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Hilbert's 14th problem and Cox rings

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Abstract

Our main result is the description of generators of the total coordinate ring of the blow-up of $\mathbb{P}^n$ in any number of points that lie on a rational normal curve. As a corollary we show that the algebra of invariants of the action of a two-dimensional vector group introduced by Nagata is finitely generated by certain explicit determinants. We also prove the finite generation of the algebras of invariants of actions of vector groups related to T-shaped Dynkin diagrams introduced by Mukai.

1. Introduction

Hilbert's 14th problem that we discuss is the following question: If an algebraic group G acts linearly on a polynomial algebra S, is the algebra of invariants S^G finitely generated? The answer is known to be affirmative if G is reductive (see [Hil90]) and if G is the simplest nonreductive group $\mathbb{G}_a$ (see [Wei32]). However, in general the answer is negative – the first counterexample was found by Nagata in 1958. Let

$$G = \mathbb{G}_a^g \subset \mathbb{G}_a^r \tag{1.1}$$

be a general linear subspace of codimension at least 3. Consider the following linear action of $\mathbb{G}_a^r$ on $S := \mathbb{C}[x_1, \ldots, x_r, y_1, \ldots, y_r]$: an element $(t_1, \ldots, t_r) \in \mathbb{G}_a^r$ acts by

$$x_i \mapsto x_i, \quad y_i \mapsto y_i + t_i x_i, \quad 1 \leqslant i \leqslant r.$$

The induced action of G on S is called *the Nagata action*. The algebra of invariants S^G is not finitely generated if $g = 13$ (see [Nag59]), $g = 6$ (see [Ste97]), and finally $g = 3$, $r = 9$ (see [Muk01]). Thus, Hilbert's 14th problem has a negative answer for $\mathbb{G}_a^3$. In [Muk01], Mukai asks what happens if $g = 2$.

Theorem 1.1. *Assume without loss of generality that $G = \mathbb{G}_a^2 \subset \mathbb{G}_a^{n+3}$ is a linear subspace spanned by rows of the matrix*

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_{n+3} \end{bmatrix},$$

where $a_1, \ldots, a_{n+3}$ are general numbers. Then S^G is generated by 2^{n+2} invariants

$$F_I = \begin{vmatrix} x_{i_1} & x_{i_2} & \ldots & x_{i_{2k+1}} \\ a_{i_1} x_{i_1} & a_{i_2} x_{i_2} & \ldots & a_{i_{2k+1}} x_{i_{2k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1}^k x_{i_1} & a_{i_2}^k x_{i_2} & \ldots & a_{i_{2k+1}}^k x_{i_{2k+1}} \\ y_{i_1} & y_{i_2} & \ldots & y_{i_{2k+1}} \\ a_{i_1} y_{i_1} & a_{i_2} y_{i_2} & \ldots & a_{i_{2k+1}} y_{i_{2k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1}^{k-1} y_{i_1} & a_{i_2}^{k-1} y_{i_2} & \ldots & a_{i_{2k+1}}^{k-1} y_{i_{2k+1}} \end{vmatrix}, \tag{1.2}$$

where $I = \{i_1, \ldots, i_{2k+1}\} \subset \{1, \ldots, n+3\}$ is any subset of odd cardinality $2k+1$.

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Of course, it is possible that the algebra of invariants of $\mathbb{G}_a^2$ is not finitely generated for actions more complicated than Nagata actions.

The ingenious insight of Nagata was to relate S^G to a Cox ring. Let X be a projective algebraic variety over $\mathbb{C}$. Assume that divisors $D_1, \dots, D_r$ freely generate the Picard group $\mathrm{Pic}(X)$. Then the *Cox ring* of X is the multigraded ring

$$\mathrm{Cox}(X) = \bigoplus_{(m_1,\dots,m_r)\in\mathbb{Z}^r} \mathrm{H}^0(X, m_1D_1 + \cdots + m_rD_r)$$

(the basis is necessary to introduce multiplication in a canonical way). This definition is a generalization of the total coordinate ring of a toric variety introduced by Cox [Cox95]. In fact, $\mathrm{Cox}(X)$ is isomorphic to a polynomial ring if and only if X is a toric variety [HK00, Proposition 2.10]. For an arbitrary variety X, Hu and Keel [HK00, Proposition 2.9] proved that $\mathrm{Cox}(X)$ is finitely generated if and only if X is a *Mori dream space*: (1) the cone of nef divisors is generated by finitely many semi-ample line bundles, and (2) the cone of moving divisors (divisors whose base locus is of codimension at least 2 in X) is the union of nef cones of small modifications of X, i.e. varieties X' isomorphic to X in codimension 1.

In recent years, an explicit description of the ring $\mathrm{Cox}(X)$ has also proved useful for applications in arithmetic algebraic geometry. Universal torsors were used for proving the Hasse principle and weak approximation for certain Del Pezzo surfaces or for the counting of rational points of bounded height [CS87, CSS87a, CSS87b, Bre02, HT04, Sal98, Hea03].

The relation to Nagata actions is as follows: If G is as in (1.1), by [Muk01] one has

$$S^G \simeq \mathrm{Cox}(\mathrm{Bl}_r\mathbb{P}^{r-g-1}),$$

where $\mathrm{Bl}_r\mathbb{P}^{r-g-1}$ is the blow-up of $\mathbb{P}^{r-g-1}$ at r distinct points. Using this isomorphism, Theorem 1.1 is equivalent to describing the Cox ring of a blow-up of $\mathbb{P}^n$ at $n+3$ points. It is a well-known fact that there is a unique rational normal curve C of degree n in $\mathbb{P}^n$ passing through $n+3$ points in general position. We generalize Theorem 1.1 as follows.

Theorem 1.2. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n and let $p_1, \dots, p_r$ be distinct points on C, $r \geqslant n+3$. Let $X = \mathrm{Bl}_{p_1,\dots,p_r}\mathbb{P}^n$. Then $\mathrm{Cox}(X)$ is finitely generated by unique (up to scalar) global sections of exceptional divisors $E_1, \dots, E_r$ and divisors*

$$E = kH - k\sum_{i\in I} E_i - (k-1)\sum_{i\in I^c} E_i \tag{1.3}$$

for each subset $I \subset \{1, \dots, r\}$, $|I| = n+2-2k$, $1 \leqslant k \leqslant 1+n/2$. Here H is the pull-back of the hyperplane class in $\mathbb{P}^n$.

Geometrically, the divisors (1.3) are proper transforms of the following hypersurfaces in $\mathbb{P}^n$ [Har92, Example 9.6]. If I is empty then (1.3) is the $(n/2)$-secant variety of C. More generally, if $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{2k-2}$ is the projection from the linear subspace spanned by the points p_i, $i \in I$, and $C' = \pi_I(C)$, then C' is a rational normal curve of degree $2k-2$ and (1.3) is the cone over the $(k-1)$-secant variety of C'.

An obvious generalization of Theorem 1.2 would be to consider the Cox ring of the iterated blow-up of $\mathbb{P}^n$ along points, lines connecting them, 2-planes, etc. A special case of this construction is $\overline{M}_{0,n}$, the Grothendieck–Knudsen moduli space of stable n-pointed rational curves. If the Cox ring of $\overline{M}_{0,n}$ is finitely generated, then results of [HK00] and [KM97] almost imply the 'Fulton conjecture' for $\overline{M}_{0,n}$ and therefore the description of the Mori cone of $\overline{M}_{g,n}$ (Gibney–Keel–Morrison [GKM01]).

Following Mukai [Muk04], we also generalize Theorem 1.1 in a different direction. Let $T_{a,b,c}$ be the T-shaped tree with legs of length a, b, and c with $a+b+c-2$ vertices. We assume that $a, c \geqslant 2$

and if $c = 2$ then $a > 2$. Let

$$X_{a,b,c} = \mathrm{Bl}_{b+c}(\mathbb{P}^{c-1})^{a-1}$$

be the blow-up of $(\mathbb{P}^{c-1})^{a-1}$ in $r = b+c$ points in general position. The effective cone $\mathrm{Eff}(X_{a,b,c})$ is the set of effective divisors in $\mathrm{Pic}(X_{a,b,c})$. Mukai proves in [Muk04] that if $T_{a,b,c}$ is not a Dynkin diagram of a finite root system then $\mathrm{Eff}(X_{a,b,c})$ is not a finitely generated semigroup and therefore $\mathrm{Cox}(X_{a,b,c})$ is not a finitely generated algebra. Mukai also shows in [Muk04] that the Cox algebra of any $X_{a,b,c}$ is isomorphic to the algebra of invariants of a certain 'extended Nagata action'. From Theorem 1.2, using a trick from commutative algebra, we deduce the following theorem.

Theorem 1.3. *The following statements are equivalent:*

(i) $\mathrm{Cox}(X_{a,b,c})$ *is a finitely generated algebra;*

(ii) $\mathrm{Eff}(X_{a,b,c})$ *is a finitely generated semigroup;*

(iii) $T_{a,b,c}$ *is a Dynkin diagram of a finite root system;*

(iv) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} > 1$.

Moreover, in these cases consider $Z = \mathrm{Proj}(\mathrm{Cox}(X))$ *with respect to the natural* $\mathbb{Z}$*-grading of* $\mathrm{Cox}(X)$ *defined in* (3.4)*. Then* Z *is a locally factorial, Cohen–Macaulay, and Gorenstein scheme with rational singularities. The Picard group* $\mathrm{Pic}(Z) = \mathbb{Z}$ *is generated by* $\mathcal{O}_Z(1)$ *and the anticanonical class is* $-K_Z = \mathcal{O}_Z(d)$*, where*

$$d = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right) > 0.$$

The proof of the 'moreover' part is exactly the same as Popov's proof [Pop04] of the analogous statement for Del Pezzo surfaces (or $X_{2,s-3,3}$ in our notation). We only sketch it for the reader's convenience.

Explicitly, Theorem 1.3 includes the following cases. Mukai [Muk04] shows that $X_{a,b,c}$ is a small modification of $X_{c,b,a}$, so we assume that $a \leqslant c$ (if X' is a small modification of X then of course $\mathrm{Pic}(X) \cong \mathrm{Pic}(X')$, $\mathrm{Eff}(X) \cong \mathrm{Eff}(X')$, and $\mathrm{Cox}(X) \cong \mathrm{Cox}(X')$).

(i) $X_{2,2,n+1} = \mathrm{Bl}_{n+3}(\mathbb{P}^n)$.

(ii) $X_{2,3,4} = \mathrm{Bl}_7\mathbb{P}^3$, $X_{2,3,5} = \mathrm{Bl}_8\mathbb{P}^4$.

(iii) $X_{3,2,3} = \mathrm{Bl}_5(\mathbb{P}^2)^2$, $X_{3,2,4} = \mathrm{Bl}_6(\mathbb{P}^3)^2$, $X_{3,2,5} = \mathrm{Bl}_7(\mathbb{P}^4)^2$.

(iv) $X_{s+1,1,n+1} = \mathrm{Bl}_{n+2}(\mathbb{P}^n)^s$. This case is well known; see Remark 3.9.

(v) Del Pezzo surfaces $X_{2,s-3,3} = \mathrm{Bl}_s\mathbb{P}^2$, $s = 4, 5, 6, 7, 8$. In this case the finite generation of the Cox ring was proved by Batyrev and Popov [BP04].

We prove Theorems 1.1–1.3 in reverse order. In § 2 we describe the effective cone of $X_{a,b,c}$. In § 3 we prove Theorem 1.3 (the finite generation of $\mathrm{Cox}(X_{a,b,c})$) in all cases, except for $X_{2,3,4}$ and $X_{2,3,5}$, for which the proof relies on the cases $n = 3$ and $n = 4$ of Theorem 1.2. The latter is proved in Section 4, which is the main section of the paper and is independent of the previous sections. Theorem 1.1 is proved in § 5. In particular, we prove the finite generation of $\mathrm{Cox}(\mathrm{Bl}_{n+3}\mathbb{P}^n)$ twice. First, we give a simple proof in the framework of Theorem 1.3. Second, we give an independent proof of the much stronger Theorem 1.2 that gives explicit generators for this ring. It is crucial for our proof to consider any number of points on a rational normal curve. For example, finding generators for $\mathrm{Cox}(\mathrm{Bl}_{n+3}\mathbb{P}^n)$ relies on finding generators for the Cox ring of the blow-up of $\mathbb{P}^{n-1}$ in $n+3$ points lying on a rational normal curve, etc., up to the blow-up of $\mathbb{P}^2$ in $n+3$ points lying on a conic. Our proof of Theorem 1.2 was inspired by the 'whole-genome shotgun' [VAM01] method of genome sequencing that involves breaking the genome up into very small pieces, sequencing the

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pieces, and reassembling the pieces into the full genome sequence. This method has some advantages (and disadvantages) over the 'clone-by-clone' approach that involves breaking the genome up into relatively large chunks.

During the final stages of the preparation of this paper, Professor Shigeru Mukai sent us his preprint [Muk05], where he proves that the Cox ring of $X_{2,b,c}$ is finitely generated when $1/2+1/b+1/c>1$ by using a completely different approach based on results of S. Bauer about parabolic bundles on curves.

2. Root systems and effective cones

From now on we assume that $T_{a,b,c}$ is a Dynkin diagram of a finite root system. It is well known that this is equivalent to $1/a+1/b+1/c>1$. Let

$$X = X_{a,b,c}.$$

The Picard group $\mathrm{Pic}(X)$ is a free $\mathbb{Z}$-module of rank $a+b+c-1$ with a basis

$$H_1,\ldots,H_{a-1} \quad \text{and} \quad E_1,\ldots,E_r,$$

where H_i is the pull-back of the hyperplane class from the ith factor of $(\mathbb{P}^{c-1})^{a-1}$ and E_j is the class of the exceptional divisor over p_j, for $j=1,\ldots,r$, $r=b+c$. We call this basis *tautological*. If $a=2$ then we write H instead of H_1 and make the appropriate modifications in all notations. The anticanonical class of X is

$$-K = c(H_1+\cdots+H_{a-1}) - (ac-a-c)(E_1+\cdots+E_r).$$

Following [Muk04], we define a symmetric bilinear form on $\mathrm{Pic}(X)$ as follows:

$$(H_i,E_j)=0, \quad (H_i,H_j)=(c-1)-\delta_{i,j}, \quad (E_i,E_j)=-\delta_{i,j}. \tag{2.1}$$

The following lemma is a straightforward calculation.

Lemma 2.1 [Muk04]. *$\mathrm{Pic}(X)$ has another $\mathbb{Z}$-basis $\alpha_1,\ldots,\alpha_{a+r-2},E_r$, where*

$$\alpha_1 = E_1-E_2,\ldots, \quad \alpha_{r-1}=E_{r-1}-E_r,$$
$$\alpha_r = H_1-E_1-\cdots-E_c,$$
$$\alpha_{r+1}=H_1-H_2,\ldots, \quad \alpha_{a+r-2}=H_{a-2}-H_{a-1}.$$

Moreover, $\alpha_1,\ldots,\alpha_{a+r-2}$ is a $\mathbb{Z}$-basis of the orthogonal complement $K^\perp$ and a system of simple roots of a finite root system with a Dynkin diagram $T_{a,b,c}$.

Let $\mathcal{W}$ be the Weyl group generated by orthogonal reflections with respect to $\alpha_1,\ldots,\alpha_{a+r-2}$. Then K is $\mathcal{W}$-invariant. Mukai calls $D\subset X$ a (-1)-divisor if there is a small modification $X \dashrightarrow X'$ such that D is the exceptional divisor for a blow-up $X'\to Y$ at a smooth point. Note that any (-1)-divisor must appear in any set of generators of $\mathrm{Eff}(X)$.

Lemma 2.2 [Muk04]. *For each transformation $w:\mathrm{Pic}(X)\to\mathrm{Pic}(X)$ of $\mathcal{W}$, there is a small modification $X \dashrightarrow X_w$ with the following property: X_w is also a blow-up of $(\mathbb{P}^{c-1})^{a-1}$ in $r=b+c$ points $q_1,\ldots,q_r$ in general position and the pull-back of the tautological basis of X_w coincides with the transformation of the tautological basis of X by w. In particular, every divisor $E\in\mathcal{W}\cdot E_r$ is a (-1)-divisor and $\mathrm{H}^0(X,E)$ is spanned by a single section x_E.*

The proof is an application of Cremona transformations. The case $a=2$ appeared in [Dol83] (where it is attributed to Coble). The case $a=2$, $c=3$ is well known from the theory of marked Del Pezzo surfaces.

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LEMMA 2.3. *The action of $\mathcal{W}$ on* $\mathrm{Pic}(X)$ *preserves* $\mathrm{Eff}(X)$.

Proof. Let $D \in \mathrm{Pic}(X)$ and $w \in \mathcal{W}$. We claim that $\mathrm{H}^0(X, D) \simeq \mathrm{H}^0(X, w \cdot D)$. We have

$$D = d_1 H_1 + \cdots + d_{a-1} H_{a-1} - m_1 E_1 - \cdots - m_r E_r.$$

Then $\mathrm{H}^0(X, D)$ can be identified with the subspace of polynomial functions on $(\mathbb{C}^c)^{a-1}$ of multidegree $(d_1, \ldots, d_{a-1})$ vanishing to the order at most m_i at the point p_i. By Lemma 2.2, $\mathrm{H}^0(X, w \cdot D)$ has the same interpretation for another choice of general points $q_1, \ldots, q_r$. Now the claim follows from semi-continuity if the points $p_1, \ldots, p_r$ are sufficiently general. □

Let $\mathrm{Eff}_{\mathbb{R}}(X) \subset \mathrm{Pic}(X) \otimes \mathbb{R}$ be the cone spanned by $\mathrm{Eff}(X)$. Let $\mathrm{N}_1(X)$ be the group generated over $\mathbb{Z}$ by 1-cycles on X modulo rational equivalence. Intersection of cycles gives a nondegenerate pairing $\mathrm{Pic}(X) \times \mathrm{N}_1(X) \to \mathbb{Z}$. For $i = 1, \ldots, a-1$, let $l_i \in \mathrm{N}_1(X)$ be the class of the proper transform of a general line in the ith copy of $\mathbb{P}^{c-1}$. For $i = 1, \ldots, r$, let $e_i \in \mathrm{N}_1(X)$ be the class of a general line in E_i. Then it is easy to check that

$$H_i \cdot l_j = \delta_{i,j}, \quad H_i \cdot e_j = 0, \quad E_i \cdot e_j = \delta_{i,j}. \tag{2.2}$$

Since the intersection pairing is nondegenerate, it follows that $\mathrm{N}_1(X)$ is generated over $\mathbb{Z}$ by the classes $l_1, \ldots, l_{a-1}, e_1, \ldots, e_r$. The action of $\mathcal{W}$ on $\mathrm{Pic}(X)$ induces an action on $\mathrm{N}_1(X)$.

A class γ in $\mathrm{N}_1(X)$ is called *nef* if, for any effective divisor D on X, $D \cdot \gamma \geqslant 0$.

LEMMA 2.4. *The classes l_i, $l_1 + \cdots + l_{a-1} - e_i$ are nef, for all $i = 1, \ldots, r$.*

Proof. Note that if a family of curves with class f covers X (i.e. through a general point of X there is an irreducible curve in the family that passes through it), then f is a nef class: if D is an effective divisor, there is an irreducible curve in the family that is not contained in D, therefore, $D \cdot f \geqslant 0$. This is obviously the case if $f = l_i$. If $f = l_1 + \cdots + l_{a-1} - e_i$, then f is the proper transform in X of a curve of multidegree $(1, \ldots, 1)$ in $(\mathbb{P}^{c-1})^{a-1}$ that passes through the point p_i. This family contains an irreducible curve by Bertini's theorem and we can use the 2-transitive action of $(\mathrm{PGL}_c)^{a-1}$ on $(\mathbb{P}^{c-1})^{a-1}$ to find a curve through any point. □

DEFINITION 2.5. Define the *degree* of $D \in \mathrm{Pic}(X)$ as an integer

$$\deg(D) = \frac{1}{ac - a - c}(D, -K).$$

Clearly, $\deg(D)$ is $\mathcal{W}$-invariant and any divisor in the orbit $\mathcal{W} \cdot E_r$ has degree 1.

DEFINITION 2.6. Let $\mathfrak{g}_{a,b,c}$ be a semisimple Lie algebra with the Dynkin diagram $T_{a,b,c}$. Let $\Lambda \subset K^{\perp} \otimes \mathbb{Q}$ be the *weight lattice* spanned by *fundamental weights* $\omega_1, \ldots, \omega_{a+r-2}$ defined by $(\omega_i, \alpha_j) = \delta_{i,j}$. For any $\omega \in \Lambda$, let L_ω be an irreducible $\mathfrak{g}_{a,b,c}$-module with the highest weight ω (see for example [VO90]). Then L_ω is called *minuscule* if weights $\mathcal{W} \cdot \omega$ are its only weights. Let $\pi : \mathrm{Pic}(X) \to K^{\perp} \otimes \mathbb{Q}$ denote the orthogonal projection.

THEOREM 2.7. $\mathrm{Eff}(X)$ *is generated as a semigroup by divisors of degree* 1. $\mathrm{Eff}_{\mathbb{R}}(X)$ *is generated as a cone by $D \in \mathcal{W} \cdot E_r$. Projection π induces a bijection between divisors of degree 1 and weights of $L_{\omega_{r-1}}$ such that divisors in $\mathcal{W} \cdot E_r$ correspond to weights in $\mathcal{W} \cdot \omega_{r-1}$. In particular, $L_{\omega_{r-1}}$ is minuscule if and only if the only effective divisors of degree 1 are $D \in \mathcal{W} \cdot E_r$.*

Remark 2.8. The classification of minuscule representations is well known. The only arising cases are

$$\mathrm{Bl}_{n+3}\mathbb{P}^n, \quad \mathrm{Bl}_{n+2}(\mathbb{P}^n)^s, \quad \text{and} \quad \mathrm{Bl}_s\mathbb{P}^2 \ (s = 4, 5, 6, 7).$$

If $X = \mathrm{Bl}_{n+3}\mathbb{P}^n$ then $L_{\omega_{r-1}}$ is a halfspinor representation of $\mathfrak{so}_{2n+6}$. Here is another example: let $X = X_{2,3,3}$ be the blow-up of $\mathbb{P}^2$ in six general points, i.e. a smooth cubic surface. Divisors of degree 1 are the 27 lines. The corresponding minuscule representation $L_{\omega_{r-1}}$ is the 27-dimensional representation of E_6 as a Lie algebra of infinitesimal norm similarities of the exceptional Jordan algebra.

Proof of Theorem 2.7. Let Γ_k be the intersection of the convex hull of $W \cdot (kE_r)$ with $\mathrm{Eff}(X)$ and let $\Gamma \subset \mathrm{Pic}(X) \otimes \mathbb{R}$ be the cone spanned by Γ_1. Since $\pi(E_r) = \omega_{r-1}$ and any element of $K^\perp$ is an integral combination of roots, it follows from the basic representation theory of semisimple Lie algebras [VO90] that $\pi(\Gamma_k)$ is the set of weights of an irreducible $\mathfrak{g}$-module $L_{k\omega_{r-1}}$ with the highest weight $k\omega_{r-1}$. Since $L_{k\omega_{r-1}} \subset L_{\omega_{r-1}}^{\otimes k}$ ($L_{k\omega_{r-1}}$ is the so-called *Cartan component* of $L_{\omega_{r-1}}^{\otimes k}$), any weight in $\pi(\Gamma_k)$ is a sum of k weights from $\pi(\Gamma_1)$, and therefore any divisor in Γ_k is a sum of k divisors from Γ_1. It follows that $\mathrm{Eff}(X) \cap \Gamma$ is generated by Γ_1 as a semigroup.

It remains to show that $\mathrm{Eff}(X)_\mathbb{R} \subset \Gamma$. We will find all faces of Γ and show that the inequalities that define them are satisfied by any effective divisor.

By Lemma 2.3, it suffices to find faces of Γ adjacent to the ray spanned by E_r up to the action of the stabilizer of E_r in $\mathcal{W}$. The algorithm for finding faces of these so-called Coxeter polytopes is explained, for example, in [Cas97, p. 9]. They are in one-to-one correspondence with connected maximal subdiagrams of $T_{a,b,c}$ that contain the support of the highest weight, i.e. the node that corresponds to the simple root α_{r-1} in our case. There are two types of such diagrams given by roots:

(1) $\alpha_2, \alpha_3, \ldots, \alpha_{a+r-2}$;

(2) $\alpha_1, \alpha_2, \ldots, \alpha_{a+r-3}$.

For each subdiagram, the linear span of the corresponding face is spanned by simple roots in the subdiagram and by E_r.

Using formulas (2.2), any face of Γ is given (up to the action of $\mathcal{W}$) by inequality

$$D \cdot f \geqslant 0, \tag{2.3}$$

where

(1) $f = l_1 + \cdots + l_{a-1} - e_1$;

(2) $f = l_{a-1}$.

By Lemma 2.4, the class f is nef. Hence, for any D effective, $D \cdot f \geqslant 0$. We conclude that (2.3) is, in fact, satisfied by any effective divisor and $\mathrm{Eff}_\mathbb{R}(X) = \Gamma$. $\square$

3. Proof of Theorem 1.3

The following is a direct generalization from [BP04, Proposition 4.4].

Proposition 3.1. *Let $\pi : X \to X'$ be the blow-up of a smooth point. Let $E \subset X$ be an exceptional divisor, and let $x_E \in \mathrm{H}^0(X, E) \subset \mathrm{Cox}(X)$ be the corresponding section. Then there is an isomorphism of rings*

$$\mathrm{Cox}(X)_{x_E} \cong \mathrm{Cox}(X')[T, T^{-1}].$$

Proof. Any divisor $D \in \mathrm{Pic}(X)$ can be uniquely written as $D = D_0 - mE$, where $D_0 \in \pi^*\mathrm{Pic}(X')$, $m \in \mathbb{Z}$. We identify $\mathrm{Pic}(X')$ with $\pi^*\mathrm{Pic}(X') \subset \mathrm{Pic}(X)$ and $\mathrm{Cox}(X')$ with $\pi^*\mathrm{Cox}(X') \subset \mathrm{Cox}(X)$. The latter embedding extends to a ring homomorphism

$$\mathrm{Cox}(X')[T, T^{-1}] \to \mathrm{Cox}(X)_{x_E}$$

by sending T to x_E. We show that this is an isomorphism by constructing an inverse to it.

If $m \geqslant 0$ and s is a section in $\mathrm{H}^0(X, D)$, then let $s_0 = s \cdot x_E^m \in \mathrm{H}^0(X', D_0)$. Define a map

$$\mathrm{H}^0(X, D) \to \mathrm{H}^0(X', D_0)T^{-m}, \quad s \mapsto s_0 T^{-m}.$$

If $m < 0$ then the canonical inclusion $\mathrm{H}^0(X, D_0) \hookrightarrow \mathrm{H}^0(X, D)$ is an isomorphism. To see this, note that for any $i \geqslant 0$ there is an exact sequence

$$0 \to \mathrm{H}^0(X, D_0 + iE) \to \mathrm{H}^0(X, D_0 + (i+1)E) \to \mathrm{H}^0(E, (D_0 + (i+1)E)|_E) = 0,$$

where the last equality follows from

$$\mathcal{O}(D_0)|_E = \mathcal{O}_E \quad \text{and} \quad \mathcal{O}(E)|_E = \mathcal{O}_E(-1).$$

Define a map $\mathrm{H}^0(X, D) \to \mathrm{H}^0(X', D_0)T^{-m}$ in the same way, by sending s to $s_0 \cdot T^{-m}$, where $s \in \mathrm{H}^0(X, D)$ is the image of a section $s_0 \in \mathrm{H}^0(X, D_0)$. This gives a map $\mathrm{Cox}(X) \to \mathrm{Cox}(X')[T, T^{-1}]$ which maps x_E to T. One can check directly that this is a ring homomorphism. The induced map $\mathrm{Cox}(X)_{x_E} \to \mathrm{Cox}(X')[T, T^{-1}]$ is the desired inverse. $\square$

Notation 3.2. In this section,

$$X = X_{a,b,c}.$$

PROPOSITION 3.3. $\mathrm{Cox}(X)$ *is a unique factorization domain (UFD).*

Proof. The Cox ring of a normal projective variety is known to be a UFD [EKW04]. We can also use a simple observation: the ring of invariants of a UFD with respect to the action of a connected algebraic group without nontrivial characters is a UFD (see [PV94]). By [Muk04], $\mathrm{Cox}(X)$ is a ring of invariants of an *extended Nagata action*. $\square$

DEFINITION 3.4. We define a $\mathbb{Z}$-*grading* of $\mathrm{Cox}(X)$ by $\deg(s) = \deg(D)$ for any $s \in \mathrm{H}^0(X, D)$. In particular, $\deg(x_E) = 1$ for any $E \in \mathcal{W} \cdot E_r$.

DEFINITION 3.5. Let $\mathrm{Cox}'(X) \subset \mathrm{Cox}(X)$ be a subalgebra generated by sections x_E, for $E \in \mathcal{W} \cdot E_r$. We say that $\mathrm{Cox}(X)$ is *minuscule* if $\mathrm{Cox}(X) = \mathrm{Cox}'(X)$.

DEFINITION 3.6. Let $\mathbb{P}(X) = \mathrm{Proj}(\mathrm{Cox}(X))$, $\mathbb{A}(X) = \mathrm{Spec}(\mathrm{Cox}(X))$, and $Z = \mathrm{Proj}(\mathrm{Cox}'(X))$, where $\mathrm{Cox}(X)$ and $\mathrm{Cox}'(X)$ are considered with their $\mathbb{Z}$-grading as in Definition 3.4 (we will show that in fact $Z \cong \mathbb{P}(X)$).

Inspecting the list of all possible $X_{a,b,c}$ given in § 1, we see that $X_{a,b-1,c}$ is contained in the following list:

(i) $X_{s+1,1,n+1} = \mathrm{Bl}_{n+2}(\mathbb{P}^n)^s$. This variety is minuscule; see Remark 3.9;

(ii) Del Pezzo surfaces $X_{2,s-3,3} = \mathrm{Bl}_s\mathbb{P}^2$, $s = 4, 5, 6, 7$. In this case $\mathrm{Cox}(X)$ is minuscule by a theorem of Batyrev and Popov [BP04];

(iii) $X_{2,2,4} = \mathrm{Bl}_6(\mathbb{P}^3)$, $X_{2,2,5} = \mathrm{Bl}_7(\mathbb{P}^4)$. These varieties are also minuscule by our Theorem 1.1 (which will be proved later).

Therefore, $X_{a,b-1,c}$ is minuscule in all cases.

Let $R = \mathrm{Cox}(X)$, $R' = \mathrm{Cox}'(X)$, and $R_0 = \mathrm{Cox}(X_{a,b-1,c})$. Let Q be the field of fractions of R. We claim that R is contained in all the localizations $R'_{x_E} \subset Q$. By Lemma 2.2, there is a small modification $\tilde{X}$ of X isomorphic to $\mathrm{Bl}_r(\mathbb{P}^{c-1})^{a-1}$, the blow-up of $(\mathbb{P}^{c-1})^{a-1}$ in $r = b + c$ points $q_1, \dots, q_r$ in general position, such that the pull-back of E is contracted to q_r. By Proposition 3.1, $R \subset (R_0)_{x_E}$. It remains to notice that $R_0 \subset R'$ because R_0 is minuscule.

CLAIM 3.7. *We claim that R is integral over R'.*

Proof. This is a standard proof; see for example [Har77, p. 123]. Let $z \in R$ be a homogeneous element of a positive degree. To show that z is integral over R', it suffices to find a faithful $R'[z]$-module M finitely generated as an R'-module. Let M be the set of elements in R' of degree greater than N, where N has to be chosen adequately. Obviously, M is an $R'[z]$-module if $zM \subset R'$. So choose N to be $kn+1$, where k is the number of generators x_i in R', and n is the maximum of integers n_i such that $zx_i^{n_i} \in R'$. Clearly, M is a finitely generated R'-module. Since R is a domain, M is of course a faithful $R'[z]$-module. $\square$

It follows that R is integral over R' and, therefore, R is finitely generated.

Now we prove the 'moreover' part of the theorem following Popov's proof [Pop04] of the analogous statement for Del Pezzo surfaces.

For each $E \in \mathcal{W} \cdot E_r$ consider the open chart $U_E(X) \subset Z$ given by $x_E \neq 0$. These charts cover Z. Let $U'_E(X) \subset \mathbb{P}(X)$ be a chart given by $x_E \neq 0$. Since R is integral over R', it is easy to see that the radical of the ideal of R generated by the x_E is the irrelevant ideal. It follows that charts $U'_E(X)$ cover $\mathbb{P}(X)$. Since $R \subset \bigcap_E R'_{x_E}$, we have $R'_{x_E} = R_{x_E}$ for any $E \in \mathcal{W} \cdot E_r$. It follows that, in fact, $U_E(X) \simeq U'_E(X)$, the inclusion $R' \subset R$ induces an isomorphism $\phi : \mathbb{P}(X) \to Z$, and $\phi^*\mathcal{O}_Z(m) \cong \mathcal{O}_{\mathbb{P}(X)}(m)$. Moreover, it is true in general that, if a graded ring R is a UFD and the irrelevant ideal is the radical of the ideal generated by degree 1 elements, then the Picard group of $\mathrm{Proj}(R)$ is $\mathbb{Z}$ and it is generated by $\mathcal{O}(1)$.

It follows from Proposition 3.1 that $U_E(X) \cong \mathbb{A}(X_{a,b-1,c})$ is factorial by Proposition 3.3. Therefore, Z is locally factorial and, in particular, Z is normal.

Arguing by induction on b, we can assume that all statements of Theorem 1.3 are satisfied for $Y = X_{a,b-1,c}$. Let $W = \mathbb{P}(Y)$. Thus W is a Cohen–Macaulay and Gorenstein scheme with rational singularities, $\mathrm{Pic}(W) = \mathbb{Z}$ is generated by $\mathcal{O}_W(1)$ and the anticanonical line bundle ω_W is ample.

LEMMA 3.8. *We have $H^i(W, \mathcal{O}(k)) = 0$ for $i \geqslant 1$, $k \geqslant 0$.*

Proof. Notice that $\mathcal{O}(k) = \omega_W \otimes L$ with L ample. Let $\pi : \tilde{W} \to W$ be a resolution of singularities. Then $H^i(\tilde{W}, \omega_{\tilde{W}} \otimes \pi^*(L)) = 0$ by Kodaira vanishing because $\pi^*(L)$ is big and nef. Now use the Leray spectral sequence and the definition of rational singularities ($R^i\pi_*\omega_{\tilde{W}} = 0$ for $i > 0$) to conclude that $H^i(W, \mathcal{O}(k)) = 0$. $\square$

Since Y is minuscule, W is projectively normal in the projective embedding given by $\mathcal{O}_W(1)$. Note that $U_E(X) \cong \mathbb{A}(Y)$ is an affine cone over Y. It follows that $\mathbb{A}(Y)$ has rational singularities by [KR87, Theorem 1] and therefore is Cohen–Macaulay [Kem73]. Since $\mathbb{A}(Y)$ is factorial and Cohen–Macaulay, it is Gorenstein [Eis95, Example 21.21].

It remains to calculate the anticanonical class of $\mathbb{P}(X)$. By [HK00], X is the GIT quotient of $\mathbb{A}(X)$ for the action of the torus $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m) = \mathbb{G}_m^{r+1}$. Moreover, X is the GIT quotient of $\mathbb{P}(X)$ for the induced action of $\mathbb{G}_m^r$. Let U be the semistable locus in $\mathbb{P}(X)$. Note that there are no strictly semistable points [HK00, Proposition 2.9]. It is easy to see by induction using charts $U_E(X)$ that $\mathbb{G}_m^r$ acts on $\mathbb{P}(X)$ with connected stabilizers. By Luna's étale slice theorem [MFK94, p. 199], this implies that $\pi : U \to X$ is a principal étale fiber bundle. In particular, U is smooth. By the general theory of Cox varieties [HK00, Proposition 2.9], $\mathbb{P}(X) \setminus U$ has codimension at least 2 in $\mathbb{P}(X)$, and therefore $\mathrm{Pic}(U) \cong \mathbb{Z}\{\mathcal{O}(1)\}$. By the GIT, the pull-back map π^* between the Picard groups is the map given by degree: $\pi^*(D) = \deg(D)$.

It is enough to prove that $K_U = \mathcal{O}_U(-d)$. Let $\mathcal{T}_X$ (respectively $\mathcal{T}_U$) be the tangent sheaf of X (respectively U). There is an exact sequence of locally free sheaves:

$$0 \to \mathcal{O}_U^r \to \mathcal{T}_U \to \pi^*\mathcal{T}_X \to 0$$

(the relative tangent sheaf of a principal étale bundle is canonically a trivial bundle with fiber isomorphic to the Lie algebra of $\mathbb{G}_m^r$). Taking Chern classes, it follows that $c_1(\mathcal{T}_U) = \pi^*(c_1(\mathcal{T}_X))$; hence, $-K_U = \pi^*(-K_X) = \mathcal{O}(d)$, where

$$d = \deg(-K_X) = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right). \qquad \square$$

Remark 3.9. Here we consider the case of $X = X_{s+1,1,n+1} = \mathrm{Bl}_{n+2}(\mathbb{P}^n)^s$. Then it is well known and easy to check that X is the GIT quotient of the Grassmannian $G(s+1, n+s+2)$. It follows from [HK00] that $\mathrm{Cox}(X)$ is isomorphic to the total coordinate ring of $G(s+1, n+s+2)$ which is generated by the $\binom{n+s+2}{s+1}$ Plücker coordinates. On the other hand, the orbit $\mathcal{W} \cdot E_r$ in this case consists of precisely $\binom{n+s+2}{s+1}$ divisors, the dimension of the minuscule representation of $\mathfrak{g}_{s+1,1,n+1} = \mathfrak{sl}_{n+s+2}$ in $L_{\omega_n} = \Lambda^{s+1}\mathbb{C}^{n+s+2}$. It follows that $\mathrm{Cox}(X)$ is minuscule.

4. Proof of Theorem 1.2

Notation 4.1. Let $X = \mathrm{Bl}_r\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ at r distinct points $p_1, \ldots, p_r$ ($r \geqslant n+3$) that lie on a rational normal curve C of degree n. Let $E_1, \ldots, E_r$ be the exceptional divisors and H the hyperplane class. Let

$$\alpha = r - n - 2.$$

Let $\tilde{C}$ be the proper transform of C on X.

Lemma 4.2. *Let $D \subset \mathbb{P}^n$ be a hypersurface of degree d that contains C with multiplicity m. If D has multiplicity m_i at p_i, $i = 1, \ldots, r$, then one has:*

$$m \geqslant \frac{\sum_{i=1}^r m_i - nd}{\alpha}.$$

Proof. Recall that the multiplicity of a divisor along a curve is the multiplicity at a general point of a curve. Let $\tilde{D}$ be the proper transform of D on X. Let $\pi' : X' \to X$ be the blow-up of X along $\tilde{C}$ and let E be the exceptional divisor. Then $E \cong \mathbb{P}(N_{\tilde{C}|X})$, where $N_{\tilde{C}|X}$ is the normal bundle of $\tilde{C}$ in X. One has

$$N_{C|\mathbb{P}^n} \cong \mathcal{O}(n+2)^{\oplus(n-1)},$$

and therefore

$$N_{\tilde{C}|X} \cong \pi^* N_{C|\mathbb{P}^n} \otimes \mathcal{O}_X(-E_1 - \cdots - E_r) \cong \mathcal{O}(n+2)^{\oplus(n-1)} \otimes \mathcal{O}(-r) \cong \mathcal{O}(-\alpha)^{\oplus(n-1)}.$$

It follows that $E \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$. Let

$$q_1 : \mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^1, \quad q_2 : \mathbb{P}^1 \times \mathbb{P}^{n-2} \to \mathbb{P}^{n-2}$$

be the two projections. Then $\mathcal{O}(E)_{|E} \cong q_1^*\mathcal{O}(-\alpha) \otimes q_2^*\mathcal{O}(-1)$.

Let D' be the proper transform of $\tilde{D}$ on X'. Then $\pi'^*\tilde{D} = D' + mE$. Denote

$$a = -\tilde{D}.\tilde{C} = \sum_{i=1}^r m_i - nd.$$

Note that $\pi'^*\mathcal{O}_X(\tilde{D})_{|E} = q_1^*\mathcal{O}(-a)$. Since $\mathcal{O}_{X'}(D')_{|E} = q_1^*\mathcal{O}(-a + m\alpha) \otimes q_2^*\mathcal{O}(m)$ is an effective divisor on E, it follows that $-a + m\alpha \geqslant 0$. Hence, $m \geqslant a/\alpha$. $\square$

Lemma 4.3. *Consider the divisor (1.3) on X. Then E is the proper transform of a unique hypersurface of degree k in $\mathbb{P}^n$ that has multiplicity k at any p_i with $i \in I$ and $k-1$ at all other points of C. In particular, $\mathrm{H}^0(X, E) \cong \mathbb{C}$ and $E - E_i$ is not effective for any $i = 1, \ldots, r$.*

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Proof. Let $J \subset I^c$ be any subset with $|J| = 2k+1$. The divisor

$$E' = kH - k\sum_{i\in I} E_i - (k-1)\sum_{i\in J} E_i$$

is an effective divisor of degree 1 on the blow-up $\mathrm{Bl}_{n+3}\mathbb{P}^n$ of $\mathbb{P}^n$ along the points p_i for $i \in I \cup J$. It follows that $\mathrm{h}^0(X, E') = 1$ and, for any $i \in I \cup J$, the divisor $E - E_i$ is not effective. It follows that E is the proper transform of a unique hypersurface Z of degree k in $\mathbb{P}^n$ such that

$$\mathrm{mult}_{p_i} Z = k \quad (i \in I) \quad \text{and} \quad \mathrm{mult}_{p_i} Z = k-1 \quad (i \in J).$$

Since Z is the image of E, and therefore does not depend on the choice of J, we have $\mathrm{mult}_{p_i} Z = k-1$ for any $i \in I^c$. If p is a point on C different from $p_1, \dots, p_r$, consider the variety $\mathrm{Bl}_{r+1}\mathbb{P}^n$ that is the blow-up of X at p. Let E_{r+1} be the exceptional divisor. By applying the same argument to the divisor $E - (k-1)E_{r+1}$ on $\mathrm{Bl}_{r+1}\mathbb{P}^n$, it follows that the multiplicity of Z at p is exactly $k-1$. $\square$

Definition 4.4. We call the divisors E in (1.3) *minimal* divisors on $\mathrm{Bl}_r\mathbb{P}^n$. We call an element in $\mathrm{Cox}(X)$ a *distinguished section* if it is a monomial in the sections $x_E \in \mathrm{H}^0(X, E)$, where E is either a minimal divisor on X or an exceptional divisor E_i. The ring $\mathrm{Cox}(X)$ is minuscule if it is generated by distinguished sections.

We prove that $\mathrm{Cox}(X)$ is minuscule by induction on n and r. Theorem 4.23 proves this for $n = 2$. Assume from now on that $n \geqslant 3$.

Definition 4.5. Let

$$D = dH - \sum_{i=1}^{r} m_i E_i \tag{4.1}$$

be any divisor on X. We call d the *H-degree* of D, denoted by $\mathrm{hdeg}(D)$.

Notation 4.6. Consider the projection $\pi_1 : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ from p_1 and let $q_i = \pi(p_i)$ for $i = 2, \dots, r$. Note that $q_2, \dots, q_r$ lie on a rational normal curve $\pi_1(C)$ of degree $n-1$ in $\mathbb{P}^{n-1}$. Let $Y = \mathrm{Bl}_{r-1}\mathbb{P}^{n-1}$ be the blow-up of $\mathbb{P}^{n-1}$ at $q_2, \dots, q_r$. Let $\overline{E}_2, \dots, \overline{E}_r$ be the exceptional divisors on Y and $\overline{H}$ the hyperplane class. Consider the linear map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ that maps (4.1) to

$$\tilde{D} = m_1\overline{H} - \sum_{i=2}^{r} (m_i + m_1 - d)\overline{E}_i. \tag{4.2}$$

Lemma 4.7. *If $\mathrm{hdeg}(D) = \mathrm{hdeg}(D')$ and $\tilde{D} = \tilde{D}'$ then $D = D'$.*

Proof. This is because, by (4.2), $\tilde{\Delta} = 0$ implies that $\Delta = e(H - \sum_{i=1}^{r} E_i)$, for some $e \in \mathbb{Z}$. Hence, if the H-degree of Δ is 0, then $\Delta = 0$. $\square$

Lemma 4.8. *There is a map r that makes the following diagram commutative.*

$$\begin{tikzcd}
\mathrm{H}^0(X, D) \arrow[r, "r"] \arrow[d, "r'"'] & \mathrm{H}^0(Y, \tilde{D}) \arrow[d, "i"] \\
\mathrm{H}^0(E_1, D_{|E_1}) \arrow[r, equal] & \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}(m_1))
\end{tikzcd}$$

Here r' is the restriction map and i is the canonical injective map given by push-forward. For any divisors D_1, D_2 on X and $s_1 \in \mathrm{H}^0(X, D_1)$, $s_2 \in \mathrm{H}^0(X, D_2)$, if $D = D_1 + D_2$, then

$$\tilde{D} = \tilde{D}_1 + \tilde{D}_2, \quad r(s_1 s_2) = r(s_1) r(s_2).$$

Proof. We can identify E_1 with the image of the projection π_1 and view r' as a map

$$r' : \mathrm{H}^0(X, D) \to \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}(m_1)) = \mathrm{H}^0(Y, m_1\overline{H}).$$

Let $x_{\overline{E}_i}$ be a generator for $\mathrm{H}^0(Y, \overline{E}_i) \cong \mathbb{C}$. Note that if for some $i = 2, \ldots, r$ one has $m_1 + m_i - d > 0$, then the image of r' lies in the linear subsystem

$$|m_1\overline{H} - (m_1 + m_i - d)\overline{E}_i| \subset |m_1\overline{H}|$$

and therefore $r'(s)$ is divisible by $x_{\overline{E}_i}^{-d+m_1+m_i}$ for any $s \in \mathrm{H}^0(X, D)$. It follows that we can formally define

$$r(s) = r'(s) \prod_{i=2}^{r} x_{\overline{E}_i}^{d-m_1-m_i}.$$

The last statement of the lemma is clear. □

Remark 4.9. The geometric interpretation for the map r is as follows. Let $l_{i,j}$ be the proper transform on X of the line in $\mathbb{P}^n$ joining the points p_i and p_j. Then $q_2, \ldots, q_r$ are the points on $E_1 \cong \mathbb{P}^{n-1}$ where $l_{1,2}, l_{1,3}, \ldots, l_{1,n}$ intersect E_1. Let $\tilde{X}$ be the blow-up of X along $l_{1,2}, \ldots, l_{1,n}$ and let $E_{1,2}, \ldots, E_{1,n}$ be the exceptional divisors. The normal bundle $N_{l_{i,j}|X}$ of $l_{i,j} \cong \mathbb{P}^1$ in X is $\mathcal{O}(-1)^{\oplus(n-1)}$. The exceptional divisors $E_{1,j}$ are given by:

$$E_{1,i} \cong \mathbb{P}(N_{l_{i,j}|X}) \cong l_{i,j} \times \mathbb{P}^{n-2} \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}.$$

For any $n \geqslant 3$, there is morphism $\tilde{X} \to X'$ that contracts all the divisors $E_{1,i}$ using the projection onto $\mathbb{P}^{n-2}$. There is an induced rational map $\psi : X \dashrightarrow X'$ that is an isomorphism in codimension 1. Let $E_1' = \psi(E_1)$. Then $E_1' \cong Y$. In fact, the rational map $X \dashrightarrow Y$ is resolved by this flip and induces a regular map $X' \to Y$ that is a $\mathbb{P}^1$-bundle, with E_1' as a section. If D is a divisor on X, let $D' = \psi(D)$. Using geometric arguments, one checks that on $E_1' \cong Y$ one has $D'_{|E_1'} = \tilde{D}$ when $D = H$, $H - E_1$, E_i, for $i = 2, \ldots, r$. Hence, the formula holds in general by linearity. Then r is the composition of the isomorphism $\mathrm{H}^0(X, D) \cong \mathrm{H}^0(X', D')$ with the restriction map $\mathrm{H}^0(X, D') \to \mathrm{H}^0(E_1', D'_{|E_1'})$.

Notation 4.10. Let $q = \tilde{C} \cap E_1$. Obviously, $q \in \pi_1(C)$. Let $Y' = \mathrm{Bl}_r\mathbb{P}^{n-1}$ be the blow-up of Y at q and let E_q be the exceptional divisor.

Lemma 4.11. *Let E be a minimal divisor on X of H-degree k. Then $E \cdot (l - e_1)$ is either 0 or 1. In the first case, $\tilde{E}$ is a minimal divisor on Y. In the second case, the divisor $E' = \tilde{E} - (k-1)E_q$ is minimal on Y', except when $k = 1$. In the latter case, one has:*

$$E = H - \sum_{i \in I} E_i, \quad \tilde{E} = \sum_{i \in I^c} \overline{E}_i, \quad I \subset \{2, \ldots, r\}, \quad |I| = n, \quad |I^c| = \alpha + 1. \tag{4.3}$$

Proof. In the first case,

$$E = kH - kE_1 - k\sum_{i \in I} E_i - (k-1)\sum_{i \in I^c} E_i, \tag{4.4}$$

where $I \subset \{2, \ldots, r\}$, $|I| = n + 1 - 2k$, and

$$\tilde{E} = k\overline{H} - k\sum_{i \in I} \overline{E}_i - (k-1)\sum_{i \in I^c} \overline{E}_i. \tag{4.5}$$

In the second case

$$E = kH - (k-1)E_1 - k\sum_{i \in I} E_i - (k-1)\sum_{i \in I^c} E_i,$$

where $I \subset \{2, \ldots, r\}$, $|I| = n + 2 - 2k$, and

$$\tilde{E} = (k-1)\overline{H} - (k-1)\sum_{i \in I} \overline{E}_i - (k-2)\sum_{i \in I^c} \overline{E}_i. \tag{4.6}$$

Let $s \in \mathrm{H}^0(Y, \tilde{E})$ be the image of the section x_E via the map r of Lemma 4.8. Let Z be the zero locus of s. By Lemma 4.3, the divisor E has multiplicity $k-1$ along $\tilde{C}$. Therefore,

$$\mathrm{mult}_q Z = \mathrm{mult}_q E \cap E_1 \geqslant \mathrm{mult}_q E \geqslant \mathrm{mult}_{\tilde{C}} E = k-1.$$

It follows that the image of r is in each case contained in the push-forward of the linear system $|E'|$ on Y'. Except in the second case when $k=1$, E' is minimal on Y'. □

We prove that $\mathrm{H}^0(X, D)$ is generated by distinguished sections for any effective divisor D.

Claim 4.12. *We may assume that* $0 < m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r$.

Proof. Indeed, if $m_i \leqslant 0$ for some i, then $\mathrm{H}^0(X, D) \cong \mathrm{H}^0(X, D_0)$, where $D_0 = D + m_i E_i$ is a divisor on $\mathrm{Bl}_{r-1}\mathbb{P}^n$. The ring $\mathrm{Cox}(\mathrm{Bl}_{r-1}\mathbb{P}^n)$ is minuscule: this follows by Remark 3.9 if $r = n+3$, and by induction if $r > n+3$. Hence, $\mathrm{H}^0(X, D_0)$ is generated by distinguished sections. □

Claim 4.13. *It suffices to prove that any distinguished section in the image of*

$$r : H^0(X, D) \to H^0(Y, \tilde{D})$$

can be lifted to a linear combination of distinguished sections.

Proof. Since $\mathrm{Cox}(Y)$ is minuscule by induction and the kernel of r is $\mathrm{H}^0(X, D - E_1)$, we are then reduced to showing that $\mathrm{H}^0(X, D - E_1)$ is generated by distinguished sections. If $D - E_1$ is effective, we may replace D with $D - E_1$ and repeat the process. The process stops only when $D - E_1$ is not effective, in which case r_{E_1} is an isomorphism onto its image. Since for any effective D, one has $D.(l - e_i) = d - m_i \geqslant 0$, for all i, the process must stop. □

Notation 4.14. We denote

$$m = \max\left\{\left\lceil \frac{\sum_{i=1}^r m_i - nd}{\alpha} \right\rceil, 0\right\}. \tag{4.7}$$

Proposition 4.15. *If* $m = 0$, *then* r *surjects onto* $\mathrm{H}^0(Y, \tilde{D})$ *and any distinguished section* $s \in \mathrm{H}^0(Y, \tilde{D})$ *can be lifted to a distinguished section.*

Proof. The section s is a monomial in the sections corresponding to minimal divisors on Y and sections $x_{\overline{E}_i}$, $i = 2, \ldots, r$; hence, it corresponds to a decomposition

$$\tilde{D} = S + \sum_{i=2}^r l_i \overline{E}_i, \tag{4.8}$$

where $l_i \geqslant 0$ and S is a sum of minimal divisors on Y. Denote

$$\beta = d - m_1 = D.(l - e_1) \geqslant 0.$$

We now need the following lemma, before completing the proof.

Lemma 4.16. *We have* $l_i \leqslant \beta$ *and* $\sum_{i=2}^r l_i \geqslant (\alpha+1)\beta$.

Proof. For each $k \geqslant 0$, let $a_k \geqslant 0$ be the number of minimal divisors of H-degree k that appear in S. Since $\tilde{D}$ and S have the same H-degree,

$$m_1 = \sum_{k \geqslant 1} k a_k.$$

By counting the number of the $\overline{E}_i$ on both sides of (4.8), one has the following formula:

$$\sum_{i=2}^r l_i = (\alpha+1)\beta + \left(nd - \sum_{i=1}^r m_i\right) + \alpha\left(m_1 - \sum_{k \geqslant 1} a_k\right). \tag{4.9}$$

Since $m = 0$ and
$$m_1 = \sum_{k\geqslant 1} k a_k \geqslant \sum_{k\geqslant 1} a_k,$$
it follows that
$$\sum_{i=2}^{r} l_i \geqslant (\alpha+1)\beta.$$
Finally,
$$d - m_i = \tilde{D}.(l - e_i) = S.(l - e_i) + l_i \geqslant l_i,$$
and therefore
$$l_i \leqslant (d - m_i) \leqslant (d - m_1) = \beta.$$ □

We lift the minimal divisors (4.5) on Y to minimal divisors (4.4) on X of the same H-degree. Let D_0 be the divisor on X equal to the sum of the lifts of all terms of S. Hence, $S = \tilde{D}_0$ and
$$\operatorname{hdeg}(D) - \operatorname{hdeg}(D_0) = \operatorname{hdeg}(D) - \operatorname{hdeg}(\tilde{D}) = \beta. \tag{4.10}$$

By Lemmas 4.16 and 4.17, we may lift $\sum_{i=2}^{r} l_i \overline{E}_i$ to an effective divisor D_1 on X, with $\operatorname{hdeg}(D_1) = \beta$. Let $D' = D_0 + D_1$. Then D' has the same H-degree as D. Since $\tilde{D}' = \tilde{D}$, it follows from Lemma 4.7 that $D = D'$. By construction, there is a distinguished section t in $\mathrm{H}^0(X, D)$ such that $r(t) = s$. This completes the proof of Proposition 4.15. □

LEMMA 4.17. *Consider the divisor $\sum_{i=2}^{r} l_i \overline{E}_i$ on Y and assume that*
$$l_i \leqslant \beta \quad (i = 2, \ldots, r), \qquad \sum_{i=2}^{r} l_i \geqslant (\alpha+1)\beta.$$
Then we may lift $\sum_{i=2}^{r} l_i \overline{E}_i$ to an effective divisor D_1 on X with $\operatorname{hdeg}(D_1) = \beta$. Moreover, there is a distinguished section $t \in \mathrm{H}^0(X, D_1)$ such that $r(t) = \prod_{i=2}^{r} x_{\overline{E}_i}^{l_i}$.

Proof. For all $i = 2, \ldots, r$, we may write $l_i = l_i' + l_i''$, for some $l_i', l_i'' \geqslant 0$, such that $0 \leqslant l_i' \leqslant \beta$ and $\sum_{i=2}^{r} l_i' = (\alpha+1)\beta$. By partitioning $\sum_{i=2}^{r} l_i' E_i$ into a sum of $(\alpha+1)$-tuples of the form $E_{i_1} + \cdots + E_{i_{\alpha+1}}$ (the precise procedure for the partitioning is explained in the proof of Lemma 4.24), we may lift $\sum_{i=2}^{r} l_i' \overline{E}_i$ using (4.3) to a divisor D_1' on X which is a sum of β 'hyperplane classes' $H - \sum E_i$. Hence, $\operatorname{hdeg}(D_1') = \beta$. Moreover, there is a distinguished section $t' \in \mathrm{H}^0(X, D_1')$ such that $r(t') = \prod_{i=2}^{r} x_{\overline{E}_i}^{l_i'}$. Let $D_1 = D_1' + \sum_{i=2}^{r} l_i'' E_i$ and $t = t' \prod_{i=2}^{r} x_{\overline{E}_i}^{l_i''}$. Since $\tilde{E}_i = \overline{E}_i$ and $r(x_{E_i}) = x_{\overline{E}_i}$, for all $i = 2, \ldots, r$, the lemma follows. □

PROPOSITION 4.18. *Let $m > 0$. Then the image of r is the push-forward of $\mathrm{H}^0(Y', \tilde{D} - mE_q)$, and we may lift any distinguished section $s \in \mathrm{H}p(Y', \tilde{D} - mE_q)$ to a section t in the subspace of $\mathrm{H}p(X, D)$ generated by distinguished sections. By lift, here we mean that $r(t) = s.x_{E_q}^m$.*

Proof. By Lemma 4.2, the multiplicity of $D_{|E_1}$ at q is at least m. Hence, the map r has image in $\mathrm{H}^0(Y', \tilde{D} - mE_q)$.

We need the following lemma before completing the proof.

LEMMA 4.19. *If E' is a minimal divisor on Y' of H-degree $k \geqslant 1$, then the multiplicity at q of a push-forward of E' to Y is either:*

(1) *$k - 1$, or*

(2) *k.*

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The push-forward is equal to $\tilde{E}$, where E is a minimal divisor on X of H-degree k in case (1) and $k+1$ in case (2).

Proof. We may lift E' using (4.5) and (4.6) to a minimal divisor E on X by:

(1) $E = kH - k\sum_{i\in I} E_i - (k-1)\sum_{i\in I^c} E_i, \quad |I| = n+2-2k, \quad 1 \in I;$

(2) $E = (k+1)H - (k+1)\sum_{i\in I} E_i - k\sum_{i\in I^c} E_i, \quad |I| = n-2k, \quad 1 \in I^c.$

Then $r(x_E) = x_{E'}x_{E_q}^{k-1}$ in case (1) and $r(x_E) = x_{E'}x_{E_q}^{k}$ in case (2). □

Let S be the sum of the minimal divisors E' on Y' whose sections $x_{E'}$ appear in s. Then

$$\tilde{D} - mE_q = S + \sum_{i=2}^{r} l_i\overline{E}_i + aE_q \tag{4.11}$$

for some integers $l_i, a \geqslant 0$. Hence, the section s in $\mathrm{H}^0(Y', \tilde{D} - mE_q)$ is of the form $s'x_{E_q}^a$, for s' a section in $\mathrm{H}^0(Y', \tilde{D} - (a+m)E_q)$. So it is enough to show that we may lift sections $s = s'x_{E_q}^a$, with s' a distinguished section in $\mathrm{H}^0(Y', \tilde{D} - (a+m)E_q)$.

The above lifting $\tilde{E} = E'$ constructs a divisor D_0 on X which lifts S, i.e. $S = \tilde{D}_0$.

Notation 4.20. We denote

$$\beta = \mathrm{hdeg}(D) - \mathrm{hdeg}(D_0).$$

If $\beta = 0$, from Lemma 4.7 and $\tilde{E}_i = \overline{E}_i$ and $r(x_{E_i}) = x_{\overline{E}_i}$, for all $i = 2, \dots, r$, it follows that $D = D_0 + \sum_{i=2}^{r} l_iE_i$ and we may lift s to a distinguished section in $\mathrm{H}^0(X, D)$. For the general case, it is enough to show that, by eventually rewriting s as a sum of distinguished sections in $\mathrm{H}^0(Y, \tilde{D})$ corresponding to different decompositions of $\tilde{D} - mE_q$, we may reduce to the case when $l_i = l_i' + l_i''$, for some $l_i', l_i'' \geqslant 0$, such that $0 \leqslant l_i' \leqslant \beta$ and $\sum_{i=2}^{r} l_i' = (\alpha+1)\beta$. Then we can finish the proof by using Lemma 4.17.

For each $k \geqslant 1$, let $a_k \geqslant 0$ (respectively $b_k \geqslant 0$), be the number of divisors E' as in case (1) (respectively case (2)) of Lemma 4.19, whose sections $x_{E'}$ appear in the monomial s (taken with multiplicities). One has the following relations:

$$0 = \mathrm{hdeg}(\tilde{D}) - \mathrm{hdeg}(S) = m_1 - \sum_{k\geqslant 1} ka_k - \sum_{k\geqslant 1} kb_k; \tag{4.12}$$

$$\beta = \mathrm{hdeg}(D) - \mathrm{hdeg}(D_0) = d - \sum_{k\geqslant 1} ka_k - \sum_{k\geqslant 1}(k+1)b_k = d - m_1 - \sum_{k\geqslant 1} b_k. \tag{4.13}$$

Note that by finding the coefficients of E_q on both sides of the expression in (4.11), one has the following relation:

$$m + a = \sum_{k\geqslant 1}(k-1)a_k + \sum_{k\geqslant 1} kb_k. \tag{4.14}$$

By counting the number of the $\overline{E}_i$ on both sides of (4.11) and using (4.12) and (4.14), one has

$$\sum_{i=1}^{r} m_i - nd = (\alpha+1)\beta + (m+a)\alpha - \sum_{i=2}^{r} l_i. \tag{4.15}$$

Claim 4.21. *We may assume that $a = 0$ or $\sum_{k\geqslant 1} b_k = 0$.*

Proof. Assume $a>0$ and $b_k>0$, for some $k\geqslant 1$. Then the monomial s contains a section $x_{E'}$, where E' is a minimal divisor of the form

$$E'=k\overline{H}-k\sum_{i\in I}\overline{E}_i-(k-1)\sum_{i\in I^c}\overline{E}_i-kE_q,$$

where $I\subset\{2,\dots,r\}$, $|I|=n-2k$. By Lemma 4.22, applied to the divisor $E'+E_q$, we may replace the section $x_{E'}x_{E_q}$ with a linear combination of sections of the form $x_{E''}x_{\overline{E}_j}$, where $j\in\{2,\dots,r\}$ and $E''=E'+E_q-\overline{E}_j$. Then E'' is a minimal divisor as in case (1) of Lemma 4.19. Hence, we may replace s with a linear combination of distinguished sections with smaller a and smaller $\sum_{k\geqslant 1}b_k$.

Assume $\sum b_k=0$. Then $\beta=d-m_1\geqslant 0$. It follows that $\beta\geqslant l_i\geqslant 0$, for all $i=2,\dots,r$. This is because one has from (4.11)

$$d-m_i=\tilde{D}.(l-e_i)=S.(l-e_i)+l_i\geqslant l_i. \tag{4.16}$$

Hence, $l_i\leqslant d-m_i\leqslant\beta$, for all $i=2,\dots,r$.

By definition (4.7), one has $0\leqslant m\alpha-(\sum_{i=1}^r m_i-nd)$. From (4.15) it follows that

$$(\alpha+1)\beta\leqslant\sum_{i=2}^r l_i.$$

We are done by Lemma 4.17.

Assume $a=0$. We show that in this case $\beta\geqslant 0$. By definition (4.7), one has $0\leqslant m\alpha-(\sum_{i=1}^r m_i-nd)<\alpha$. From (4.15) it follows that

$$0\leqslant\sum_{i=2}^r l_i-(\alpha+1)\beta<\alpha.$$

It follows that $\beta\geqslant 0$. We find $l_i',l_i''\geqslant 0$ such that $l_i=l_i'+l_i''$ and $l_i'\leqslant\beta$, for all $i=2,\dots,r$ and $\sum_{i=2}^r l_i'=(\alpha+1)\beta$. First, randomly choose l_i',l_i'' with $l_i=l_i'+l_i''$, $l_i',l_i''\geqslant 0$ and $\sum_{i=2}^r l_i'=(\alpha+1)\beta$. We show that, by eventually replacing s with a linear combination of distinguished sections (with smaller l_i'), we may reduce to the case when $l_i'\leqslant\beta$, for all i. First take the case when $i\in\{2,\dots,r\}$ is such that in S there is no minimal divisor E' of the form

$$E'=k\overline{H}-k\overline{E}_i-k\sum_{j\in I}\overline{E}_j-(k-1)\sum_{j\in J}\overline{E}_j-kE_q, \tag{4.17}$$

where $I\subset\{2,\dots,r\}$, $i\notin I$ and $|I|=n-1-2k$, $J=\{2,\dots,r\}\setminus(\{i\}\cup I)$. We claim that $l_i\leqslant\beta$. Since, in each E' appearing in S, the divisor $\overline{E}_i$ appears with coefficient $-(k-1)$, one has

$$d-m_i=\tilde{D}.(l-e_i)=S.(l-e_i)+l_i\geqslant\sum_{k\geqslant 1}b_k+l_i.$$

It follows that $l_i\leqslant(d-m_i)-\sum_{k\geqslant 1}b_k\leqslant\beta$.

Assume now that $i\in\{2,\dots,r\}$ is such that $l_i'>\beta$. By the previous observation, S contains at least one minimal divisor E' of the form (4.17). By Lemma 4.22 applied to the divisor $E'+\overline{E}_i$, we may replace the section $x_{E'}x_{\overline{E}_i}$ with a linear combination of sections of the form $x_{E''}x_{\overline{E}_j}$, where $j\in J$ and $E''=E'+E_q-\overline{E}_j$ is a minimal divisor on Y'. Moreover, we claim that we may choose only indices $j\in J$ with $l_j'<\beta$. Let us call $j\in\{2,\dots,r\}$ a *good* index if $l_j'<\beta$. We claim that there are at least $k+1$ good indices in J. Clearly, $|J|=r-n+2k-1\geqslant k+1$. Assume there are at most k good indices in J. Then there are at least $(r-n+k-1)=(\alpha+k+1)$ indices in J that are not good. Since $l_i'>\beta$ and $i\notin J$, it follows that

$$(\alpha+1)\beta=\sum_{i=2}^r l_i'>(\alpha+k+1)\beta+\beta\geqslant(\alpha+1)\beta,$$

which is a contradiction. Hence, the claim follows. By repeating the process, we end up with $l_i' \leqslant \beta$, for all $i = 2, \dots, r$, and we are done by Lemma 4.17. $\square$

This completes the proof of Proposition 4.18. $\square$

Lemma 4.22. *Let $X = \mathrm{Bl}_r\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ in $r \geqslant n+4$ points on a rational normal curve C of degree n. For any $1 \leqslant k \leqslant (n+1)/2$ and any $I \subset \{1, \dots, r\}$, $|I| = n+1-2k$, let*

$$D = kH - k\sum_{i\in I} E_i - (k-1)\sum_{i\in I^c} E_i.$$

Then $h^0(X, D) = k+1$. For any $i \in I^c$, the divisor $D - E_i$ is minimal and, for any choice of $k+1$ indices $i \in I^c$, the sections $x_{D-E_i}x_{E_i}$ generate $\mathrm{H}^0(X, D)$.

Proof. Consider the exact sequence

$$0 \to \mathrm{H}^0(X, D - E_i) \to \mathrm{H}^0(X, D) \to \mathrm{H}^0(E_i, D_{|E_i}). \tag{4.18}$$

We argue by induction on $n \geqslant 2$. If $n = 2$, then $k = 1$ and $D = H - E_j$, for some $j \in \{1, \dots, r\}$. Clearly, for any $i \neq j$, the divisor $H - E_i - E_j$ is minimal. Since $(H - E_j).E_i = 0$, one has $\mathrm{H}^0(E_i, D_{|E_i}) \cong \mathbb{C}$. For any $l \neq i, j$, the section $x_{H-E_j-E_l}x_{E_l}$ has nonzero restriction to E_i. Hence, the map $\mathrm{H}^0(X, D) \to \mathrm{H}^0(E_i, D_{|E_i})$ is surjective and $\mathrm{H}^0(X, D)$ is generated by the sections $x_{H-E_i-E_j}x_{E_i}$ and $x_{H-E_j-E_l}x_{E_l}$.

Assume $n \geqslant 3$. Fix some $i \in I^c$. From Lemma 4.3 the divisor $E = D - E_i$ is a minimal divisor. Let $Y = \mathrm{Bl}_{r-1}\mathbb{P}^{n-1}$ be the blow-up of $\mathbb{P}^{n-1}$ at $r-1$ points corresponding to the projection from p_i and let $Y' = \mathrm{Bl}_r\mathbb{P}^{n-1}$ be the blow-up of Y at the extra point q. Then the restriction map in (4.18) factors through the map $r_{E_i} : \mathrm{H}^0(X, D) \to \mathrm{H}^0(Y, \tilde{D})$, where

$$\tilde{D} = (k-1)H - (k-1)\sum_{j\in I} E_j - (k-2)\sum_{j\in I^c\setminus\{i\}} E_j.$$

Note that, by Lemma 4.2, the multiplicity at q of any divisor in the linear system $|D|$ is at least

$$\frac{k(n+1-2k) + (k-1)(r+2k-n-1) - nk}{r-n-2} = k - 1 - \frac{1}{r-n-2}.$$

Since $r \geqslant n+4$, the map r_{E_i} has image in $\mathrm{H}^0(Y', D')$, where $D' = \tilde{D} - (k-1)E_q$. By induction, $\mathrm{H}^0(Y', D')$ has dimension k and it is generated by any distinct k sections of the form $x_{E'}x_{E_j}$, where $E' = D' - E_j$ and $j \in I^c \setminus \{i\}$. On X, the divisor $E = D - E_j$ is minimal. By Lemma 4.19, $r_{E_i}(x_E) = x_{E'}x_{E_q}^{k-1}$. Since $r_{E_i}(x_{E_j}) = x_{E_j}$, it follows that $r_{E_i}(x_E x_{E_j}) = x_{E'}x_{E_j}x_{E_q}^{k-1}$. Hence, the map r_{E_i} has image $\mathrm{H}^0(Y', D')x_{E_q}^{k-1}$. Therefore, $\mathrm{H}^0(X, D)$ has dimension $k+1$ and it is generated by any $k+1$ sections of the form $x_E x_{E_i}$, where $i \in I^c$, $E = D - E_i$. $\square$

Theorem 4.23. *Let $r \geqslant 5$ and let $X = \mathrm{Bl}_r\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at r distinct points $p_1, \dots, p_r$ that lie on an irreducible conic. Then $\mathrm{Cox}(X)$ is minuscule.*

Proof. Let C be the proper transform on X of the conic in $\mathbb{P}^2$ that contains the points $p_1, \dots, p_r$. Then $C = 2H - \sum_{i=1}^r E_i$. For any $i, j \in \{1, \dots, r\}$ with $i \neq j$, let $L_{i,j}$ be the proper transform on X of the line that passes through p_i, p_j. The classes C and $L_{i,j}$ are the minimal divisors on X. Let x_C (respectively $x_{L_{i,j}}$) be the corresponding sections. A distinguished section on X is a monomial in x_C, $x_{L_{i,j}}$ and x_{E_i}, for all i, j.

We prove by induction on r that $\mathrm{Cox}(\mathrm{Bl}_r\mathbb{P}^2)$ is generated by distinguished sections. The case $r = 5$ was proved in [BP04]. Assume $r \geqslant 6$.

Let D be an effective divisor (4.1) on X. If $m_i = D.E_i \leqslant 0$ for some $i \in \{1, \dots, r\}$, then $\mathrm{H}^0(X, D) \cong \mathrm{H}^0(X, D_0)$, where $D_0 = D + m_iE_i$ is a divisor on $\mathrm{Bl}_{r-1}\mathbb{P}^n$ and $\mathrm{H}^0(X, D_0)$ is generated by

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distinguished sections by induction. It follows that $\mathrm{H}^0(X,D)$ is generated by distinguished sections (obtained by multiplying sections of $\mathrm{H}^0(X,D_0)$ by $x_{E_i}^{-m_i}$). Hence, we may assume that $d, m_i > 0$ and argue by induction on d.

From the exact sequence

$$0 \to \mathrm{H}^0(X, D-C) \to \mathrm{H}^0(X,D) \to \mathrm{H}^0(C, D_{|C})$$

it follows that, if $D.C = -a < 0$, then $\mathrm{H}^0(C, D_{|C}) = 0$ and $\mathrm{H}^0(X,D) \cong \mathrm{H}^0(X, D-C)$ is generated by global sections by induction.

Assume now $D.C = 2d - \sum_{i=1}^r m_i \geqslant 0$ and $m_i > 0$ for all $i = 1, \dots, r$. Without loss of generality, we may assume $m_1 \leqslant m_i$, for all i. Consider the exact sequence

$$0 \to \mathrm{H}^0(X, D-E_1) \to \mathrm{H}^0(X,D) \to \mathrm{H}^0(E_1, D_{|E_1}).$$

Note that $\mathrm{H}^0(E_1, D_{|E_1}) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m_1))$. For $i = 2, \dots, r$, let $q_i = L_{1,i} \cap E_1$. Let $x_i \in \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(1))$ be the section vanishing at q_i. The divisor $D_{|E_1}$ has multiplicity at least $m_1 + m_i - d$ at q_i. Let $I \subset \{2, \dots, r\}$ be the set of indices i for which $m_1 + m_i - d \geqslant 0$. It follows that the image of the restriction map

$$r : \mathrm{H}^0(X,D) \to \mathrm{H}^0(E_1, D_{|E_1}) \tag{4.19}$$

lies in the subspace

$$V = \prod_{i \in I} x_i^{m_1+m_i-d} \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(e)) \subset \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m_1)),$$

where

$$e = m_1 - \sum_{i \in I} (m_1 + m_i - d). \tag{4.20}$$

We claim that one may lift any section in V to a section in $\mathrm{H}^0(X,D)$ that is generated by distinguished sections. Then we are reduced to showing that $\mathrm{H}^0(X, D-E_1)$ is generated by distinguished sections. If $D - E_1$ is not effective, we are done; if not, we replace D with $D - E_1$ and repeat the process until either $D - E_1$ is not effective or $D.E_i \leqslant 0$.

Clearly, $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(e))$ is generated by sections $\prod_{i=2}^r x_i^{k_i}$, where $k_i \geqslant 0$ and $\sum k_i = e$ (of course, we may assume that, for example, $k_4 = k_5 = \dots = 0$). Note that $r(x_{L_{1,j}}) = x_j$, for all $j = 2, \dots, r$. Consider the following divisor on X:

$$\begin{aligned} D_0 &= \sum_{i \in I^c} k_i L_{1,i} + \sum_{i \in I} (k_i + m_1 + m_i - d) L_{1,i} \\ &= m_1 H - m_1 E_1 - \sum_{i \in I^c} k_i E_i - \sum_{i \in I} (k_i + m_1 + m_i - d) E_i. \end{aligned}$$

The restriction map r maps the section

$$t' = \prod_{i=2}^r x_{L_{1,i}}^{k_i} \prod_{i \in I} x_{L_{1,i}}^{m_1+m_i-d} \in \mathrm{H}^0(X, D_0)$$

to the section

$$s = \prod_{i=2}^r x_i^{k_i} \prod_{i \in I} x_i^{m_1+m_i-d} \in \mathrm{H}^0(E_1, D_{0|E_1}) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(m_1)).$$

Consider

$$D - D_0 = (d - m_1) H - \sum_{i \in I^c} (m_i - k_i) E_i - \sum_{i \in I} (d - m_1 - k_i) E_i.$$

Since

$$d \geqslant m_1, \quad m_i \geqslant m_1 \geqslant e \geqslant k_i, \quad d - m_1 \geqslant m_1 \geqslant e \geqslant k_i$$

and using (4.20) one has

$$\sum_{i \in I^c}(m_i - k_i) + \sum_{i \in I}(d - m_1 - k_i) = \sum_{i \in I^c} m_i + (d - m_1)|I| - e = \sum_{i=1}^{r} m_i - 2m_1 \leqslant 2(d - m_1).$$

It follows from Lemma 4.24 that $D - D_0$ is an effective divisor on X. Since $(D - D_0).E_1 = 0$, the space $\mathrm{H}^0(X, D - D_0)$ is generated by distinguished sections by induction. Let $t'' \in \mathrm{H}^0(X, D - D_0)$ be any distinguished section not zero on E_1. Then $t't''$ is a distinguished section in $\mathrm{H}^0(X, D)$ that maps to s. □

Lemma 4.24. *Let X be the blow-up of $\mathbb{P}^n$ in any r distinct points. Let $D = dH - \sum_{i=1}^{r} m_i E_i$, with $d, m_i \geqslant 0$, be a divisor class with $\sum_{i=1}^{r} m_i \leqslant nd$ and $d \geqslant m_i$, for all $i = 1, \ldots, r$. Then D is an effective divisor.*

Proof. We claim that D is an effective combination of (effective) classes $H - (E_{i_1} + \cdots + E_{i_l})$, for $i_1, \ldots, i_l \in \{1, \ldots, r\}$ and $0 \leqslant l \leqslant n$. Consider the table with n rows and d columns filled with E_i in the following way. Start in the upper left corner and write E_1 a total of m_1 times in the first row. Then write E_2 a total of m_2 times, passing to the second row if necessary, and so on. Fill the remaining entries with zeros. In the following example $n = 3$ and $D = 5H - 3E_1 - 3E_2 - 2E_3 - 5E_4 - E_5$:

$$\begin{matrix} E_1 & E_1 & E_1 & E_2 & E_2 \\ E_2 & E_3 & E_3 & E_4 & E_4 \\ E_4 & E_4 & E_4 & E_5 & 0 \end{matrix}$$

Our conditions guarantee that all entries of a given column are different. Therefore D is the sum of classes $H - (E_{i_1} + \cdots + E_{i_l})$, one for each column, where $E_{i_1}, \ldots, E_{i_l}$ are entries of the column. In the example above

$$D = (H - E_1 - E_2 - E_4) + (H - E_1 - E_3 - E_4) + \cdots + (H - E_2 - E_4). \qquad \square$$

5. Proof of Theorem 1.1

By [Muk01], there is an isomorphism $\phi : S^G \to \mathrm{Cox}(X)$ where X is the blow-up of $\mathbb{P}^n$ in $n+3$ points $p_1, \ldots, p_{n+3}$ in general position. By Theorem 1.3, the ring $\mathrm{Cox}(X)$ is generated by the sections x_{E_i}, for each exceptional divisor E_i, $i = 1, \ldots, n+3$, and the sections x_E, corresponding to the minimal divisors

$$E = kH - k\sum_{i \in I} E_i - (k-1)\sum_{i \in I^c} E_i \tag{5.1}$$

for each subset $I \subset \{1, \ldots, n+3\}$, $|I| = n + 2 - 2k$, $1 \leqslant k \leqslant 1 + n/2$. Then $|I^c| = 2k + 1$. Note that if $k = 0$ in (5.1), then $E = E_i$.

The polynomials F_I in (1.2) are clearly invariant (just use the rule of differentiating a determinant). We claim that, for all $0 \leqslant k \leqslant 1 + n/2$ one has $\phi(F_{I^c}) = x_E$, where E is as in (5.1). It is clear from [Muk01] that $\phi(x_i) = x_{E_i}$. Following [Muk01], if $F_0 = \cdots = F_n = 0$ are $n + 1$ linear equations (in $t_1, \ldots, t_r$) that cut G in $\mathbb{G}^{n+3}$, let $J_0, \ldots, J_n$ be the polynomials in S given by $J_i = F_i(y_1/x_1, \ldots, y_r/x_r)x_1 \ldots x_r$. Then sections of the divisor $D = dH - \sum_{i=1}^{n+3} m_i E_i$ on X, for $d, m_i \geqslant 0$, correspond by ϕ to an invariant polynomial of the form

$$Q = \frac{P(J_0, \ldots, J_n)}{\prod_{i=1}^{n+3} x_i^{m_i}},$$

where $P(z_0, \ldots, z_n)$ is a homogeneous polynomial of degree d in variables $z_0, \ldots, z_n$, such that $P(J_0, \ldots, J_n)$ is divisible by $\prod_{i=1}^{n+3} x_i^{m_i}$. If we let $\deg_x(Q)$ (respectively $\deg_y(Q)$), be the degree of Q in the x_i (respectively in the y_i), then

$$\deg_y(Q) = d, \quad \deg_x(Q) = (n+2)d - \sum_{i=1}^{n+3} m_i, \quad \deg(D) = \deg_x(Q) - \deg_y(Q). \tag{5.2}$$

Hence, $\phi(F_{I^c})$ is a section in $\mathrm{H}^0(X, D)$, where D is a divisor with $d = k$ and $\deg(D) = 1$. To show that $D = E$, consider the following action of the torus $\mathbb{G}_m^r$ on S: $(\lambda_1, \ldots, \lambda_r) \in \mathbb{G}_m^r$ acts by $x_i \mapsto \lambda_i x_i$, $y_i \mapsto \lambda_i y_i$. The action of $\mathbb{G}_m^r$ on S is compatible with the action of $\mathbb{G}_a^r$ on S. Hence, there is an induced action of $\mathbb{G}_m^r$ on S^G. Since $(\lambda_1, \ldots, \lambda_r) \in \mathbb{G}_m^r$ maps J_i onto $\lambda_1 \ldots \lambda_r J_i$, it follows that Q is mapped to $\prod_{i=1}^r \lambda_i^{d-m_i}$. Since F_{I^c} is mapped to $\prod_{i \in I^c} \lambda_i$, it follows that $D = E$.

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