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A TAUBERIAN THEOREM OF EXPONENTIAL TYPE

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1. Introduction. We will be interested in Tauberian theorems concerning the limiting behaviour of a monotone function U and its Laplace transform

$$\hat{U}(s) = s \int_0^\infty U(t)e^{-st}dt, \quad s > 0.$$

A famous theorem of Karamata concerns the case in which the function U is regularly varying (i.e., $U(tx)/U(t) \to x^\alpha(t \to \infty)$ for $x > 0$). Here we will consider functions U that grow faster, in fact our conditions will be in terms of log U rather than on U itself. So it is convenient to write the Laplace transform in terms of $q = \log U$. For a function $q:\mathbf{R}^+ \to \mathbf{R}^+$ such that exp q is locally integrable and

$$\lim_{t\to\infty} q(t)/t = 0$$

we define the function $\tilde{q}$ by the relation

$$\tilde{q}(s): = \log \hat{U}(s) = \log s \int_0^\infty e^{q(t)-st}dt, \quad s > 0.$$

This transform was considered in a paper by Kohlbecker [**9**], who proved the following result.

THEOREM 1.1. *Assume that* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing and integrable on* $(0, 1)$ *and* $s(\infty) = 0$. *Let* α *and* β *be real constants,* $0 < \beta < 1$ *and let* α *be related to* β *by* $\alpha^{-1} + \beta^{-1} = 1$. *Consider the statements*

$$(1.1) \qquad q(t) \sim \int_0^t s(u)du \quad (t \to \infty)$$

with s satisfying

$$s(tx)/s(t) \to x^{\beta-1} \quad (t \to \infty) \quad x > 0$$

and (with $s^{\leftarrow}$ *an inverse function of s)*

$$(1.2) \qquad \tilde{q}(s) \sim \int_s^\infty s^{\leftarrow}(x)dx \quad (s \to 0+)$$

with $s^{\leftarrow}$ *satisfying*

$$s^{\leftarrow}(tx)/s^{\leftarrow}(t) \to x^{\alpha-1} \quad (t \to 0+), \quad x > 0.$$

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If $\exp q(x)$ *is locally integrable, then* (1.1) *implies that* $\tilde{q}(s)$ *is finite* $(s > 0)$ *and satisfies* (1.2). *If* q *is non-decreasing*, (1.2) *implies* (1.1).

Remark that (1.1) is equivalent to

$$\lim_{t\to\infty} q(tx)/q(t) = x^{\beta} \quad \text{for } x > 0$$

and (1.2) is equivalent to

$$\lim_{s\to 0+} \tilde{q}(sx)/\tilde{q}(s) = x^{\alpha} \quad \text{for } x > 0$$

(see [**2**]).

Note that the function

$$S(t): = \int_0^t s(u)du$$

is concave and that the function

$$T(s): = \int_s^{\infty} s^{\leftarrow}(u)du$$

is its complementary concave function (cf. [**11**]). Another way to get the complementary function T from S is by the relation

(1.3) $$T(s) = \sup\{S(x) - xs, x > 0\} \quad s > 0.$$

This alternative definition can be extended to the class of functions D^* defined as follows.

Definition. The set D^* consists of all functions $q:\mathbf{R}^+ \to \mathbf{R}^+$ which are locally bounded, $q(\infty) = \infty$ and $q(t) = o(t)(t \to \infty)$. The set D_* consists of all functions $q:\mathbf{R}^+ \to \mathbf{R}^+$ which are bounded in every interval (a, ∞) for $a > 0$ and for which $q(0+) = \infty$.

Definition. For $q \in D^*$ the *complementary function* q^* is defined by

(1.4) $$q^*(y) = \sup\{q(x) - xy, x > 0\} \quad y > 0.$$

The inverse transform is defined as follows.

Definition. For $q \in D_*$ we define

(1.5) $$q_*(x) = \inf\{q(y) + xy, y > 0\}.$$

Note that if

$$q(x): = \int_0^x s(u)du$$

with $s:\mathbf{R}^+ \to \mathbf{R}^+$ decreasing, then for any point $(x, y = s(x))$ on the graph of the function s we have

$$q^*(y) = \int_y^{\infty} s^{\leftarrow}(u)du = \int_0^x s(t)dt - xy = q(x) - xy.$$

Theorem 1.1 says that (1.1) implies $\tilde{q}(s) \sim T(s)(s \to 0+)$ and (1.2) implies $q(t) \sim S(t)(t \to \infty)$. But under the assumptions of Theorem 1.1 we have $q \in D^*$ and $\tilde{q} \in D_*$. In fact an analogue of Theorem 1.1 can be proved for the * transform:

THEOREM 1.2. *Theorem* 1.1 *holds with* $\tilde{q}$ *replaced by* q^*. *In particular then*

$$\tilde{q}(s) \sim q^*(s) \quad (s \to 0+).$$

Similar results can be proved in case $\beta < 0$ or $\beta > 1$ (see e.g. [**10**]). The case $\beta = 0$ has been discussed in [**2**].

In this paper we consider the case $\beta = 1$, i.e., we consider functions q with

$$t^{-\alpha}q(t) \to \infty \quad (t \to \infty) \quad \text{for all } \alpha < 1$$

but $t^{-1}q(t) \to 0$ (otherwise the transform $\tilde{q}(s)$ would not be finite for $s > 0$). In order to be able to formulate these results we need the following definitions.

Definition (cf [**5**]). Suppose $s:(0, \infty) \to \mathbf{R}$ is measurable. We say that $s \in \Pi^-$ if there exists a positive function a such that for all $x > 0$

$$(1.6) \qquad \lim_{t \to \infty} \frac{s(tx) - s(t)}{a(t)} = -\ln x.$$

The function a is called the auxiliary function. It is defined only up to asymptotic equivalence.

Definition (cf [**5**]). Suppose $g:(0, \infty) \to (0, \infty)$ is nonincreasing. We say $g \in \Gamma^-$ if there exists a positive function f such that

$$(1.7) \qquad \lim_{t \to 0+} \frac{g(t + xf(t))}{g(t)} = e^{-x} \quad \text{for } x > 0.$$

If h is non-increasing, we have $h \in \Pi^-$ if and only if $h^{\leftarrow} \in \Gamma^-$ (see [**5**]). We will prove basically that $q(t)/t \in \Pi^-$ is equivalent to $q^*(s) \in \Gamma^-$ or $\tilde{q}(s) \in \Gamma^-$.

Definition. Suppose $q_1, q_2:(0, \infty) \to (0, \infty)$ and $q_i(t) = o(t)(t \to \infty)$ for $i = 1, 2$. We say $q_1(t) \stackrel{\bullet}{\sim} q_2(t) \quad (t \to \infty)$ if for all $c > 1$ there exists a $t_o = t_o(c)$ such that for all $t > t_o$

$$(1.8) \qquad q_1(tc) \leqq cq_2(t) \quad \text{and} \quad q_2(tc) \leqq cq_1(t).$$

Note that if $q_i(t)/t$ is non-increasing for $i = 1, 2$, then (1.8) is equivalent to : the inverse of $q_1(t)/t$ is asymptotically equivalent to the inverse of $q_2(t)/t$. Note also that

$$q_1(t) \stackrel{\bullet}{\sim} q_2(t)(t \to \infty)$$

if and only if

$$q_1(t)/t \overset{*}{\sim} q_2(t)/t$$

with the relation $\overset{*}{\sim}$ as defined in [**2**] but with inverse inequalities.

Hence if $q_1(t)/t \in \Pi^-$ has auxiliary function $a(t)$, then

$$q_1(t) \overset{\bullet}{\sim} q_2(t) \quad (t \to \infty)$$

if and only if

$$q_1(t) - q_2(t) = o(ta(t)) \quad (t \to \infty).$$

See also [**2**], Lemma 2.

THEOREM 1.3. *Suppose* $q:(0, \infty) \to (0, \infty)$ *is locally bounded,* $q(t) \to \infty \quad (t \to \infty)$ *and* $q(t) = o(t) \quad (t \to \infty)$. *Consider the statements*

(1.9) $\qquad q(t)/t \in \Pi^-$

and

(1.10) $\qquad q^* \in \Gamma^-.$

Under the stated conditions (1.9) *implies* (1.10). *Conversely if* $q(t)/t$ *is non-increasing*, (1.10) *implies* (1.9).

Moreover if $s:(0, \infty) \to (0, \infty)$ *is non-increasing*

(1.11) $\qquad q(t) \overset{\bullet}{\sim} \int_0^t s(x)dx \quad (t \to \infty) \quad$ *with* $s \in \Pi^-$

implies

(1.12) $\qquad q^*(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+) \quad$ *with* $s^{\leftarrow} \in \Gamma^-.$

Conversely if $q(t)/t$ *is non-increasing* (1.12) *implies* (1.11).

Note that (1.11) is true for some $s \in \Pi^-$ if and only if (1.9) holds and similarly for (1.12) and (1.10). This will be shown in Lemma 2.3.

THEOREM 1.4. *Suppose* q *satisfies the basic assumptions in Theorem* 1.3. *Consider the statements*

(1.13) $\qquad q(t)/t \in \Pi^-$

and

(1.14) $\qquad \tilde{q} \in \Gamma^-.$

Relation (1.13) *implies* (1.14). *Conversely if* q *is non-decreasing and* $q(t)/t$ *is non-increasing* (1.14) *implies* (1.13).

Moreover in that situation with $s:(0, \infty) \to (0, \infty)$ *non-increasing*

(1.15) $\qquad q(t) \overset{\bullet}{\sim} \int_0^t s(x)dx \quad (t \to \infty) \quad$ *with* $s \in \Pi^-$

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if and only if

$$(1.16) \qquad \tilde{q}(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+) \quad \textit{with } s^{\leftarrow} \in \Gamma^-.$$

Let us state some sufficient conditions for (1.13) and (1.14) that are easier to verify.

(i) In case q is twice differentiable (1.13) is implied by

$$(1.13a) \qquad \frac{t^2 q''(t)}{tq'(t) - q(t)} \to 1 \quad (t \to \infty)$$

(define $\rho(t) := -t^2(q(t)/t)'$; relation (1.13a) is the same as

$$t\rho'(t)/\rho(t) \to 1 \quad (t \to \infty)$$

implying

$$\rho(tx) \sim x\rho(t) \quad (t \to \infty, x > 0),$$

which in turn implies (1.13). See [**5**]).

(ii) Relation (1.14) is implied by

$$(1.14a) \qquad \frac{(\tilde{q})''(s)\tilde{q}(s)}{\{(\tilde{q})'(s)\}^2} \to 1 \quad (s \to 0+).$$

For a proof see [**5**], Theorem 1.5.3.

We next give two explicit examples, showing the scope of applicability of the theorem.

1. If

$$q(t) = t/\log^\alpha t + o(t/\log^{\alpha+1} t) \quad (t \to \infty)$$

(or, equivalently, $q(t) \stackrel{\bullet}{\sim} t/\log^\alpha t$, $t \to \infty$) for some $\alpha > 0$, then by Theorem 1.4 this implies

$$\tilde{q}(s) \sim \alpha s^{1+1/\alpha} \exp(-1 + s^{-1/\alpha}) \quad (s \to 0+).$$

The converse statement holds under the assumptions q non-decreasing and $q(t)/t$ non-increasing.

2.

$$\begin{aligned} q(t) = {} & t/(\log t)^{1/\alpha} + \alpha^{-1}(1 + \alpha^{-1})t \log\log t/(\log t)^{1+1/\alpha} \\ & + \alpha^{-1}(1 + \log \alpha)t/(\log t)^{1+1/\alpha} \\ & + o(t/(\log t)^{1+1/\alpha}) \quad (t \to \infty) \end{aligned}$$

for some $\alpha > 0$ if and only if

$$\tilde{q}(s) \sim \exp(1/s^\alpha) \quad (s \to 0+).$$

We will show by an example that the converse parts of Theorems 1.3 and 1.4 fail to hold if the condition $q(t)/t$ non-increasing is dropped. It is somewhat surprising that the class Γ^- plays an essential role in a Tauberian theorem concerning the Laplace transform since in the defining relation (1.7) we use addition rather than multiplication of the arguments.

The outline of the paper is as follows. First we study some function classes (Section 2). These are essential in the formulation of the Abelian and Tauberian theorems we prove next (Sections 3 and 4). These theorems (Theorems 3.1, 3.3, 4.1 and 4.2) contain Theorems 1.3 and 1.4 as special cases. The statement of Theorem 4.2 resembles the "Satz" of Wagner [**12**]. Both Wagner's conditions and his conclusion are weaker. Condition (3) in Wagner's theorem, namely: there exists a function g with

$$0 < \delta \leqq g(s)g''(s)/(g'(s))^2 \leqq 1$$

as $s \to 0+$ and $g(s) \sim \tilde{q}(s)$ as $s \to 0+$ is related to the condition $\tilde{q} \in \Gamma^-$ (compare [**5**] Theorems 2.5.2, 2.7.4). But observe that $\tilde{q} \in \Gamma^-$ does not imply (3) and furthermore for nondecreasing functions the relation $\stackrel{\bullet}{\sim}$ is stronger than the relation $\sim$.

Some methods of proof are adapted from [**2**].

2. Some function classes. For the formulation and proof of the main theorems we need to discuss the definition and some properties of certain classes of functions.

Definition. A measurable function a is *0-regularly varying* if for all $x > 0$

(2.1) $$a(tx) \asymp a(t) \quad \text{for } t \to \infty$$

(we say that $g(t) \asymp h(t)$ for $t \to \infty$ if g and h are positive and $\log g(t)/h(t)$ is bounded for $t \to \infty$).

It can be proved (see [**1**]) that a is 0-regularly varying if and only if the *upper* and *lower index* of a defined by

$$\overline{\text{index}}\ a = \lim_{x\to\infty} \frac{\log \varlimsup_{t\to\infty} a(tx)/a(t)}{\log t}$$

and

$$\underline{\text{index}}\ a = \lim_{x\to\infty} \frac{\log \varliminf_{t\to\infty} a(tx)/a(t)}{\log t}$$

respectively, exist and are finite.

Definition. A non-increasing function $f:(0, \infty) \to \mathbf{R}$ is *asymptotically* *balanced*$^+$ (notation: $f \in \text{As Bal}^+$) if there exists a positive function a such that for all $x > 1$

$$f(t) - f(tx) \asymp a(t) \quad \text{for } t \to \infty.$$

The function a is called an auxiliary function for f. It is defined up to $\asymp$ – equivalence. Asymptotically balanced functions are defined in [7]. There it is shown that any auxiliary function a is 0-regularly varying. In the particular case when $a \asymp 1$ and the function U is defined by

$$U(t) = \exp\{-f(t)\}$$

we say that $U \in BI \cap PI^+$. This notation is chosen for reasons which would become obvious to the reader upon reading [6].

The class Π^- introduced in Section 1 is an important subclass of the asymptotically balanced$^+$ functions.

LEMMA 2.1. *Suppose*

$$q(t) = \int_0^t s(u)du$$

with s positive, non-increasing and

(2.2) $\quad s(u) \to 0 \quad (u \to \infty).$

Define

(2.3) $\quad r(t) := q(t)/t \quad (t > 0)$

and

(2.4) $\quad b(x) := \int_0^x s(u)du - xs(x).$

Then the following statements are equivalent.

(i) *s is asymptotically balanced$^+$ with an auxiliary function a satisfying* $\underline{\text{index}}\ a > -1$.

(ii) $b \in BI \cap PI^+$.

(iii) *r is asymptotically balanced$^+$ with an auxiliary function a satisfying* $\underline{\text{index}}\ a > -1$ *and for all* $0 < \epsilon < 1$

(2.5) $$\overline{\lim_{t\to\infty}} \frac{(q(t(1+\epsilon)) - 2q(t) + q(t(1-\epsilon)))}{ta(t)} < 0.$$

Proof. Note that b is non-negative and non-decreasing,

$$s(x) = -\frac{b(x)}{x} + \int_x^\infty \frac{b(u)}{u^2}du \quad \text{and}$$

$$r(x) = \int_x^\infty \frac{b(u)}{u^2}du.$$

Note that

$$\int_1^\infty b(u)du/u^2 < \infty$$

hence $\underline{\text{index}}\ b \leqq 1$.

(i) $\Rightarrow$ (ii). As in [**8**], Lemma 2 we find

$$ta(t) \asymp b(t) \quad (t \to \infty).$$

Since s is non-increasing we have for $t > 1$, $v > 0$

$$b(vt) - b(v) = v(s(v) - s(vt)) - v(t - 1)s(vt) + \int_v^{vt} s(u)du \geqq v(s(v) - s(vt)).$$

Hence

$$\varliminf_{v\to\infty} \frac{b(vt) - b(v)}{b(v)} \geqq \varliminf_{v\to\infty} \frac{s(v) - s(vt)}{b(v)/v} > 0$$

for all $t > 1$.

(ii) $\Rightarrow$ (i). Since

$$s(x) = -\frac{b(x)}{x} + \int_x^\infty \frac{b(u)}{u^2}du \quad (x > 0)$$

we have by Fatou's lemma

$$\varliminf_{t\to\infty} \frac{s(t) - s(tx)}{b(t)/t} \geqq \int_1^x \varliminf_{t\to\infty} \frac{b(ut)}{b(t)}\frac{du}{u^2} + \frac{1}{x}\varliminf_{t\to\infty}\frac{b(tx)}{b(t)} - 1$$

$$= \int_1^x \left(\varliminf_{t\to\infty} \frac{b(ut)}{b(t)} - 1\right)\frac{du}{u^2} + \frac{1}{x}\left(\varliminf_{t\to\infty}\frac{b(tx)}{b(t)} - 1\right) > 0.$$

(ii) $\Rightarrow$ (iii). Since

$$r(x) = \int_x^\infty b(u)\frac{du}{u^2}$$

we have

$$\frac{r(t) - r(tx)}{b(t)/t} = \int_1^x \frac{b(ut)}{b(t)}\frac{du}{u^2}$$

and all limit points of the right-hand side ($t \to \infty$) are finite and $> 1 - x^{-1}$ for all $x > 1$ since $b \in BI \cap PI^+$. Hence r is asymptotically balanced$^+$ and its auxiliary function $a(t) := b(t)/t$ satisfies $\underline{\text{index}}\ a > -1$ since $\underline{\text{index}}\ b > 0$. Moreover

$$(2.6) \qquad [q(t(1 + \epsilon)) - 2q(t) + q(t(1 - \epsilon))]/b(t)$$
$$= t(1 + \epsilon)[r(t(1 + \epsilon)) - r(t)]/b(t)$$

$$+ t(1 - \epsilon)[r(t(1 - \epsilon)) - r(t)]/b(t)$$
$$= -(1 + \epsilon)\int_1^{1+\epsilon} \frac{b(ut)}{b(t)}\frac{du}{u^2} + (1 - \epsilon)\int_{1-\epsilon}^1 \frac{b(ut)}{b(t)}\frac{du}{u^2}.$$

Hence

$$\varlimsup_{t\to\infty} \{q(t(1 + \epsilon)) - 2q(t) + q(t(1 - \epsilon))\}/b(t)$$
$$\leqq -(1 + \epsilon)\int_1^{1+\epsilon} \varliminf_{t\to\infty} \frac{b(ut)}{b(t)}\frac{du}{u^2}$$
$$+ (1 - \epsilon)\int_{1-\epsilon}^1 \varlimsup_{t\to\infty} \frac{b(ut)}{b(t)}\frac{du}{u^2}$$
$$< -(1 + \epsilon)\int_1^{1+\epsilon} \frac{du}{u^2} + (1 - \epsilon)\int_{1-\epsilon}^1 \frac{du}{u^2} = 0.$$

(iii) $\Rightarrow$ (ii). Since b is non-decreasing we have for $x > 1$ and $t > 0$

$$\frac{b(t)}{ta(t)}\left(1 - \frac{1}{x}\right) \leqq \frac{r(t) - r(tx)}{a(t)}$$
$$= \int_1^x \frac{b(ut)}{ta(t)}\frac{du}{u^2} \leqq \frac{b(tx)}{txa(tx)}\frac{txa(tx)}{ta(t)}\left(1 - \frac{1}{x}\right).$$

Since

$$\{r(t) - r(tx)\} \asymp a(t) \quad \text{and} \quad a(tx) \asymp a(t)$$

for some $x > 1$, $b(t) \asymp ta(t)$ and hence

$$\varlimsup_{t\to\infty} b(tx)/b(t) < \infty \quad \text{for all } x > 1.$$

Now suppose $b \notin BI \cap PI^+$, i.e., for some $\delta > 0$ and some sequence $t'_n \to \infty \quad (n \to \infty)$

$$\frac{b(t'_n x)}{b(t'_n)} \to 1 \quad (n \to \infty) \quad \text{for all } x \in [1, 1 + \delta].$$

Then

$$\frac{b(t'_n(1 + \delta/2)s)}{b(t'_n(1 + \delta/2))}$$
$$= \frac{b(t'_n(1 + \delta/2)s)}{b(t'_n)}\frac{b(t'_n)}{b(t'_n(1 + \delta/2))} \to 1 \quad (n \to \infty)$$

for all

$$s \in I: = [(1 + \delta/2)^{-1}, (1 + \delta)/(1 + \delta/2)].$$

Hence

$$\frac{b(t_n s)}{b(t_n)} \to 1 \quad (n \to \infty)$$

for all $s \in I$ with $t_n = t_n'(1 + \delta/2)$. Now take $\epsilon > 0$ such that

$$[1 - \epsilon, 1 + \epsilon] \subset [\,(1 + \delta/2)^{-1}, (1 + \delta)/(1 + \delta/2)\,].$$

Then for some $\Delta > 0$ and n sufficiently large we have

$$\begin{aligned} -\Delta &\geqq \frac{q(t_n(1 + \epsilon)) - 2q(t_n) + q(t_n(1 - \epsilon))}{b(t_n)} \\ &= -(1 + \epsilon) \int_1^{1+\epsilon} \frac{b(t_n u)}{b(t_n)} \frac{du}{u^2} \\ &\quad + (1 - \epsilon) \int_{1-\epsilon}^{1} \frac{b(t_n u)}{b(t_n)} \frac{du}{u^2} \to 0 \quad (n \to \infty). \end{aligned}$$

This contradiction finishes the proof.

The next lemma shows that the asymptotic concavity property (2.5) is also true for any q_1 with $q_1 \stackrel{\bullet}{\sim} q$ at infinity.

LEMMA 2.2. *Suppose a, s, b, r and q are as in Lemma* 2.1. *If $q_1 \stackrel{\bullet}{\sim} q$ and one of the statements in Lemma* 2.1 *holds, then statement* (iii) *of Lemma* 2.1 *holds with q replaced by q_1 and r replaced by $r_1(t) := q_1(t)/t$.*

Proof. Suppose q satisfies (iii) in Lemma 2.1. Define $r_1(t) := q_1(t)/t$. Since for any $c > 1$ we have $r(t/c) \geqq r_1(t) \geqq r(tc)$ for t sufficiently large we find that r_1 is asymptotically balanced$^+$. Suppose $\epsilon > 0$ is arbitrary. Since $q_1 \stackrel{\bullet}{\sim} q$ at infinity we have for ϵ, $\delta > 0$ with $(1 + \delta)^2 < 1 + \epsilon$ and t sufficiently large

$$\begin{aligned} &q_1(t(1 + \epsilon)) - 2q_1(t) + q_1(t(1 - \epsilon)) \\ &\leqq (1 + \delta)q\left(\frac{t(1 + \epsilon)}{1 + \delta}\right) - \frac{2}{1 + \delta} q(t(1 + \delta)) \\ &+ (1 + \delta)q\left(\frac{t(1 - \epsilon)}{1 + \delta}\right) \\ &= t(1 + \epsilon)\left\{r\left(\frac{t(1 + \epsilon)}{1 + \delta}\right) - r(t(1 + \delta))\right\} \\ &+ t(1 - \epsilon)\left\{r\left(\frac{t(1 - \epsilon)}{1 + \delta}\right) - r(t(1 + \delta))\right\}. \end{aligned}$$

Since

$$r(t) = \int_t^\infty b(u)du/u^2,$$

we have

$$\{q_1(t(1 + \epsilon)) - 2q_1(t) + q_1(t(1 - \epsilon))\}/b(t)$$

$$\leqq -(1 + \epsilon) \int_{1+\delta}^{(1+\epsilon)/(1+\delta)} \frac{b(tu)}{b(t)} \frac{du}{u^2}$$

$$+ (1 - \epsilon) \int_{(1-\epsilon)/(1+\delta)}^{1+\delta} \frac{b(tu)}{b(t)} \frac{du}{u^2}.$$

Hence

$$\overline{\lim_{t\to\infty}} \frac{q_1(t(1 + \epsilon)) - 2q_1(t) + q_1(t(1 - \epsilon))}{b(t)}$$

$$\leqq -(1 + \epsilon) \int_{1+\delta}^{(1+\epsilon)/(1+\delta)} \underline{\lim_{t\to\infty}} \frac{b(tu)}{b(t)} \frac{du}{u^2}$$

$$+ (1 - \epsilon) \int_{(1-\epsilon)/(1+\delta)}^{1+\delta} \overline{\lim_{t\to\infty}} \frac{b(tu)}{b(t)} \frac{du}{u^2} = :I(\epsilon, \delta).$$

We have $I(\epsilon, 0) < 0$ for $\epsilon > 0$ since $b \in BI \cap PI^+$ as we have seen before. Also $I(\epsilon, \delta)$ is continuous in δ for fixed $\epsilon > 0$. It follows that for fixed $\epsilon > 0$ there is a $\delta > 0$ such that $I(\epsilon, \delta) < 0$.

For an understanding of the formulation of Theorem 3.3 the following lemma is helpful.

LEMMA 2.3. (i) *Suppose* $q:(0, \infty) \to (0, \infty)$. *Then* $q(t)/t \in \Pi^-$ *if and only if there is a nonincreasing* $s \in \Pi^-$ *such that*

$$q(t) \overset{\bullet}{\sim} \int_0^t s(x)dx \quad (t \to \infty).$$

(ii) *Suppose* $g:\mathbf{R} \to (0, \infty)$ *is non-increasing. Then* $g \in \Gamma^-$ *if and only if there is a* $t \in \Gamma^-$ *such that*

$$g(s) \sim \int_s^\infty t(y)dy \quad (s \to 0+).$$

Proof. (i) If $q_1(t)/t \in \Pi^-$, then

$$q(t) \overset{\bullet}{\sim} e^{-1} \int_0^{te} q(s)ds/s$$

$$\overset{\bullet}{\sim} e^{-2} \int_0^{te} \int_0^{se} q(u)du/uds/s,$$

(cf. [**5**] Theorem 1.4.1; the monotonicity is not required, compare [**4**] Theorem 17).

Conversely, if $s \in \Pi^-$, also

$$t^{-1}\int_0^t s(x)dx \in \Pi^-$$

(cf. [4]). It is easy to see that if $q(t) \stackrel{\bullet}{\sim} q_1(t)$ and $q_1(t)/t \in \Pi^-$, then also $q(t)/t \in \Pi^-$.

(ii) If $g \in \Gamma^-$ we have the representation (see [3])

$$g(s) \sim \exp \int_s^\infty \frac{du}{f(u)} \quad (s \to 0+),$$

with

$$f:(0, \infty) \to (0, \infty) \quad \text{and} \quad f'(s) \to 0 \quad (s \to 0+).$$

The right-hand side is an integral and its derivative is

$$t(s) = -\exp \int_s^\infty \frac{1 + f'(u)}{f(u)} du$$

which is in Γ^- by the criterion of Theorem 2.5.2 from [5]. The converse is a direct consequence of Lemma 2.5.1 from [5].

3. The complementary function q^*.

LEMMA 3.1. *If $q_1, q_2 \in D^*$ and*

(3.1) $\quad q_1(t) \stackrel{\bullet}{\sim} q_2(t)(t \to \infty)$

then

(3.2) $\quad q_1^*(s) \sim q_2^*(s)(s \to 0+).$

If $p_1, p_2 \in D_*$ and

(3.3) $\quad p_1(s) \sim p_2(s) \quad (s \to 0+)$

then

(3.4) $\quad p_{1*}(t) \stackrel{\bullet}{\sim} p_{2*}(t) \quad (t \to \infty).$

Proof. Note that the *-transform has the following properties:

(i) $(cq)^*(t) = cq^*(t/c)$,

(ii) if $\rho(t) = q(tc)$ then $\rho^*(t) = q^*(t/c)$,

(iii) if $q_1 \leqq q_2$ then $q_1^* \leqq q_2^*$,

(iv) if $q_1 = q_2$ on a neighbourhood of ∞ then $q_1^* = q_2^*$ on a right neighbourhood of zero.

In order to prove (3.2) remark that $cq_1(t) \geqq q_2(tc)$ for t sufficiently large implies

$$cq_1^*(s/c) = (cq_1)^*(s) \geqq q_2^*(s/c)$$

for s sufficiently small. The inequality

$$q_1^*(s) \leqq cq_2^*(s) \quad (s < s_o)$$

follows similarly. The proof of (3.4) is analogous.

THEOREM 3.1. *Suppose* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing, integrable on* $(0, 1)$, $s(\infty) = 0$ *and* $q \in D^*$. *Then*

$$q(t) \stackrel{\bullet}{\sim} \int_0^t s(u)du \quad (t \to \infty) \tag{3.5}$$

implies

$$q^*(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+) \tag{3.6}$$

where $s^{\leftarrow}$ *is an inverse function of* s.

THEOREM 3.2. *Suppose* $t:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing, integrable on* $(1, \infty)$ *and* $p \in D_*$. *Then*

$$p(s) \sim \int_s^\infty t(x)dx \quad (s \to 0+) \tag{3.7}$$

implies

$$p_*(x) \stackrel{\bullet}{\sim} \int_0^x t^{\leftarrow}(u)du \quad (x \to \infty). \tag{3.8}$$

These theorems follow immediately from Lemma 3.1.

COROLLARY. *If* $q(t)/t \in \Pi^-$ *and* $q \in D^*$ *then* $q^* \in \Gamma^-$.

This corollary follows from Theorem 3.1 by application of Lemma 2.3.

The combination of Theorems 3.1 and 3.2 yields a Tauberian result: If q is concave and non-decreasing, then (3.2) implies (3.1) (this follows since q_*^* is the concave non-decreasing hull of q and satisfies (3.1) by Theorem 3.2). Monotonicity of q is not a sufficient Tauberian condition.

Next we prove a Tauberian converse of Theorem 3.1 for the * transform.

THEOREM 3.3. *Suppose* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing, integrable on* $(0, 1)$, $s(\infty) = 0$, s *is asymptotically balanced*$^+$ *with auxiliary function* a *satisfying* index $a > -1$ *and* $q \in D^*$. *If* $q(t)/t$ *is non-increasing* (3.6) *implies* (3.5).

Proof. Application of Theorem 3.2 to the function q^* gives

$$(q^*)_*(t) \stackrel{\bullet}{\sim} \int_0^t s(u)du \quad (t \to \infty)$$

where $(q^*)_* \geqq q$ is the concave upper hull of q. Since s is asymptotically balanced$^+$ Lemmas 2.1 and 2.2 show that

$$\overline{\lim_{t\to\infty}} \frac{(q^*)_*(t(1+\epsilon)) - 2(q^*)_*(t) + (q^*)_*(t(1-\epsilon))}{ta(t)} < 0.$$

This means that for fixed $c > 1$ any interval $[t, ct]$ contains a point x such that $(q^*)_*(x) = q(x)$ provided t is sufficiently large (indeed otherwise the function $(q^*)_*$ is linear on this interval). Since $(q^*)_*$ is concave $q^*(t)/t$ is non-increasing. Hence

$$\frac{(q^*)_*(ct)}{ct} \leqq \frac{(q^*)_*(x)}{x} = \frac{q(x)}{x} \leqq \frac{q(t)}{t}$$

for t sufficiently large.

On the other hand we find

$$\frac{(q^*)_*(t)}{t} \geqq \frac{(q^*)_*(ct)}{ct} \geqq \frac{q(ct)}{ct}.$$

This proves $q \stackrel{\bullet}{\sim} (q^*)_*$ at infinity, hence (3.5).

Theorem 3.3 does not hold without the condition $q(t)/t$ monotone, as the following example shows.

Example. Define $q_0(t){:} = t/\log t$ for $t \geqq e$ and $: = 0$ for $0 < t < e$. Write

$$t_n{:} = \exp\{(n+1)/\log(n+1)\}, \quad n \geqq 1$$

and define

$$q(t){:} = \begin{cases} t_{n-1}/\log t_{n-1} & \text{for } t_{n-1} < t \leqq t_n,\ n = 3, 4, \dots \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $q(t)/t$ is not monotone. Since

$$\log t_n = (n+1)/\log(n+1) = n/\log n + \{1 + o(1)\}/\log n,$$

we get

$$q_0(et_n) = \frac{e \cdot \exp\left\{\dfrac{n}{\log n}\right\} \cdot \exp\left\{\dfrac{1+o(1)}{\log n}\right\}}{\dfrac{n}{\log n}\left\{1 + \dfrac{1+o(1)}{n/\log n}\right\}}$$

$$= \frac{e \cdot \exp(n/\log n)}{n/\log n} \cdot \left\{1 + \frac{1+o(1)}{\log n}\right\}\left\{1 - \frac{1+o(1)}{n/\log n}\right\}$$

$$> e \cdot q_0(t_{n-1}) = eq(t_n)$$

for sufficiently large n. Hence the relation

$$q_0(t) \stackrel{\bullet}{\sim} q(t) \quad (t \to \infty)$$

does not hold. Since $q_0(t)/t \in \Pi^-$, $q(t_n) = q_0(t_{n-1})$ and $t_n \to \infty$, $t_{n+1}/t_n \to 1$, it follows that

$$q(t)/t \in \Pi^-.$$

But, as we now prove,

$$q^*(s) \sim q_0^*(s) \quad (s \to 0+),$$

so that $q^* \in \Gamma^-$. Observe that

$$q_0^*(s) = q_0(t(s)) - st(s)$$

with

$$t(s) = (q_0')^{\leftarrow}(s) = \exp\{1/s - 1 + o(1)\} \quad (s \to 0+).$$

Further for any $s > 0$ there is $n(s) \in \mathbf{N}$ with

$$t_{n(s)} < t(s) \leqq t_{n(s)+1},$$

so that, since $q_0(t) - st$ is concave for fixed $s > 0$,

$$q^*(s) = \max\{q_0(t_{n(s)}) - s \cdot t_{n(s)}, q_0(t_{n(s)+1}) - st_{n(s)+1}\}.$$

Now $t_{n+1}/t_n \to 1$, so for $s \to 0+$

$$\tau_1(s): = t_{n(s)+1}/t(s) \to 1$$

and

$$\tau_2(s): = t_{n(s)}/t(s) \to 1.$$

Hence

$$\begin{aligned}\frac{q^*(s)}{q_0^*(s)} &= \max_{i=1,2} \frac{q_0(t(s)\tau_i(s)) - st(s)\tau_i(s)}{q_0(t(s)) - st(s)} \\ &= \max_{i=1,2} \frac{t(s)\tau_i(s)/\log\{t(s)\tau_i(s)\} - st(s)\tau_i(s)}{t(s)/\log t(s) - st(s)} \\ &= \max_{i=1,2} \tau_i(s)\frac{1/s - \log t(s) - \log \tau_i(s)}{1/s - \log t(s)} \\ &\times \frac{\log t(s)}{\log t(s) + \log \tau_i(s)}.\end{aligned}$$

Now $t(s) \to \infty$, $s^{-1} - \log t(s) \to 1 \quad (s \to 0+)$, hence indeed

$$q^*(s) \sim q_0^*(s) \quad (s \to 0+).$$

4. Laplace transforms and complementary functions. In this section we consider the transform

$$\tilde{q}(s) := \log s \int_0^\infty e^{q(t)-st}dt$$

with q positive and non-decreasing. We are going to prove an Abelian and a Tauberian result for this transform.

LEMMA 4.1.

(4.1) *If q is non-decreasing, then $q^* \leqq \tilde{q}$.*

Proof. See [**2**], Lemma 6.

LEMMA 4.2. *Suppose $s:\mathbf{R}^+ \to \mathbf{R}^+$ is non-increasing,*

$$\int_0^1 s(u)du < \infty \quad \textit{and} \quad \lim_{u\to\infty} s(u) = 0.$$

Define

$$q_o(t) := \int_0^t s(u)du \quad (t > 0).$$

If

(4.2) $s \in \text{As Bal}^+$ *with auxiliary function a satisfying* <u>index</u> $a > -1$

then

(4.3) $q_o^*(s) \sim \tilde{q}_o(s) \quad (s \to 0+)$.

Proof. Assume that (t, s) is on the graph of s, i.e., $s(t-) \leqq s \leqq s(t+)$. Then

$$q_o(t) - st = q_o^*(s).$$

The function

$$\Delta(u) := us - q_o(t + u) + q_o(t)$$

is convex and non-negative for $u \geqq -t$ since s is non-increasing. By (4.2) we have for some $\gamma > 0$ and all $t \geqq t_o$

$$\Delta'(t) = s(t) - s(2t) \geqq \gamma a(t).$$

Hence

$$\Delta(u + 2t) \geqq \Delta(t) + (u + t)\Delta'(t) \geqq \gamma u a(t)$$

for $u > 0$, $t \geqq t_o$. Using Lemma 4.1 we find

$$\begin{aligned} q_o^*(s) \leqq \tilde{q}_o(s) &= q_o(t) - st + \log s \int_{-t}^\infty e^{-\Delta(u)}du \\ &\leqq q_o^*(s) + \log\left\{s \int_{-t}^{2t} 1\, du + s \int_0^\infty e^{-\Delta(2t+u)}du\right\} \\ &\leqq q_o^*(s) + \log(3st + s/\{\gamma a(t)\}). \end{aligned}$$

We finish the proof by showing that

$$\log[3st + s/\{\gamma a(t)\}] = o(q_o^*(s)) \quad (t \to \infty).$$

Since $ta(t) \to \infty \quad (t \to \infty)$ it is sufficient to prove

$$\log ts(t) = o\left(\int_0^t s(u)du - ts(t)\right) \quad (t \to \infty).$$

With

$$b(t) := \int_0^t s(u)du - ts(t) \to \infty \quad (t \to \infty)$$

we have as before

$$ts(t) = -b(t) + t\int_t^\infty b(u)du/u^2,$$

hence

$$\log ts(t) = \log b(t) + \log \int_1^\infty \left(\frac{b(tu)}{b(t)} - 1\right)\frac{du}{u^2}.$$

Now on the one hand since $b \in BI \cap PI^+$ (Lemma 2.1 (ii))

$$\varliminf_{t\to\infty} \int_1^\infty \left(\frac{b(tu)}{b(t)} - 1\right)\frac{du}{u^2} > 0,$$

so the argument of the second logarithm is bounded away from zero $(t \to \infty)$ and on the other hand

$$\int_1^\infty \frac{b(tu)}{b(t)}\frac{du}{u^2} = t\left\{\int_t^\infty b(u)du/u^2\right\}/b(t),$$

a function with finite upper index so that

$$\log \int_1^\infty \frac{b(tu)}{b(t)}\frac{du}{u^2}$$

has non-positive upper index $(t \to \infty)$. Since $\underline{\text{index}}\ b(t) > 0$, the functions

$$\left\{\log \int_1^\infty \frac{b(tu)}{b(t)}\frac{du}{u^2}\right\}/b(t) \quad \text{and} \quad \frac{\log b(t)}{b(t)}$$

have negative upper index and hence converge to zero $(t \to \infty)$. This completes the proof.

In order to extend the result of Lemma 4.2 to functions q with $q(t) \stackrel{\bullet}{\sim} q_o(t)$ we first prove

LEMMA 4.3. *Under the assumptions of Lemma* 4.2 *and* $q_o(t) \to \infty$ $(t \to \infty)$ *we have as* $s \to 0+$ *for all* $c > 1$

(4.4) $\quad (q_o/c)^\sim(s) - \tilde{q}_o(cs)/c^2 \to \infty$

and

(4.5) $$(cq_o)^{\sim}(s) - c^2\tilde{q}_o(s/c) \to -\infty.$$

Here e.g.

$$(q_o/c)^{\sim}(s) := \log s \int_0^\infty \exp\{q_o(t)/c - ts\}dt.$$

Proof. Fix $c > 1$. From the Lemmas 4.1 and 4.2 it follows

$$(q_o/c)^{\sim}(s) \geqq (q_o/c)^*(s) = q_o^*(cs)/c \sim \tilde{q}_o(cs)/c \quad (s \to 0+).$$

This proves (4.4) since $\tilde{q}_o(s) \to \infty$ as $s \to 0+$. In order to prove the second relation note that

$$c\tilde{q}_o(s/c) \geqq cq_o^*(s/c) = (cq_o)^*(s) \sim (cq_o)^{\sim}(s) \quad (s \to 0+).$$

THEOREM 4.1. *Suppose* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing,* $s(\infty) = 0$ *and* q_o *is defined by*

$$q_o(t) = \int_0^t s(u)du \quad (t > 0).$$

If $q_o(\infty) = \infty$, $s \in$ As Bal$^+$ *with auxiliary function a satisfying* index $a > -1$ *and* $q:\mathbf{R}^+ \to \mathbf{R}^+$ *locally bounded, then*

(4.6) $$q(t) \overset{\bullet}{\sim} q_o(t) \quad (t \to \infty)$$

implies

(4.7) $$\tilde{q}(u) \sim \int_u^\infty s^{\leftarrow}(x)dx \quad (u \to 0+).$$

Proof. Fix $c > 1$ and define

$$q_1(t) = \min\{q_o(t)/c, q(t/c)\}.$$

Since $q \overset{\bullet}{\sim} q_o$ it follows that

$$q_1(t) = q_o(t)/c \quad \text{for } t > t_o.$$

Hence

(4.8) $$\begin{aligned}\exp((q_o/c)^{\sim}(s)) &- \exp(\tilde{q}_1(s)) \\ &= s\int_0^{t_o} \{\exp(q_o(t)/c) - \exp(q_1(t))\}e^{-st}dt \\ &= O(1) \quad (s \to 0+).\end{aligned}$$

Now (4.4) and (4.8) with $q_1(t) \leqq q(t/c)$, imply

$$\tilde{q}(cs) \geqq \tilde{q}_1(s) > \tilde{q}_o(cs)/c^2$$

for s sufficiently small. Similarly we find

$$c^2\tilde{q}_o(s/c) \geqq \tilde{q}(s/c)$$

by introducing the function

$$q_2(t) = \max(cq_o(t), q(tc)).$$

This proves

$$\tilde{q}(s) \sim \tilde{q}_0(s) \quad (s \to 0+),$$

hence by Lemma 4.2 and Theorem 3.1 we find (4.7).

In order to prove a Tauberian result concerning the transform $\tilde{q}$ we need the following lemma.

LEMMA 4.4. *Suppose* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing,* $s(\infty) = 0$ *and* q_o *is defined by*

$$q_o(t) = \int_0^t s(u)du \quad (t > 0).$$

If $s \in$ As Bal$^+$ *with auxiliary function a satisfying* index $a > -1$, *then for all* $0 < \alpha < 1$ *there exists* $c > 1$ *and* $x_o(c) = x_o$ *such that for all* (x, y) *with* $x > x_o$, $s(x+) \leqq y \leqq s(x-)$ *we have*

$$\log y \int_{I^c} \exp\{q_o(t) - yt\}dt \leqq \tilde{q}_o(y)/c \tag{4.9}$$

where $I = (x - \alpha x, x + \alpha x)$.

Proof. (as in [**2**]). Define

$$\Delta(u) = uy - q_o(x + u) + q_o(x).$$

Now by assumption there is a $\lambda > 0$ such that

$$\Delta'(u) \geqq s(x+) - s(x + u) \geqq 2\lambda a(x)$$

for $u \geqq \alpha x/2$, $x \geqq x_0$. Hence

$$\Delta(\alpha x) = \Delta(\alpha x/2) + \int_{\alpha x/2}^{\alpha x} \Delta'(u)du \geqq \lambda\alpha x a(x).$$

Consequently, since $\Delta(0) = 0$ and $\Delta(u)$ is convex

$$\begin{aligned} &y \int_{x(1+\alpha)}^{\infty} \exp(q_o(t) - yt)dt \\ &= [\exp(q_o(x) - xy)] \cdot y \int_{\alpha x}^{\infty} e^{-\Delta(u)}du \\ &\leqq e^{q_o(x) - xy} y e^{-c_o\alpha x a(x)} \int_0^{\infty} e^{-\Delta(u)}du \\ &= e^{-c_o\alpha x a(x)} y \int_x^{\infty} e^{q_o(t) - yt}dt. \end{aligned} \tag{4.10}$$

Similarly $\Delta(-\alpha x) \geqq c_1\alpha x a(x)$ and

$$y \int_0^{x(1-\alpha)} e^{q_o(t) - yt}dt \leqq e^{-c_1\alpha x a(x)} y \int_0^x e^{q_o(t) - yt}dt. \tag{4.11}$$

By Lemmas 4.2 and 2.1 we have

$$\tilde{q}_o(y) \sim q_o^*(y) = \int_y^\infty s^{\leftarrow}(u)du = -\int_0^x u ds(u) \tag{4.12}$$
$$= x[-s(x) + \frac{1}{x}\int_0^x s(u)du] \asymp xa(x).$$

Combination of (4.10), (4.11) and (4.12) gives (4.9).

THEOREM 4.2. *Suppose* $s:\mathbf{R}^+ \to \mathbf{R}^+$ *is non-increasing, integrable on* (0, 1) *and* $s \in$ as bal$^+$ *with auxiliary function a satisfying* <u>index</u> $a > -1$. *If q is non-decreasing,* $q(t)/t$ *non-increasing then*

$$\tilde{q}(u) \sim \int_u^\infty s^{\leftarrow}(y)dy \quad (u \to 0+) \tag{4.13}$$

implies

$$q(t) \overset{\bullet}{\sim} \int_0^t s(u)du \quad (t \to \infty). \tag{4.14}$$

Proof. Set $h(t) = (\tilde{q})_*(t)$. Then by Theorem 3.2 we have

$$h(t) \overset{\bullet}{\sim} \int_0^t s(u)du \quad (t \to \infty)$$

and by Lemmas 2.1 and 2.2 this implies $h(t)/t$ is asymptotically balanced$^+$ with auxiliary function a and (2.5) holds for h. Since $h = (\tilde{q})_*$ is concave we may write

$$h(t) = \int_0^t s_1(u)du$$

(note that s_1 is non-increasing). The function s_1 then is asymptotically balanced$^+$ by Lemma 2.1. Since $h(t)/t$ is non-increasing we have for $c > 1$ and t sufficiently large (use Lemma 4.1)

$$\frac{h(t/c)}{t/c} \geqq \frac{h(t)}{t} \geqq \frac{(q^*)_*(t)}{t} \geqq \frac{q(t)}{t}.$$

We claim $h(t) \overset{\bullet}{\sim} q(t)$. If not, there exists a sequence $t'_n \to \infty$, a constant $c > 1$ such that

$$\frac{q(t'_n)}{t'_n} < \frac{h(t'_n c)}{t'_n c}.$$

Now $q(t)/t$ and $h(t)/t$ are non-increasing, so for $t'_n < t < t'_n \sqrt{c}$ we have

$$q(t)/t \leqq q(t'_n)/t'_n < h(t'_n c)/t'_n c \leqq h(t\sqrt{c})/t\sqrt{c}.$$

Hence there is a sequence $t_n \to \infty$ and a constant $\alpha \in (0, 1)$ such that

$$\frac{q(t)}{t} < \frac{h(tc)}{tc} \quad \text{on} \quad I_n = (t_n - \alpha t_n, t_n + \alpha t_n).$$

Define $s_n = s_1(t_n)$ and apply Lemma 4.4 to the function h. For n sufficiently large

$$\tilde{q}(s_n) \leqq \log\left\{ s_n \int_{I_n} \exp(h(tc)/c - ts_n)dt \right.$$

$$\left. + s_n \int_{I_n^c} \exp(h(t) - s_n t)dt \right\}$$

$$\leqq \log\{ (\exp(h/c)^{\sim}(s_n/c) + \exp(\tilde{h}(s_n)/c) \}.$$

Since

$$(h/c)^{\sim}(s_n/c) \sim (h/c)^*(s_n/c) = h^*(s_n)/c \sim \tilde{h}(s_n)/c \quad (n \to \infty)$$

by Lemma 4.2 we find

$$\tilde{q}(s_n) \leqq h^{\sim}(s_n)/\sqrt{c} \quad \text{for } n \geqq n_o.$$

Since

$$h(t) \stackrel{\bullet}{\sim} \int_0^t s(u)du$$

implies

$$\tilde{h}(u) \sim h^*(u) \sim \int_u^\infty \overleftarrow{s}(x)dx \sim \tilde{q}(u)(u \to 0+)$$

by Lemma 4.2 and Theorem 3.1 we have a contradiction.

An interesting special case of our theorems is obtained by replacing the As bal$^+$ condition by the condition that $s \in BI \cap PI^+$.

We show by an example that Theorem 4.2 does not hold without the condition $q(t)/t$ monotone.

Example. Take q and q_0 as in the example after Theorem 3.3. We shall show that

$$\tilde{q}(s) \sim q^*(s) \quad (s \to 0+).$$

First $q(t) \leqq q_0(t) \quad (t > 0)$ implies

$$\tilde{q}(s) \leqq \tilde{q}_0(s) \quad (s > 0).$$

On the other hand by Lemma 4.1

$$\tilde{q}(s) \geqq q^*(s) \sim q_0^*(s) \sim \tilde{q}_0(s) \quad (s \to 0+).$$

The statement follows.

Remarks. 1) It can be verified that the condition $q(t)/t$ nonincreasing can be weakened to: for all $c_o > 1$ there exists $t_o(c_o)$ such that for $t \geqq t_o$ and $c \geqq c_o$ $q(ct) \leqq cq(t)$.

2) Theorems 1.3 and 1.4 can be easily deduced from Lemma 2.3 and the Abelian and Tauberian theorems above.

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