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On an Identity due to Bump and Diaconis, and Tracy and Widom

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Abstract. A classical question for a Toeplitz matrix with given symbol is how to compute asymptotics for the determinants of its reductions to finite rank. One can also consider how those asymptotics are affected when shifting an initial set of rows and columns (or, equivalently, asymptotics of their minors). Bump and Diaconis obtained a formula for such shifts involving Laguerre polynomials and sums over symmetric groups. They also showed how the Heine identity extends for such minors, which makes this question relevant to Random Matrix Theory. Independently, Tracy and Widom used the Wiener–Hopf factorization to express those shifts in terms of products of infinite matrices. We show directly why those two expressions are equal and uncover some structure in both formulas that was unknown to their authors. We introduce a mysterious differential operator on symmetric functions that is very similar to vertex operators. We show that the Bump–Diaconis–Tracy–Widom identity is a differentiated version of the classical Jacobi–Trudi identity.

1 Introduction

1.1 Origin: Toeplitz Determinants

Fix $\sigma(t)$ to be a function of the unit circle $\mathbb{T}$ in $\mathbb{C}$ that can be written in the form

$$\sigma(t) = \exp\Big(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k}\Big)$$

for the sets of constants $\{p_k \in \mathbb{C}\}$ and $\{\tilde{p}_k \in \mathbb{C}\}$.[1] This requires σ to have winding number 0 around the origin (since $\log \sigma(t)$ is defined, see [BS99, pp. 15–17] for more details). This also defines a set of constants $\{d_k\}$ so that $\sum_{k\in\mathbb{Z}} d_k t^k := \sigma(t)$ (*i.e.,* the d_k's are the Fourier coefficients of $\sigma(t)$). We will further assume that the $|p_k|$'s and $|\tilde{p}_k|$'s decrease fast enough, *i.e.,* that all of the sums $\sum_k \frac{|p_k|}{k}$, $\sum_k \frac{|\tilde{p}_k|}{k}$, and $\sum_k \frac{|p_k \tilde{p}_k|}{k}$ are bounded.

We now construct a matrix M_n having constant entries on diagonals parallel to

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[1] This implies that $\sigma(0) = 1$, a condition that is merely there for exposition.

the main diagonal (Toeplitz property with *symbol* σ):

$$M_n(\sigma) = M_n = \begin{pmatrix} d_0 & d_1 & \cdots & \cdots & d_{n-1} \\ d_{-1} & d_0 & d_1 & \cdots & d_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & d_1 \\ d_{1-n} & \cdots & \cdots & d_{-1} & d_0 \end{pmatrix}_{n\times n} = \left(d_{i-j}\right)_{n\times n}.$$

A classical problem for Toeplitz matrices is then to consider the asymptotics of the determinant $\det(M_n)$ as n goes to infinity. Our identity will stem from the same question for a slightly altered version of M_n.

For λ and μ partitions of length less than or equal to n, look at

$$M_n^{\lambda\,\mu}(\sigma) := \left(d_{\lambda_i-\mu_j-i+j}\right)_{n\times n}.$$

Those new matrices are not Toeplitz, but at least they are minors of the Toeplitz matrix $M_m(\sigma)$, for some m larger than n. This is clear once illustrated. For example, set $n := 3$, $m := 5$, $\lambda := (2,1)$, $\mu := (1)$. We then have the matrices

$$M_3^{\lambda\,\mu}(\sigma) = \begin{pmatrix} d_1 & d_3 & d_4 \\ d_{-1} & d_1 & d_2 \\ d_{-3} & d_{-1} & d_0 \end{pmatrix} \quad \text{and} \quad M_5(\sigma) = \begin{pmatrix} d_0 & d_1 & d_2 & d_3 & d_4 \\ d_{-1} & d_0 & d_1 & d_2 & d_3 \\ d_{-2} & d_{-1} & d_0 & d_1 & d_2 \\ d_{-3} & d_{-2} & d_{-1} & d_0 & d_1 \\ d_{-4} & d_{-3} & d_{-2} & d_{-1} & d_0 \end{pmatrix}.$$

Observe that $M_3^{\lambda\,\mu}(\sigma)$ is the minor of $M_5(\sigma)$ obtained by striking its first and third columns and its second and fourth rows. If m had been bigger, we would only have needed to strike more rows and columns.

The asymptotics of the determinants of $M^{\lambda\mu}(\sigma)$ are well known through the Szegö limit theorem, so it is natural to look at the ratios

$$R^{\lambda\mu}(\sigma) := \lim_{n\to\infty} \frac{\det M_n^{\lambda\,\mu}(\sigma)}{\det M_n(\sigma)}.$$

These ratios have indeed been studied by two pairs of researchers, independently.

Tracy and Widom [TW02] obtained the asymptotics $R^{\lambda\mu}(\sigma)$ as determinants involving the Fourier coefficients in the Wiener–Hopf factorization

$$\begin{aligned} \sigma(t) &= \exp\Big(\sum_{k>0} \frac{p_k}{k} t^k\Big) \cdot \exp\Big(\sum_{k>0} \frac{\tilde{p}_k}{k} t^{-k}\Big) \\ &=: \sum_{k\geq 0} h_k t^k \cdot \sum_{k\geq 0} \tilde{h}_k t^{-k} \end{aligned} \tag{1.1}$$

of $\sigma(t)$. The second line serves as definition of the h_ks and $\tilde{h}_k$s. We will present the full expression they obtain in equation (4.1). Meanwhile, we refer to that expression as $\mathrm{TW}^{\lambda\,\mu}(\sigma)$.

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Bump and Diaconis [BD02] instead generalized the Heine identity. This classical identity gives

$$\det M_n(\sigma) \sim_{n\to\infty} \int_{U(n)} \sigma(g)\,\mathrm{d}g,$$

with $\sigma(g) := \prod \sigma(t_i)$, t_i being the eigenvalues of g. They extended this to

$$\det M_n^{\lambda\,\mu}(\sigma) \sim_{n\to\infty} \int_{U(n)} \sigma(g) s_\lambda(g)\overline{s_\mu(g)}\,\mathrm{d}g,$$

with s_λ, s_μ being the usual Schur polynomials applied to the eigenvalues of g. Thus all results presented here for Toeplitz matrices apply for twisted integrals as well (hence the interest for Random Matrix Theory), and Bump and Diaconis derived independently from Tracy and Widom a second expression (presented in Section 3) for the following limit:

$$\mathrm{BD}^{\lambda\,\mu}(\sigma) := \lim_{n\to\infty} \frac{\int_{U(n)} \sigma(g) s_\lambda(g)\overline{s_\mu(g)}\,\mathrm{d}g}{\int_{U(n)} \sigma(g)\,\mathrm{d}g}.$$

Tracy and Widom's theorems are valid under slightly more general conditions than Bump and Diaconis'. Lyons [Lyo03] discusses this point in detail.

We now wish to state the theorem alluded to in the title of this article.

Theorem 1.1 ([BD02, TW02]) *Let λ, μ be partitions. Given $\sigma(t)$ such that $\sum_k \frac{|p_k|}{k}$, $\sum_k \frac{|\tilde{p}_k|}{k}$, and $\sum_k \frac{|p_k \tilde{p}_k|}{k}$ are bounded, we have*

$$\mathrm{BD}^{\lambda\,\mu}(\sigma) = \mathrm{R}^{\lambda\,\mu}(\sigma) = \mathrm{TW}^{\lambda\,\mu}(\sigma).$$

The proof of this theorem thus comes from two entirely disjoint papers.

1.2 Concept

Theorem 1.1 raises an immediate question. If one forgets its origins, Theorem 1.1 is a mysterious combinatorial identity $\mathrm{BD}^{\lambda\,\mu}(\sigma) = \mathrm{TW}^{\lambda\,\mu}(\sigma)$. Our main goal for this paper is to prove this identity more directly, without relying on Toeplitz determinants (*i.e.*, $\mathrm{R}^{\lambda\,\mu}(\sigma)$).

Let $\varnothing$ be the trivial partition. We will show how this identity is a differentiated version of the Jacobi–Trudi identity. We proceed along the following stages:

(i) Both $\mathrm{BD}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}$ are functions of σ, but can also be seen as functions of the Fourier coefficients of $\log\sigma$ $\{p_1, p_2, \dots, \tilde{p}_1, \tilde{p}_2, \dots\}$. Those functions turn out to be power series in those Fourier coefficients. This is present in [BD02] and partly in [TW02].

(ii) As explained in Section 2, the variable set $\{p_1, p_2, \dots, \tilde{p}_1, \tilde{p}_2, \dots\}$ can be replaced by $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \dots\}$, the union of the two sets of symmetric power sums in two separate sets of variables (say $\overline{X}$ and $\overline{Y}$). Notationally, this will replace $\mathrm{BD}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}$ with $\mathbf{BD}^{\lambda\,\mu}$ and $\mathbf{TW}^{\lambda\,\mu}$.

(iii) There are two related differential operators, Δ and $\boldsymbol{\Delta}$, that act respectively on $\mathrm{BD}^{\lambda\,\mu}$ or $\mathrm{TW}^{\lambda\,\mu}$ and on $\mathbf{BD}^{\lambda\,\mu}$ or $\mathbf{TW}^{\lambda\,\mu}$ (see Section 2.5).

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(iv) ***Theorem 1.2***

$$\Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right)=\mathrm{BD}^{\lambda\,\mu}\quad\textit{and}\quad\boldsymbol{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right)=\mathbf{BD}^{\lambda\,\mu}.$$

(v) ***Theorem 1.3***

$$\Delta\left(\mathrm{TW}^{\lambda\,\varnothing}\cdot\mathrm{TW}^{\varnothing\,\mu}\right)=\mathrm{TW}^{\lambda\,\mu}\quad\textit{and}\quad\boldsymbol{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right)=\mathbf{TW}^{\lambda\,\mu}.$$

(vi) $\mathbf{BD}^{\lambda\,\varnothing}=\mathbf{TW}^{\lambda\,\varnothing}$ (Jacobi–Trudi identity, a classical identity in symmetric function theory).

As we can see, everything is proved through analogues in symmetric function theory that specialize to the objects of original interest. This can only work by ignoring the Toeplitz determinant origin of the expressions $\mathrm{BD}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}$, but still gives a (new) corollary about the structure of the determinants.

Corollary 1.4 $\Delta\left(\mathrm{R}^{\lambda\,\varnothing}\cdot\mathrm{R}^{\varnothing\,\mu}\right)=\mathrm{R}^{\lambda\,\mu}$.

1.3 Organization of this Paper

In Section 2, we will review the notions of symmetric function theory that we need. In Section 3, we will define $\mathrm{BD}^{\lambda\,\mu}$ and prove Theorem 1.2. The next section accomplishes the same for the Tracy–Widom side and Theorem 1.3. Section 5 will be devoted to the proof of Theorem 1.1. We give in Section 6 a couple of noteworthy relations on the $\mathrm{R}^{\lambda\,\mu}$'s. Finally, we discuss in Section 7 how this paper fits into a more general program.

2 General Definitions and Notations

We summarize the definitions and notations employed in this paper.

2.1 Partitions and Symmetric Groups

A partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ is a finite weakly decreasing sequence of non-negative integers. We define the weight $|\lambda|$ of λ to be the sum $\sum\lambda_i$. If this weight is k, we also use the notation $\lambda\vdash k$. If $k=0$, we denote the trivial partition $(0,0,0,0,\dots)$ by $\varnothing$. The length $l(\lambda)$ of λ is the maximal i such that $\lambda_i\neq 0$.

There is a partial ordering on partitions: $\lambda\subseteq\mu$ if and only if $\lambda_i\leq\mu_i$ for all i. In a probable break of standard notation, $\lambda(i)$ counts the number of λ_j's equal to i, so that $(i^{\lambda(i)})_{i=1}^n=(\lambda_1,\lambda_2,\dots,\lambda_n)$. In an even greater offense, if π is a permutation, we will use $\pi(i)$ for the number of elements of i's in the cycle type of π, not for the image of point i under π (with no risk of notational confusion in the whole paper).

As usual, partitions of fixed weight k index conjugacy classes in the symmetric group on k points $\mathcal{S}_k$. We set $z_\lambda:=\prod_i i^{\lambda(i)}i!$. This is the order of the centralizer of a permutation in $\mathcal{S}_{|\lambda|}$ of cycle-type λ.

In order to present the formula of Bump and Diaconis, we will also need the irreducible characters of the symmetric groups. For a fixed k, all irreducible representations of $\mathcal{S}_k$ are indexed by partitions of weight k (see the book by Sagan [Sag01] for a friendly introduction). If $\lambda \vdash k$, we will use χ^λ for the character of the representation corresponding to λ.

2.2 Symmetric Functions

We now introduce a few functions in the graded algebras $\mathbf{\Lambda}(\overline{X})$ and $\mathbf{\Lambda}(\overline{Y})$ of symmetric functions in countably many independent variables $\overline{X} := \{x_1, x_2, x_3, \dots\}$ and $\overline{Y} := \{y_1, y_2, y_3, \dots\}$ over $\mathbb{Q}$. The former can be most directly thought of as the ring of formal sums $S(x_1, \dots)$ of monomials in the variables x_i that have the symmetry property $S(x_{\rho(1)}, x_{\rho(2)}, \dots) = S(x_1, x_2, \dots)$ for all $\rho \in \mathcal{S}_\infty$. The most classic reference on the topic is Macdonald's book [Mac95, Sections 1.2–1.5].

We will use the notation $\mathbf{p}_\lambda$, $\mathbf{h}_\lambda$, $\mathbf{s}_\lambda$, and $\mathbf{s}_{\lambda/\mu}$ for the various interesting functions living in $\mathbf{\Lambda}(\overline{X})$. They will be the power sum, complete, Schur, and skew Schur functions in the variables $\{x_i\}$ associated to the partition λ (to the skew partition λ/μ for the latter), respectively. Similarly, we use $\tilde{\mathbf{p}}_\lambda$, $\tilde{\mathbf{h}}_\lambda$, $\tilde{\mathbf{s}}_\lambda$ and $\tilde{\mathbf{s}}_{\lambda/\mu}$ for the same functions in $\mathbf{\Lambda}(\overline{Y})$. We remind the reader that boldface font will be used for functions in $\mathbf{\Lambda}(\,\cdot\,)$. A tilde indicates the variable set $\overline{Y}$, while the default (when there is no tilde) is to assume that the variable set is $\overline{X}$.

One can define an inner product on $\mathbf{\Lambda}(\,\cdot\,)$ by setting the Schur polynomials to be orthonormal: $\langle \mathbf{s}_\lambda, \mathbf{s}_\mu \rangle_{\mathbf{\Lambda}(\overline{X})} = \delta_{\lambda\mu}$. The $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{\Lambda}(\overline{X})}$ indicates that this inner product is for $\mathbf{\Lambda}(\overline{X})$. We will need the fact that the $\mathbf{p}_\lambda$'s form an orthogonal base: $\langle \mathbf{p}_\lambda, \mathbf{p}_\mu \rangle_{\mathbf{\Lambda}(\overline{X})} = z_\lambda \delta_{\lambda\mu}$.

We will also need to consider the algebra of symmetric functions in two sets of variables

$$\mathbf{\Lambda}(\overline{X}, \overline{Y}) = \mathbf{\Lambda}(\overline{X}) \otimes_{\mathbb{Q}} \mathbf{\Lambda}(\overline{Y}).$$

This comes equipped with an induced inner product defined by extending linearly

$$\langle \mathbf{a} \cdot \tilde{\mathbf{a}}, \mathbf{b} \cdot \tilde{\mathbf{b}} \rangle_{\mathbf{\Lambda}(\overline{X},\overline{Y})} = \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{\Lambda}(\overline{X})} \cdot \langle \tilde{\mathbf{a}}, \tilde{\mathbf{b}} \rangle_{\mathbf{\Lambda}(\overline{Y})}.$$

2.3 The Derivations $\mathbf{p}_n^\perp$ and $\tilde{\mathbf{p}}_n^\perp$

Let us first consider just the set of variables $\overline{X}$.

Following Macdonald [Mac95, Example 3, Section 1.5, page 75], we define the algebra homomorphism ${}^\perp \colon \mathbf{\Lambda}(\overline{X}) \longrightarrow \operatorname{End}(\mathbf{\Lambda}(\overline{X}))$ in such a way that

$$\langle \mathbf{f}^\perp \mathbf{u}, \mathbf{v} \rangle_{\mathbf{\Lambda}(\overline{X})} = \langle \mathbf{u}, \mathbf{f}\mathbf{v} \rangle_{\mathbf{\Lambda}(\overline{X})}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{\Lambda}(\overline{X})$. This is the adjoint of multiplication in the algebra $\mathbf{\Lambda}(\overline{X})$.

Macdonald (following Foulkes) shows that $\mathbf{p}_n^\perp = n\partial_{\mathbf{p}_n}$, and so $\mathbf{p}_n^\perp$ is a derivation.

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Indeed, we have

$$\begin{aligned}\langle \mathbf{p}_n^\perp(\mathbf{p}_\lambda), \mathbf{p}_\mu\rangle_{\mathbf{\Lambda}(\overline{X})} &= \langle \mathbf{p}_\lambda, \mathbf{p}_\mu \mathbf{p}_n\rangle_{\mathbf{\Lambda}(\overline{X})}\\ &= \begin{cases} 0 & \text{if } \lambda \neq (n)\cup\mu,\\ z_\lambda & \text{if } \lambda = (n)\cup\mu \end{cases}\\ &= \begin{cases} 0 & \text{if } \mu \neq \lambda\setminus(n),\\ z_\lambda & \text{if } \mu \neq \lambda\setminus(n) \end{cases}\\ &= \langle z_\lambda z_\mu^{-1}\mathbf{p}_{\lambda\setminus(n)}, \mathbf{p}_\mu\rangle_{\mathbf{\Lambda}(\overline{X})}.\end{aligned}$$

But $z_\lambda z_{\lambda\setminus(n)}^{-1} = n\lambda(n)$, so $\mathbf{p}_n^\perp(\mathbf{p}_\lambda) = n\partial_{\mathbf{p}_n}(\mathbf{p}_\lambda)$, and we get our claim that $\mathbf{p}_n^\perp = n\partial_{\mathbf{p}_n}$.

A similar result is of course true for $\mathbf{\Lambda}(\overline{Y})$ (for the adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbf{\Lambda}(\overline{Y})}$).

Observe that $(\mathbf{a}\cdot\tilde{\mathbf{a}})^\perp = \mathbf{a}^\perp \otimes \tilde{\mathbf{a}}^\perp$, and so $^\perp$ is a homomorphism $\mathbf{\Lambda}(\overline{X},\overline{Y}) \to \operatorname{End}(\mathbf{\Lambda}(\overline{X},\overline{Y}))$.

2.4 Specializing Symmetric Objects

Let $P = \{p_1, p_2, \dots\}$ and $\tilde{p} = \{\tilde{p}_1, \tilde{p}_2, \dots\}$ be sets of variables. We define $V_P = \mathbb{Q}[[P]]$, $V_{\tilde{p}} = \mathbb{Q}[[\tilde{P}]]$ and $V = \mathbb{Q}[[\tilde{P}\cup P]]$.

Any $\sigma(t) = \exp(\sum_{k>0} \frac{p_k}{k}t^k + \frac{\tilde{p}_k}{k}t^{-k})$ induces an evaluation map $E_\sigma\colon V \to \mathbb{C}$ obtained by replacing the variables in V by the values of the Fourier coefficients of $\log\sigma$. This is of course only convergent on a subset of V, but we will limit ourselves to that subset.

We define algebra homomorphisms

$$\begin{aligned} F_{\overline{X}}\colon \mathbf{\Lambda}(\overline{X}) &\longrightarrow V_P \subset V \quad (\text{resp. for } \overline{Y}, \tilde{P})\\ \mathbf{p}_k &\longmapsto p_k\\ F\colon \mathbf{\Lambda}(\overline{X},\overline{Y}) &\longrightarrow V\\ \mathbf{p}_k &\longmapsto p_k\\ \tilde{\mathbf{p}}_k &\longmapsto \tilde{p}_k. \end{aligned}$$

Clearly, F restricts to $F_{\overline{X}}$ and $F_{\overline{Y}}$, and merely forgets that the range was a vector space of symmetric polynomials. The variables $\overline{X}$ and $\overline{Y}$ are completely lost.

For a given $\sigma(t)$, we observe that the generating function for the h_k is the same as the generating function for the $\mathbf{h}_k$, *i.e.*, compare Equation (1.1) with the generating function identity

$$\exp\Big(\sum_{k>0}\frac{\mathbf{p}_k}{k}t^k\Big) = \sum_{k\geq 0}\mathbf{h}_k t^k.$$

This very classical identity (Newton's identity describing the roots of a polynomial) was already discussed in the context of Pólya's enumeration theory in the paper by Bump and Diaconis [BD02]. In any case, this guarantees that

$$E_\sigma \circ F_{\overline{X}}(\mathbf{h}_k) = h_k.$$

Of course, a similar map $E_\sigma \circ F_{\overline{Y}} \colon \mathbf{\Lambda}(\overline{Y}) \to \mathbb{C}$ exists, and both maps together induce a third one, $E_\sigma \circ F \colon \mathbf{\Lambda}(\overline{X}, \overline{Y}) \to \mathbb{C}$. We will call this whole process *specialization*: start with a series in symmetric functions of countably many variables, forget through F that each symmetric function is a function itself (and thus assign a new variable for each function), and finally replace each of these new variables by a complex number through E_σ.

The advantage in setting up specialization in this way is that derivations are sent to derivations by F. Thanks to Section 2.3, we know that for $\mathbf{f} \in \mathbf{\Lambda}(\overline{X}, \overline{Y})$,

$$k\partial_{p_k}(F(\mathbf{f})) = F(k\partial_{\mathbf{p}_k}(\mathbf{f})) = F(\mathbf{p}_k^\perp(\mathbf{f})).$$

We can use this property to create differential operators and specialize them from one algebra to another.

2.5 Differential Operators Δ and $\mathbf{\Delta}$

Consider still $V = \mathbb{Q}[[P \cup \tilde{P}]]$. We define a (generalized) differential operator Δ as

$$\Delta = \exp\Big(\sum_k k\partial_{p_k}\partial_{\tilde{p}_k}\Big) = \prod_{k>0}\sum_{i\geq 0}\frac{k^i}{i!}(\partial_{p_k}\partial_{\tilde{p}_k})^i,$$

where $(\partial_{p_k}\partial_{\tilde{p}_k})^i$ is composition. Note that sums and products will be finite for any element of V, but that the order of Δ is not uniformly bounded on V.

We define the operator $\mathbf{\Delta}$ on $\mathbf{\Lambda}(\overline{X}, \overline{Y})$ in the same way (merely replacing ∂_{p_k} by $\partial_{\mathbf{p}_k}$). This implies the commutation relation

$$F \circ \mathbf{\Delta} = \Delta \circ F. \tag{2.1}$$

It follows from the previous sections that

$$\mathbf{\Delta} = \exp\Big(\sum_k \frac{\mathbf{p}_k^\perp \tilde{\mathbf{p}}_k^\perp}{k}\Big).$$

3 The Bump–Diaconis Side

Assume $\lambda \vdash l$ and $\mu \vdash m$. Then Bump and Diaconis define for each

$$\sigma(t) = \exp\Big(\sum_{k>0}\frac{p_k}{k}t^k\Big)\cdot\exp\Big(\sum_{k>0}\frac{\tilde{p}_k}{k}t^{-k}\Big)$$

an expression $\mathrm{BD}^{\lambda\mu}(\sigma) = E_\sigma(\mathrm{BD}^{\lambda\mu})$, where they have

$$\mathrm{BD}^{\lambda\mu} = \frac{1}{l!}\sum_{\pi\in\mathfrak{S}_l}\frac{1}{m!}\sum_{\rho\in\mathfrak{S}_m}\chi^\lambda(\pi)\chi^\mu(\rho)\prod_{k>0}\mathrm{F}_k(\pi,\rho),$$

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with

$$\mathrm{F}_k(\pi,\rho) = \begin{cases} k^{\rho(k)}\rho(k)!L_{\rho(k)}^{(\pi(k)-\rho(k))}\left(-\frac{p_k\tilde{p}_k}{k}\right)p_k^{\pi(k)-\rho(k)} & \text{if } \pi(k)\geq\rho(k),^2 \\ k^{\pi(k)}\pi(k)!L_{\pi(k)}^{(\rho(k)-\pi(k))}\left(-\frac{p_k\tilde{p}_k}{k}\right)\tilde{p}_k^{\rho(k)-\pi(k)} & \text{if } \rho(k)\geq\pi(k), \end{cases}$$

and where

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-t)^k}{k!} = \sum_{k=0}^{n}\binom{n+\alpha}{k}\frac{(-t)^{n-k}}{(n-k)!}$$

is the usual Laguerre polynomial (the former expression is the standard definition, while the latter formula is only a reindexing of it that will be more useful here).

We define $\mathbf{BD}^{\lambda\,\mu}$ and $\mathbf{F}_k(\pi,\rho)$ similarly.

Lemma 3.1 *Let* $\max_k = \max(\pi(k),\rho(k))$ *and* $\min_k = \min(\pi(k),\rho(k))$. *Then*

$$\mathbf{F}_k(\pi,\rho) = \sum_{i=0}^{\min_k} k^i i! \binom{\max_k}{i}\binom{\min_k}{i}\mathbf{p}_k^{\pi(k)-i}\tilde{\mathbf{p}}_k^{\rho(k)-i}.$$

Proof We just need to expand the Laguerre polynomial in the definition of $\mathbf{F}_k$ while keeping track of the degrees in $\mathbf{p}_k$ and $\tilde{\mathbf{p}}_k$. The key is to observe that all the monomials will have the correct degrees, *i.e.*, will be $\mathbf{p}_k^{\pi(k)-i}\tilde{\mathbf{p}}_k^{\rho(k)-i}$ for $0\leq i\leq\min(\rho(k),\pi(k))$. ■

Proof of Theorem 1.2 When one of the partitions is trivial, the $\mathbf{BD}^{\lambda\,\mu}$ reduce[3] to

$$\mathbf{BD}^{\lambda\,\varnothing} = \frac{1}{l!}\sum_{\pi\in\mathfrak{S}_l}\chi^\lambda(\pi)\mathbf{p}_\pi \quad\text{and}\quad \mathbf{BD}^{\varnothing\,\mu} = \frac{1}{m!}\sum_{\rho\in\mathfrak{S}_m}\chi^\mu(\rho)\tilde{\mathbf{p}}_\rho.$$

We thus need to evaluate

$$\begin{aligned}\boldsymbol{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right) &= \boldsymbol{\Delta}\left(\frac{1}{l!}\sum_{\pi\in\mathfrak{S}_l}\chi^\lambda(\pi)\mathbf{p}_\pi\cdot\frac{1}{m!}\sum_{\rho\in\mathfrak{S}_m}\chi^\mu(\rho)\tilde{\mathbf{p}}_\rho\right)\\ &= \frac{1}{l!}\sum_{\pi\in\mathfrak{S}_l}\frac{1}{m!}\sum_{\rho\in\mathfrak{S}_m}\chi^\lambda(\pi)\chi^\mu(\rho)\boldsymbol{\Delta}\left(\mathbf{p}_\pi\tilde{\mathbf{p}}_\rho\right).\end{aligned}$$

Each term is of the form

$$\boldsymbol{\Delta}\left(\mathbf{p}_\pi\tilde{\mathbf{p}}_\rho\right) = \Big[\prod_{k>0}e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}\Big]\Big(\prod_{k>0}\mathbf{p}_k^{\pi(k)}\tilde{\mathbf{p}}_k^{\rho(k)}\Big) = \prod_{k>0}\big[e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}(\mathbf{p}_k^{\pi(k)}\tilde{\mathbf{p}}_k^{\rho(k)})\big],$$

[2]We remind the reader of our unconventional usage of $\pi(k)$ for the number of k-cycles in π.

[3]Here we use here permutations as an index for the power sum functions. We mean by $\mathbf{p}_\pi$ the function $\mathbf{p}_\lambda$, where λ is the cycle-type of π.

where

$$
\begin{aligned}
[e^{k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}}](\mathbf{p}_k^{\pi(k)}\tilde{\mathbf{p}}_k^{\rho(k)}) &= \sum_{i\geq 0} \frac{(k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k})^i}{i!}(\mathbf{p}_k^{\pi(k)}\tilde{\mathbf{p}}_k^{\rho(k)}) \\
&= \sum_{i\geq 0} k^i i! \binom{\pi(k)}{i} \mathbf{p}_k^{\pi(k)-i} \binom{\rho(k)}{i} \tilde{\mathbf{p}}_k^{\rho(k)-i} \\
&= \mathbf{F}_k(\pi,\rho)
\end{aligned}
$$

by Lemma 3.1.

Summing over all terms, we have

$$
\begin{aligned}
\mathbf{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right) &= \frac{1}{l!}\sum_{\pi\in\mathcal{S}_l}\frac{1}{m!}\sum_{\rho\in\mathcal{S}_m}\chi^\lambda(\pi)\chi^\mu(\rho)\mathbf{\Delta}\left(\mathbf{p}_\pi\tilde{\mathbf{p}}_\rho\right) \\
&= \frac{1}{l!}\sum_{\pi\in\mathcal{S}_l}\frac{1}{m!}\sum_{\rho\in\mathcal{S}_m}\chi^\lambda(\pi)\chi^\mu(\rho)\prod_{k>0}\mathbf{F}_k(\pi,\rho) \\
&= \mathbf{BD}^{\lambda\,\mu}.
\end{aligned}
$$

The identity $\Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right) = \mathrm{BD}^{\lambda\,\mu}$ follows from

$$
\begin{aligned}
\mathrm{BD}^{\lambda\,\mu} &= F(\mathbf{BD}^{\lambda\,\mu}) \\
&= F\left(\mathbf{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right)\right) \\
&= \Delta\left(F(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu})\right) \qquad \text{(equation (2.1))} \\
&= \Delta\left(\mathrm{BD}^{\lambda\,\varnothing}\cdot\mathrm{BD}^{\varnothing\,\mu}\right),
\end{aligned}
$$

completing our proof of Theorem 1.2. ■

We will need an additional lemma later.

Lemma 3.2 $\mathbf{BD}^{\lambda\,\varnothing} = \mathbf{s}_\lambda$ *and* $\mathbf{BD}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu$.

Proof This is immediate from the definitions of $\mathbf{BD}^{\lambda\,\varnothing}$ and $\mathbf{BD}^{\varnothing\,\mu}$. We get the classical expansions[4]

$$
\frac{1}{|\lambda|!}\sum_{\pi\in\mathcal{S}_{|\lambda|}}\chi^\lambda(\pi)\mathbf{p}_\pi = \mathbf{s}_\lambda \quad \text{and} \quad \frac{1}{|\lambda|!}\sum_{\pi\in\mathcal{S}_{|\lambda|}}\chi^\lambda(\pi)\tilde{\mathbf{p}}_\pi = \tilde{\mathbf{s}}_\lambda
$$

for Schur polynomials in terms of power sums, a fact that was already pointed out by Bump and Diaconis in their paper. ■

[4]Again, we use permutations as an index for the power sum functions.

4 The Tracy–Widom Side

Since $\sigma(t) = \exp(\sum_{k>0} \frac{p_k}{k} t^k + \frac{\tilde{p}_k}{k} t^{-k})$, it is reasonable to consider the functions

$$\sigma^+(t) := \sum_{k\geq 0} h_k t^k := \exp\Big(\sum_{k>0} \frac{p_k}{k} t^k\Big)$$

$$\sigma^-(t) := \sum_{k\geq 0} \tilde{h}_k t^{-k} := \exp\Big(\sum_{k>0} \frac{\tilde{p}_k}{k} t^{-k}\Big).$$

It is a classical theorem from operator theory for Toeplitz matrices (see Böttcher and Silbermann's book [BS99, page 15]) that we then have

$$\lim_{n\to\infty} (M_n(\sigma^+) \cdot M_n(\sigma^-))_{ij} = \lim_{n\to\infty} M_n(\sigma)_{ij}.$$

This is called the Wiener–Hopf factorization of the symbol σ.

Tracy and Widom use the Fourier coefficients h_k's and $\tilde{h}_k$'s of σ^+ and σ^- to formulate their result.

We are now ready to define $\mathbf{TW}^{\lambda\,\mu}$ for the partitions $\lambda \vdash m$ and $\mu \vdash p$. Let d be an integer large enough that $\lambda_{d+1} = \mu_{d+1} = 0$. Obviously, $d = \max(l(\lambda), l(\mu))$ would do, but d could be taken larger without affecting the result. Then we set

(4.1)

$$\mathbf{TW}^{\lambda\,\mu} := \det\Big(\big(\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}}\big)_{d\times\infty} \cdot \big(\mathbf{h}_{j-i+\lambda_{d-j+1}}\big)_{\infty\times d}\Big)$$

$$= \det\left(\left(\begin{array}{cccccccccccc}
\tilde{\mathbf{h}}_{\mu_d} & & & \succ & \tilde{\mathbf{h}}_{1-d+\mu_d} & \tilde{\mathbf{h}}_{-d+\mu_d} & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\ \cdots \\
& \nwarrow & & \succ & & & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\ \cdots \\
& & \tilde{\mathbf{h}}_{\mu_{d-i+1}} & & & & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\ \cdots \\
& & & \nwarrow & & & \succ & \succ \tilde{\mathbf{h}}_0 & 0 & \cdots & & 0\ \cdots \\
\tilde{\mathbf{h}}_{d-1+\mu_1} & & & \succ & \tilde{\mathbf{h}}_{\mu_1} & \tilde{\mathbf{h}}_{-1+\mu_1} & \succ & & \succ \tilde{\mathbf{h}}_0 & 0\ \cdots & & 0\ \cdots
\end{array}\right)_{d\times\infty} \cdot \left(\begin{array}{ccccc}
\mathbf{h}_{\lambda_d} & & & & \mathbf{h}_{d-1+\lambda_1} \\
\curlyvee & \nwarrow & & & \curlyvee \\
& & \mathbf{h}_{\lambda_{d-i+1}} & & \\
\curlyvee & & & \nwarrow & \curlyvee \\
\mathbf{h}_{1-d+\lambda_d} & & & & \mathbf{h}_{\lambda_1} \\
\mathbf{h}_{-d+\lambda_d} & & & & \mathbf{h}_{-1+\lambda_1} \\
\curlyvee & \curlyvee & \curlyvee & \curlyvee & \curlyvee \\
\curlyvee & \curlyvee & & & \\
\mathbf{h}_0 & \mathbf{h}_0 & \curlyvee & \curlyvee & \\
0 & 0 & \mathbf{h}_0 & \mathbf{h}_0 & \\
\vdots & \vdots & 0 & 0 & \curlyvee \\
& & \vdots & \vdots & \mathbf{h}_0 \\
& & & & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)_{\infty\times d}\right).$$

The structure of the matrices is important. We now attempt to describe it in words.

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We have here the determinant of a product of two "half-strip" matrices of sizes $d \times \infty$ and $\infty \times d$. The entries along the main diagonal (marked by the arrows $\nwarrow$) are all of the form $\mathbf{h}_{\lambda_i}$ or $\tilde{\mathbf{h}}_{\mu_i}$. The first matrix (resp. second) has a privileged direction, $\succ$ (resp. $\curlyvee$), in which the indices of $\mathbf{h}_\star$ (resp. $\tilde{\mathbf{h}}_\star$) are decreasing. This guarantees that the product is well defined. Each line on the first column has only finitely many non-zero entries.[5]

We define $\mathrm{TW}^{\lambda\,\mu} := F(\mathbf{TW}^{\lambda\,\mu})$, and note that indeed the expression $\mathrm{TW}^{\lambda\,\mu}(\sigma) = E_\sigma(\mathrm{TW}^{\lambda\,\mu})$ is what appears in [TW02]. We make the pedantic distinction here between $\mathrm{TW}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}(\sigma)$ to highlight that the former is an element of V, *i.e.,* a power series in the variable set $P \cup \tilde{P}$, and can thus be differentiated, unlike the latter which is only a complex number.

The matrices involved in the definitions of $\mathbf{TW}^{\lambda\,\mu}$ and $\mathrm{TW}^{\lambda\,\mu}$ are obviously very similar to the Jacobi–Trudi matrix. We remind the reader that the Jacobi–Trudi matrix of dimension $d \times d$ for the partition λ ($d \geq l(\lambda)$) is the matrix

$$\mathbf{JT}^d_\lambda = \begin{pmatrix} \mathbf{h}_{\lambda_1} & \prec & & \prec & \mathbf{h}_{d-1+\lambda_1} \\ & \searrow & & \prec & \\ & & \mathbf{h}_{\lambda_i} & & \\ & \prec & & \searrow & \\ \mathbf{h}_{1-d+\lambda_d} & \prec & & \prec & \mathbf{h}_{\lambda_d} \end{pmatrix}_{d\times d},$$

where we respected the same conventions with arrows. We define $\tilde{\mathbf{JT}}^d_\lambda$ in a totally analogous way (*i.e.,* using $\tilde{\mathbf{h}}$'s). It is a central theorem of the theory of symmetric functions that $\det(\mathbf{JT}^d_\lambda) = \mathbf{s}_\lambda$ (see [Bum04, Theorem 37.1]) and is thus independent of d (as long as $d \geq l(\lambda)$). Similarly, $\det(\tilde{\mathbf{JT}}^d_\lambda) = \tilde{\mathbf{s}}_\lambda$.

We are now ready to comment on the result of Tracy and Widom a bit further.

Lemma 4.1 $\mathbf{TW}^{\lambda\,\varnothing} = \mathbf{s}_\lambda$ *and* $\mathbf{TW}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu$.

Proof We will only prove the case $\mu = \varnothing$. Pick $d \geq l(\lambda)$. The matrix on the left-hand side in the definition of $\mathbf{TW}^{\lambda\,\varnothing}$ is then lower triangular, with 1's on the main diagonal. Without affecting the final determinant, we can row-reduce this matrix to $(\delta_{ij})_{d\times\infty}$, with δ_{ij} the Kronecker delta.

Hence we easily compute

$$\begin{aligned}\mathbf{TW}^{\lambda\,\varnothing} &= \begin{pmatrix} \mathbf{h}_{\lambda_d} & & & & \mathbf{h}_{d-1+\lambda_1} \\ \curlyvee & \nwarrow & & & \curlyvee \\ & & \mathbf{h}_{\lambda_{d-i+1}} & & \\ \curlyvee & & & \nwarrow & \curlyvee \\ \mathbf{h}_{1-d+\lambda_d} & & & & \mathbf{h}_{\lambda_1} \end{pmatrix}_{d\times d} \\ &= \det\big((\mathbf{JT}^d_\lambda)_{d+1-j,d+1-i}\big) = \det \mathbf{JT}^d_\lambda = \mathbf{s}_\lambda.\end{aligned}$$

[5]This is not important, but there is also a "cascading effect" among non-zero entries. In the first matrix for instance, the last non-zero entry on each row (*i.e.,* $\tilde{\mathbf{h}}_0$) has to be (weakly) to the right of any non-zero entry on the rows above.

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The key observation is thus that the $d \times d$ truncation of the matrix on the right-hand side in the Tracy–Widom determinant is the anti-transpose[6] of the Jacobi–Trudi matrix, and that a determinant is not affected under anti-transposition. ■

We can now get started on the proof of Theorem 1.3.

Proof of Theorem 1.3 We need to compute $\boldsymbol{\Delta}(\mathbf{TW}^{\lambda\,\varnothing} \cdot \mathbf{TW}^{\varnothing\,\mu})$. We have

$$\boldsymbol{\Delta} = \exp\Big(\sum_k k\partial_{\mathbf{p}_k}\partial_{\tilde{\mathbf{p}}_k}\Big) = \exp\Big(\sum_k \frac{\mathbf{p}_k\tilde{\mathbf{p}}_k}{k}\Big)^{\perp}.$$

The exponential can easily be expanded to obtain

$$\boldsymbol{\Delta} = \Big(\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu\Big)^{\perp},$$

where the sum is over all partitions ν. We now make use of the Cauchy identity

$$\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu = \prod_{\substack{x_i\in\overline{X}\\ y_j\in\overline{Y}}} \frac{1}{1-x_iy_j} = \sum_\nu \mathbf{s}_\nu\tilde{\mathbf{s}}_\nu$$

and obtain our final expression:

$$\boldsymbol{\Delta} = \Big(\sum_\nu \frac{1}{z_\nu}\mathbf{p}_\nu\tilde{\mathbf{p}}_\nu\Big)^{\perp} = \Big(\sum_\nu \mathbf{s}_\nu\tilde{\mathbf{s}}_\nu\Big)^{\perp}.$$

Coming back to our original computation, we just obtained

$$\boldsymbol{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) = \sum_\nu \mathbf{s}_\nu^{\perp}(\mathbf{s}_\lambda)\tilde{\mathbf{s}}_\nu^{\perp}(\tilde{\mathbf{s}}_\mu). \tag{4.2}$$

Observe that

$$\mathbf{s}_\nu^{\perp}(\mathbf{s}_\lambda) = \sum_\mu \langle \mathbf{s}_\nu^{\perp}(\mathbf{s}_\lambda), \mathbf{s}_\mu\rangle \mathbf{s}_\mu = \sum_\mu \langle \mathbf{s}_\lambda, \mathbf{s}_\mu\cdot\mathbf{s}_\nu\rangle\mathbf{s}_\mu = \sum_\mu c^\lambda_{\mu\nu}\mathbf{s}_\mu = \mathbf{s}_{\lambda/\nu}.$$

The last sum, which involves the Littlewood–Richardson coefficients, is precisely the definition of $\mathbf{s}_{\lambda/\nu}$.

Armed with this observation, we can thus rework equation (4.2) into

$$\boldsymbol{\Delta}\big(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\big) = \sum_\nu \mathbf{s}_{\lambda/\nu}\tilde{\mathbf{s}}_{\mu/\nu}.$$

When ν runs through all partitions, the skew function $\mathbf{s}_{\lambda/\nu}$ runs through all $d\times d$ minors $(\mathbf{h}_{j-i-\nu_i+\lambda_{d-j+1}})_{d\times d}$ of the matrix $(\mathbf{h}_{j-i+\lambda_{d-j+1}})_{\infty\times d}$. Similarly, $\tilde{\mathbf{s}}_{\mu/\nu}$ will run

[6] The anti-transpose of a matrix is its transposed along the main anti-diagonal.

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through the minors $(\tilde{\mathbf{h}}_{i-j-\nu_j+\mu_{d-i+1}})_{d\times d}$ of $(\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}})_{d\times\infty}$. Moreover, the minors obtained with $\mathbf{s}_\nu^\perp$ and $\tilde{\mathbf{s}}_\nu^\perp$ are paired up just as in the Cauchy–Binet identity. Therefore, we obtain

$$\begin{aligned}\boldsymbol{\Delta}\big(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\big) &= \det\big((\tilde{\mathbf{h}}_{i-j+\mu_{d-i+1}})_{d\times\infty}\cdot(\mathbf{h}_{j-i+\lambda_{d-j+1}})_{\infty\times d}\big)\\ &= \mathbf{TW}^{\lambda\,\mu},\end{aligned}$$

and we are done. The proof for $\mathrm{TW}^{\lambda\,\mu}$ simply follows from applying the homomorphism F. ■

5 The Proof of Theorem 1.1

Proof We have from Lemmas 3.2 and 4.1 that

$$\mathbf{BD}^{\lambda\,\varnothing} = \mathbf{s}_\lambda = \mathbf{TW}^{\lambda\,\varnothing} \quad \text{and} \quad \mathbf{BD}^{\varnothing\,\mu} = \tilde{\mathbf{s}}_\mu = \mathbf{TW}^{\varnothing\,\mu}.$$

Tracing back to those lemmas, this is a direct consequence of the Jacobi–Trudi identity.

The theorem now follows. We have

$$\begin{aligned}\mathbf{BD}^{\lambda\,\mu} &= \boldsymbol{\Delta}\left(\mathbf{BD}^{\lambda\,\varnothing}\cdot\mathbf{BD}^{\varnothing\,\mu}\right) && \text{(Theorem 1.2)}\\ &= \boldsymbol{\Delta}\left(\mathbf{TW}^{\lambda\,\varnothing}\cdot\mathbf{TW}^{\varnothing\,\mu}\right) && \text{(Lemmas 3.2 and 4.1)}\\ &= \mathbf{TW}^{\lambda\,\mu} && \text{(Theorem 1.3).}\end{aligned}$$ ■

6 Some Relations among $\mathrm{R}^{\lambda\,\mu}$'s

We now consider $\mathrm{R}^{\lambda\,\mu}$ as an element of V, and immediately see that Corollary 1.4 is a consequence of the previous theorems. Two very natural properties of $\mathrm{R}^{\lambda\,\mu}$ also pop out of the presentation due to Tracy and Widom. The proofs rely only on basic properties of determinants and the Tracy–Widom expression $\mathbf{TW}^{\lambda\,\mu}$, and their statement does not involve differential operators. Unlike Corollary 1.4, they could thus be stated by evaluation at a specific σ.

Proposition 6.1 *Let (r) and (s) denote partitions with just one part each, of size $r \geq 1$ and $s \geq 1$, and let λ, μ be partitions with $\max(l(\lambda), l(\mu)) \leq d$. Then*

$$\mathrm{R}^{(r)\,(s)} = \mathrm{R}^{(r)\,\varnothing}\cdot\mathrm{R}^{\varnothing\,(s)} + \mathrm{R}^{(r-1)\,(s-1)} \tag{6.1}$$

and

$$\mathrm{R}^{\lambda\,\mu} = \det\big(\mathrm{R}^{(\lambda_i+d-i)\,(\mu_j+d-j)}\big)_{1\leq i,j\leq d}. \tag{6.2}$$

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Proof Both results follow from the same fact:

$$\begin{aligned}\mathbf{TW}^{(r)\,(s)} &= \det\big((\tilde{\mathbf{h}}_{1-j+s})_{1\times\infty}\cdot(\mathbf{h}_{1-i+r})_{\infty\times 1}\big)\\ &= \tilde{\mathbf{h}}_s\mathbf{h}_r + \tilde{\mathbf{h}}_{s-1}\mathbf{h}_{r-1} + \cdots\\ &= \tilde{\mathbf{h}}_s\mathbf{h}_r + \mathbf{TW}^{(r-1)\,(s-1)}\\ &= \mathbf{TW}^{(r)\,\varnothing}\mathbf{TW}^{\varnothing\,(s)} + \mathbf{TW}^{(r-1)\,(s-1)},\end{aligned}$$

which proves equation (6.1).

For equation (6.2), we just need to observe that $\mathbf{TW}^{\lambda\,\mu}$ is defined as the determinant of a matrix $\mathbf{M}$ that itself is a product of two matrices. The coefficient on the i^{th} row and the j^{th} column of $\mathbf{M}$ is given by

$$\mathbf{M}_{ij} = \sum_{k=0}^{\infty}\tilde{\mathbf{h}}_{i-1-k+\mu_{d+1-i}}\mathbf{h}_{j-1-k+\lambda_{d+1-j}},$$

where this sum is actually finite (because the terms eventually vanish).

By the reasoning of equation (6.1), we actually know that

$$\mathbf{M}_{ij} = \mathbf{TW}^{(j-1+\lambda_{d+1-j})\,(i-1+\mu_{d+1-i})}.$$

Equation (6.2) then follows from the invariance of determinants under transposition and anti-transposition. ■

7 Conclusion and Speculation

To summarize this paper, we have reproved the Bump–Diaconis, Tracy–Widom identity (and proved Corollary 1.4) through specialization from the deeper symmetric function identity

$$\boldsymbol{\Delta}\left(\mathbf{s}_\lambda\cdot\tilde{\mathbf{s}}_\mu\right) = \sum_\nu \mathbf{s}_{\lambda/\nu}\tilde{\mathbf{s}}_{\mu/\nu}.$$

We feel that this more axiomatic approach to random matrix theory integrals through the theory of symmetric functions has a lot of potential. The backbone of symmetric function theory is common with much of the work of Fauser and Jarvis [FJK, FJKW06, FJ04] on group branchings (which they sometimes specialize for perturbative quantum field theory). In particular, the operator Δ appears as a twisted product or "Cliffordization" in [FJ04], and results generalizing Theorems 1.2 and 1.3 to other groups have been obtained in [FJKW06]. Our methods, however, seem to be much simpler, mostly because the differential operator Δ allows us to encode the Newell–Littlewood formula in a generating series form. We thus hope the techniques presented here will naturally expand to all classical compact Lie groups. Note that this generalization for expressions of the type $R^{\varnothing\lambda}$ has already been achieved (independently) in [Deh07].

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