | 256 | 254 | | | |
| 259 | | 1694450 | 105288 | 2 | 6542 | 72 | 20 | | | |
| 262 | | 209961034 | 12971436 | 1 | 801377 | 260 | 258 | | | |
| # 263 | | 278256 | 17158 | 1 | 1058 | 2 | 0 | | | |
| 265 | | 12144 | 746 | −2 | 45 | 219 | 181 | | | |
| 266 | | 1370 | 84 | 2 | 5 | 40 | 6 | | | |
| 267 | | 4804 | 294 | 2 | 17 | 265 | 263 | | | |
| # 269 | | 164 | 10 | −1 | | | | 1 | 64 | 41 |
| # 271 | | * | | 1 | | | | | | |
| 273 | | 1454 | 88 | 2 | 5 | 89 | 29 | | | |
| 274 | | 2814 | 170 | −4 | 10 | 74 | 20 | | | |
| # 277 | | 2613 | 157 | −1 | 9 | 120 | 52 | | | |
| 278 | | 5002 | 300 | 1 | 17 | 276 | 274 | | | |
| # 281 | | 2127064 | 126890 | −1 | 7569 | 175 | 109 | | | |
| 282 | | 4702 | 280 | 2 | 16 | 190 | 128 | | | |
| # 283 | | 276548164 | 16439082 | 1 | 977201 | 281 | 279 | | | |
| 285 | −4 | 17 | 1 | 2 | | | | 17 | 0 | −4 |
| 286 | | 1123670 | 66444 | 2 | 3928 | 262 | 240 | | | |
| 287 | −2 | 576 | 34 | 2 | 2 | 2 | 0 | | | |
| 290 | 1 | 34 | 2 | −4 | | | | 8 | 2 | 2 |
| 291 | 2 | 580 | 34 | 4 | 1 | 289 | 287 | | | |
| # 293 | 4 | 17 | 1 | −1 | | | | 17 | 0 | 4 |
| 295 | | 4049998 | 235800 | 2 | 13728 | 238 | 192 | | | |
| 298 | | 819114 | 47450 | −2 | 2748 | 210 | 148 | | | |
| 299 | | 830 | 48 | 2 | 2 | 232 | 180 | | | |
| 301 | | 22745 | 1311 | 1 | 75 | 170 | 96 | | | |
| 302 | | 8553246 | 492184 | 1 | 28322 | 2 | 0 | | | |
| 303 | | 5048 | 290 | 2 | 16 | 200 | 132 | | | |
| 305 | | 978 | 56 | 2 | 3 | 63 | 13 | | | |
| # 307 | | 177058564 | 10105266 | 1 | 576737 | 305 | 303 | | | |
| 309 | | 5045 | 287 | 1 | 16 | 101 | 33 | | | |
| 310 | | 1697438 | 96408 | 2 | 5475 | 188 | 114 | | | |
| # 311 | | 33767760 | 1914794 | 1 | 108578 | 2 | 0 | | | |
| # 313 | | 253724736 | 14341370 | −1 | 810622 | 50 | 8 | | | |
| 314 | | 886 | 50 | −2 | 2 | 258 | 212 | | | |
| # 317 | | 89 | 5 | −1 | | | | 3 | 14 | 8 |
| 318 | −6 | 214 | 12 | 2 | | | | 1 | 70 | 34 |

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| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 319 | | 25803560 | 1444722 | 2 | 80888 | 288 | 260 | | | |
| 321 | −3 | 430 | 24 | 3 | 1 | 109 | 37 | | | |
| 322 | −2 | 646 | 36 | 4 | 2 | 2 | 0 | | | |
| 323 | −1 | 36 | 2 | 4 | | | | 9 | 0 | −1 |
| 326 | 2 | 650 | 36 | 3 | 1 | 324 | 322 | | | |
| 327 | 3 | 434 | 24 | 2 | 1 | 107 | 35 | | | |
| 329 | | 4752830 | 262032 | 1 | 14446 | 96 | 28 | | | |
| 330 | 6 | 218 | 12 | 4 | | | | 1 | 74 | 38 |
| # 331 | | * | | 1 | | | | | | |
| 334 | | * | | 1 | | | | | | |
| 335 | | 1208 | 66 | 2 | 3 | 203 | 123 | | | |
| # 337 | | 2031654672 | 70671282 | −1 | * | | | | | |
| 339 | | 195940 | 10642 | 2 | 577 | 337 | 335 | | | |
| 341 | | 277 | 15 | 1 | | | | 1 | 52 | 12 |
| 345 | | 13522 | 728 | 2 | 39 | 67 | 13 | | | |
| 346 | | 186 | 10 | −6 | | | | 1 | 86 | 74 |
| # 347 | | 1283204 | 68886 | 1 | 3697 | 345 | 343 | | | |
| # 349 | | 18420 | 986 | −1 | 52 | 272 | 212 | | | |
| # 353 | | 142528 | 7586 | −1 | 403 | 269 | 205 | | | |
| 354 | | 516130 | 27432 | 2 | 1457 | 352 | 350 | | | |
| 355 | | 1909618 | 101352 | 2 | 5379 | 73 | 15 | | | |
| 357 | −4 | 19 | 1 | 2 | | | | 19 | 0 | −4 |
| 358 | | * | | 1 | | | | | | |
| # 359 | −2 | 720 | 38 | 3 | 2 | 2 | 0 | | | |
| 362 | 1 | 38 | 2 | −2 | | | | 9 | 2 | 2 |
| 365 | 4 | 19 | 1 | −2 | | | | 19 | 0 | 4 |
| 366 | | 1815850 | 94916 | 2 | 4961 | 124 | 42 | | | |
| # 367 | | * | | 1 | | | | | | |
| 370 | | 654 | 34 | −4 | 1 | 284 | 218 | | | |
| 371 | | 3390 | 176 | 2 | 9 | 51 | 7 | | | |
| # 373 | | 10236 | 530 | −1 | 27 | 165 | 73 | | | |
| 374 | | 6730 | 348 | 2 | 17 | 372 | 370 | | | |
| 377 | | 466 | 24 | 2 | 1 | 89 | 21 | | | |
| # 379 | | * | | 1 | | | | | | |
| 381 | | 2030 | 104 | 1 | 5 | 125 | 41 | | | |
| 382 | | * | | 1 | | | | | | |
| # 383 | | 37536 | 1918 | 1 | 98 | 2 | 0 | | | |
| 385 | | 191662 | 9768 | 2 | 497 | 317 | 261 | | | |
| 386 | | 223110 | 11356 | 2 | 578 | 2 | 0 | | | |
| # 389 | | 2564 | 130 | −1 | 6 | 230 | 136 | | | |
| 390 | −10 | 158 | 8 | 4 | | | | 2 | 30 | 14 |

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 391 | | 14677360 | 14677360 | 2 | 37538 | 2 | 0 | | | |
| 393 | | 92874286 | 4684888 | 1 | 236321 | 133 | 45 | | | |
| 394 | | 790046070 | 39801946 | − 2 | 2005193 | 28 | 2 | | | |
| 395 | − 5 | 318 | 16 | 2 | | | | 1 | 62 | 15 |
| # 397 | | 3447 | 173 | − 1 | 8 | 271 | 185 | | | |
| 398 | − 2 | 798 | 40 | 1 | 2 | 2 | 0 | | | |
| 399 | − 1 | 40 | 2 | 8 | | | | 10 | 0 | − 1 |
| # 401 | 1 | 40 | 2 | − 5 | | | | 10 | 0 | 1 |
| 402 | 2 | 802 | 40 | 2 | 1 | 400 | 398 | | | |
| 403 | | 1339756 | 66738 | 2 | 3324 | 184 | 84 | | | |
| 406 | | 118936190 | 5902704 | 2 | 292946 | 114 | 32 | | | |
| 407 | | 5326 | 264 | 2 | 13 | 35 | 3 | | | |
| # 409 | | * | | − 1 | | | | | | |
| 410 | | 162 | 8 | 4 | | | | 2 | 34 | 18 |
| 411 | | 99460 | 4906 | 2 | 241 | 409 | 407 | | | |
| 413 | | 61 | 3 | 1 | | | | 6 | 7 | 5 |
| 415 | | 36825608 | 1807698 | 2 | 88736 | 168 | 68 | | | |
| 417 | | 170645294 | 8356536 | 1 | 409221 | 137 | 45 | | | |
| 418 | | 67714 | 3312 | 2 | 161 | 416 | 414 | | | |
| # 419 | | 540349940 | 26397822 | 1 | 1289617 | 417 | 415 | | | |
| # 421 | | 444939 | 21685 | − 1 | 1056 | 363 | 313 | | | |
| 422 | | 14045002 | 683700 | 1 | 33281 | 420 | 418 | | | |
| 426 | | 177502 | 8600 | 2 | 416 | 286 | 192 | | | |
| 427 | − 14 | 124 | 6 | 6 | | | | 3 | 16 | 7 |
| 429 | − 12 | 145 | 7 | 2 | | | | 2 | 47 | 45 |
| 430 | | 5724502 | 276060 | 2 | 13312 | 342 | 272 | | | |
| # 431 | | 303121440 | 14600846 | 1 | 703298 | 2 | 0 | | | |
| # 433 | | * | | − 1 | | | | | | |
| 434 | − 7 | 250 | 12 | 4 | | | | 1 | 106 | 78 |
| 435 | − 6 | 292 | 14 | 4 | | | | 1 | 96 | 47 |
| 437 | − 4 | 21 | 1 | 1 | | | | 21 | 0 | − 4 |
| 438 | − 3 | 586 | 28 | 4 | 1 | 148 | 50 | | | |
| # 439 | − 2 | 880 | 42 | 5 | 2 | 2 | 0 | | | |
| 442 | 1 | 42 | 2 | − 8 | | | | 10 | 2 | 2 |
| # 443 | 2 | 884 | 42 | 3 | 1 | 441 | 439 | | | |
| 445 | 4 | 21 | 1 | − 4 | | | | 21 | 0 | 4 |
| 446 | | 220332030 | 10433024 | 1 | 494018 | 2 | 0 | | | |
| 447 | 6 | 296 | 14 | 2 | | | | 1 | 100 | 51 |
| # 449 | | 378942664 | 17883410 | − 1 | 843970 | 134 | 40 | | | |
| 451 | | 92942980 | 4376514 | 2 | 206081 | 449 | 447 | | | |
| 453 | 12 | 149 | 7 | 1 | | | | 3 | 2 | 0 |

https://doi.org/10.1017/S0027763000004700 Published online by Cambridge University Press

| $D$ | $r$ | $t_D$ | $u_D$ | $h_D$ | $m_D$ | $a_D$ | $b_D$ | $n_D$ | $v_D$ | $w_D$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 454 | | * | | 1 | | | | | | |
| 455 | 14 | 128 | 6 | 4 | | | | 3 | 20 | 11 |
| # 457 | | * | | − 1 | | | | | | |
| 458 | | 214 | 10 | − 2 | | | | 2 | 14 | 2 |
| # 461 | | 365 | 17 | − 1 | | | | 1 | 76 | 20 |
| 462 | 21 | 86 | 4 | 4 | | | | 5 | 6 | 2 |
| # 463 | | * | | 1 | | | | | | |
| 465 | | 31742 | 1472 | 2 | 68 | 122 | 32 | | | |
| 466 | | 1876638850 | 86933616 | 2 | * | | | | | |
| # 467 | | 3251252 | 450450 | 1 | 6961 | 465 | 463 | | | |
| 469 | | 65 | 3 | 3 | | | | 7 | 2 | 0 |
| 470 | | 3382 | 156 | 2 | 7 | 92 | 18 | | | |
| 471 | | 15677390 | 722376 | 2 | 33285 | 155 | 51 | | | |
| 473 | − 11 | 174 | 8 | 3 | | | | 2 | 46 | 33 |
| 474 | | 387098 | 17780 | 2 | 816 | 314 | 208 | | | |
| 478 | | * | | 1 | | | | | | |
| # 479 | | 5978880 | 273182 | 1 | 12482 | 2 | 0 | | | |
| 481 | | 1928280 | 87922 | − 2 | 4008 | 432 | 388 | | | |
| 482 | − 2 | 966 | 44 | 2 | 2 | 2 | 0 | | | |
| 483 | − 1 | 44 | 2 | 4 | | | | 11 | 0 | − 1 |
| 485 | 1 | 44 | 2 | − 2 | | | | 11 | 0 | 1 |
| # 487 | | * | | 1 | | | | | | |
| 489 | | * | | 1 | | | | | | |
| # 491 | | * | | 1 | | | | | | |
| 493 | | 111 | 5 | − 2 | | | | 4 | 11 | 5 |
| 494 | | 146070 | 6572 | 2 | 295 | 340 | 234 | | | |
| 497 | | 2403774 | 197824 | 1 | 4836 | 282 | 160 | | | |
| 498 | | 359554 | 16112 | 2 | 721 | 496 | 494 | | | |
| # 499 | | 8980 | 402 | 5 | 17 | 497 | 495 | | | |

# indicates prime number.

$h_D = -n$ means that $N\varepsilon_D = -1$ and $h_D = n$.

$r$ representes the integer such that $D = k^2 + r$, $-k < r \leqq k$ and $4k \equiv 0 \pmod{r}$ for real quadratic field $\mathbf{Q}(\sqrt{D})$ of R-D type.

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