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NOTES ON WEAKLY-SEMISIMPLE RINGS

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Responding to a question on right weakly semisimple rings due to Jain, Lopez-Permouth and Singh, we report the existence of a non-right-Noetherian ring R for which every uniform cyclic right R-module is weakly-injective and every uniform finitely generated right R-module is compressible. We show that a ring R is a right Noetherian ring for which every cyclic right R-module is weakly R-injective if and only if R is a right Noetherian ring for which every uniform cyclic right R-module is compressible if and only if every cyclic right R-module is compressible. Finally, we characterise those modules M for which every finitely generated (respectively, cyclic) module in $\sigma[M]$ is compressible.

0. INTRODUCTION AND NOTATION

All rings R are associative with identity and all modules are unitary right R-modules. For a module M and a submodule N, $N \leqslant_e M$ denotes that N is essential in M. As usual, $E(M)$ and $Z(M)$ indicate the injective hull and the singular submodule of M respectively. The module M is said to be tight (respectively, R-tight) if every finitely generated (respectively, cyclic) submodule of $E(M)$ is embeddable in M, while M is defined to be weakly-injective (respectively, weakly R-injective) if for any finitely generated (respectively, cyclic) submodule Y of $E(M)$ there exists a submodule X of $E(M)$ such that $Y \subseteq X \cong M$ (see [**3**] and [**5**]). A module is called a compressible module if it is embeddable in each of its essential submodules [**9**]. Rings for which all modules are weakly-injective, called right weakly semisimple rings, were introduced by Jain, Lopez-Permouth and Singh [**7**] and studied in [**1, 6, 7, 11, 12**].

The following characterisations of right weakly-semisimple rings were obtained in [**7**].

THEOREM 0. *The following are equivalent for a ring R:*

(i) *R is a right weakly-semisimple ring;*
(ii) *Every finitely generated R-module is weakly-injective and R is right Noetherian;*
(iii) *Every cyclic R-module is weakly-injective and R is right Noetherian;*

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(iv) *Every uniform cyclic R-module is weakly-injective and R is right Noetherian;*

(v) *Every uniform finitely generated R-module is compressible and R is right Noetherian.*

It is open whether or not one may remove the Noetherian condition from any of the equivalent statements (ii) through (v) in the above theorem (see [**6**] or [**7**]). Here we shall report an example of a ring R which is not a right Noetherian ring but for which every uniform cyclic module is weakly-injective and every uniform finitely generated module is compressible. Motivated by Theorem 0, we characterise the right Noetherian rings R for which every cyclic R-module is weakly R-injective and the right Noetherian rings R for which every uniform cyclic R-module is compressible. These rings are also mentioned as worthy of study in [**7**]. We shall show that the two classes of rings coincide and each can be characterised as being those rings for which every cyclic module is compressible. We shall call a ring R a right CC-ring if R satisfies one of these equivalent conditions. [This terminology has been used differently in [**8**].] In Section 3, we consider analogues of right weakly semisimple rings and right CC-rings to modules. For any module M, we denote by $\sigma[M]$ the full subcategory of Mod$-R$, whose objects are the submodules of M-generated modules, and by $E_M(N)$ the M-injective hull of a module N which is the trace of M in $E(M)$, that is, $E_M(N) = \sum\{f(M) : f \in \mathrm{Hom}(M, E(N))\}$ (see [**14**]). We present characterisations for those modules M for which every finitely generated (respectively, cyclic) module in $\sigma[M]$ is compressible. These characterisations extend naturally the corresponding results of right weakly semisimple rings and right CC-rings.

1. The example

Example 1. Let $Q = \prod_i^{\infty} F_i$, where each $F_i = \mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$, be the full product of rings $\mathbf{Z}_2$, R the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q, and I an arbitrary right ideal of R. Then R is not a right Noetherian ring and

(a) $Soc(R_R) = \bigoplus_{i=1}^{\infty} F_i$ is the only proper essential right ideal of R and $R/Soc(R_R)$ is a two-element field;

(b) R/I is R-injective if $I \subseteq Soc(R_R)$ and $Soc(R)/I$ is finitely generated;

(c) R/I is R-injective if $I \not\subseteq Soc(R)$;

(d) R/I is R-injective and simple if R/I is uniform. Therefore every uniform R-module is simple and compressible.

Proof: It is easy to check (a). We let $\pi_i(\alpha)$ denote the ith component of the element α in Q. Note that, for $\alpha \in Q$, $\alpha \in R$ if and only if $\pi_i(\alpha) = a$ for all but finitely many i's with $a = \bar{0}$ or $\bar{1}$.

(b) Suppose that $I \subseteq Soc(R_R)$ and $Soc(R)/I$ is finitely generated. We can write $I = \bigoplus_{i \in V} F_i$, where V is a subset of $\mathbf{N}$. Let $f : Soc(R_R) \longrightarrow R/I$ be an R-homomorphism. Clearly $I \subseteq \operatorname{Ker}(f)$, and we may write $\operatorname{Ker}(f) = \bigoplus_{i \in W} F_i$, where $V \subseteq W \subseteq \mathbf{N}$. It can be easily checked that the restriction of f on $\bigoplus_{i \in \mathbf{N} \setminus W} F_i$ coincides with $\pi \circ \varepsilon$, where ε is the identity map on $\bigoplus_{i \in \mathbf{N} \setminus W} F_i$ and $\pi : R \longrightarrow R/I$ is the canonical homomorphism. Let β be in Q satisfying $\pi_i(\beta) = \overline{0}$ for precisely $i \in W$, and $= \overline{1}$ if $i \in \mathbf{N} \setminus W$. Since $Soc(R)/I$ is finitely generated, the set $\mathbf{N} \setminus V$ and hence the set $\mathbf{N} \setminus W$ is finite. Therefore, β is in R. Define $g : R \longrightarrow R/I$ by $g(1) = \beta + I$. Then g extends f.

(c) Suppose that $I \not\subseteq Soc(R_R)$. Then there exists a finite subset U of $\mathbf{N}$ such that $\alpha \in I$ if and only if $\pi_i(\alpha) = 0$ for every $i \in U$. So we have $R = I \oplus \left(\bigoplus_{i \in U} F_i\right)$ and $R/I \cong \bigoplus_{i \in U} F_i$. Therefore, to show R/I is R-injective, it suffices to show that each F_i is R-injective. Let $f : Soc(R_R) \longrightarrow F_i$ be a nonzero R-homomorphism. Then $\operatorname{Ker}(f) = \bigoplus_{j \neq i} F_j$ and $f|_{F_i} = 1$. Define $g : R \longrightarrow F_i$ by $\pi_j(g(1)) = \overline{0}$ if $j \neq i$ and $\pi_i(g(1)) = \overline{1}$. Then g extends f.

(d) Suppose that R/I is uniform. Case 1: $I \not\subseteq Soc(R_R)$. As above, we have $R = I \oplus \left(\bigoplus_{i \in U} F_i\right)$, where U is a finite subset of $\mathbf{N}$. Then $R/I \cong \bigoplus_{i \in U} F_i$. Since R/I is uniform, $|U| = 1$ and hence R/I is simple. It follows from (c) that R/I is injective. Case 2: $I \subseteq Soc(R_R)$. Since R/I is uniform, $Soc(R_R)/I$ must be zero or simple, and hence finitely generated. By (b), R/I is injective. If $I = Soc(R)$, then $R/I = R/Soc(R)$ is simple. If $I \subset Soc(R)$, then $Soc(R)/I$ is uniform. Thus $I = \bigoplus_{i \neq j} F_i$ for some j. Then $R/I = [Soc(R)/I] \oplus (J/I)$, where $J = \{\alpha \in R : \pi_j(\alpha) = \overline{0}\}$. This contradicts the uniformness of R/I. ∎

2. Rings whose cyclics are compressible

A module M is called a V-module if every simple R-module is M-injective. The following lemma is an easy corollary of Shock [**13**, Thorem 3.8].

Lemma 2. *For a V-module M, M is Noetherian if and only if every factor module of M has finitely generated socle.*

Theorem 3. *The following are equivalent for a ring R:*

(a) *Every R-module is weakly R-injective (or R-tight);*

(b) *R is right Noetherian and every finitely generated R-module is weakly R-injective (or R-tight);*

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(c) *R is right Noetherian and every cyclic R-module is weakly R-injective (or R-tight);*
(d) *R is right Noetherian and every uniform cyclic R-module is weakly R-injective (or R-tight);*
(e) *R is right Noetherian and every uniform cyclic R-module is compressible;*
(f) *Every cyclic R-module is compressible.*

PROOF: $(f) \Rightarrow (e)$. For any simple R-module M, each $xR \subseteq E(M)$ is embeddable in M since $M \cap xR \leqslant_e xR$. This implies that $E(M)$ is simple. Thus $M = E(M)$ is injective. Therefore, R_R is a V-module. Next we show that every cyclic R-module has finitely generated socle. For a cyclic R-module N, we have a submodule X of N maximal with respect to $Soc(N) \cap X = 0$ and hence $Soc(N)$ is essentially embeddable in N/X. Since N/X is compressible, we have an embedding $N/X \hookrightarrow Soc(N)$, implying that N/X is a semisimple module. This shows that $Soc(N) \cong N/X$ is finitely generated. Therefore, by Lemma 2, R is a right Noetherian ring.

$(e) \Rightarrow (f)$. Let $M = xR$ be a cyclic module and $N \leqslant_e M$. Since R is right Noetherian, M has finite Goldie dimension. Then there exist cyclic uniform submodules x_iR $(i = 1, \cdots, n)$ of M such that $x_1R + \cdots + x_nR = x_1R \oplus \cdots \oplus x_nR \leqslant_e N \subseteq M$. Therefore, $E(M) \overset{f}{\cong} E(x_1R) \oplus \cdots \oplus E(x_nR)$. Write $f(x) = y_1 + \cdots + y_n$ with $y_i \in E(x_iR)$. Note that each $y_i \neq 0$. Then $M \cong f(M) = (y_1 + \cdots + y_n)R \subseteq y_1R \oplus \cdots \oplus y_nR$. Since $x_iR \cap y_iR \leqslant_e y_iR$ and y_iR is uniform, we have $y_iR \overset{g_i}{\hookrightarrow} x_iR$. Define $g : y_1R \oplus \cdots \oplus y_nR \longrightarrow x_1R \oplus \cdots \oplus x_nR$ by $g(y_1r_1 + \cdots + y_nr_n) = g_1(y_1)r_1 + \cdots + g_n(y_n)r_n$ (all $r_i \in R$). Then g is one-to-one and we have $M \overset{g \circ f}{\hookrightarrow} N$, showing that M is compressible.

To complete the proof, we note two facts. First, it is straightforward to verify that every cyclic (respectively, uniform cyclic) R-module is compressible if and only if every R-module (respectively, uniform R-module) is R-tight. Next, from [**6**, 2.8], we see that for a ring R for which every cyclic R-module has finitely generated socle, an R-module M is tight (respectively, R-tight) if and only if M is weakly-injective (respectively, weakly R-injective). Therefore, the remaining equivalences follow from these facts and the equivalence $(e) \Leftrightarrow (f)$. ∎

From now on, we call a ring R a right CC-ring if R satisfies any one of the equivalent conditions in Theorem 3.

It is known that, for a semiprime right Goldie ring R, R is left Goldie if and only if every finitely generated non-singular R-module is embeddable in a free module [**10**] if and only if every finitely generated non-singular R-module is compressible [**9**, 2.2.15]. We need the following proposition for the next characterisation of right CC-rings.

PROPOSITION 4. *The following are equivalent for a semiprime right Goldie ring*

R:

(a) *Every cyclic non-singular R-module is embeddable in a free module;*
(b) *Every cyclic non-singular R-module is compressible;*
(c) *R_R is R-tight.*

PROOF: The implication $(a) \Rightarrow (b)$ follows from [**15**, Theorem 5], while $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$. Let $M = xR$ be non-singular. We let Q be the right classical quotient ring of R. Then Q is a semisimple ring. Consider the right Q-module $M \bigotimes_R Q$. Clearly $M \bigotimes_R Q = (x \otimes 1)Q$ is cyclic. Since Q is a semisimple ring, $M \bigotimes_R Q$ is a semisimple Q-module, implying that $M \bigotimes_R Q$ is of finite length as a Q-module. By [**4**, Exercise 6E, p.104], M has finite Goldie dimension and so there exist uniform cyclic submodules x_iR $(i = 1, \cdots, n)$ of M such that $x_1R + \cdots + x_nR = x_1R \oplus \cdots \oplus x_nR \leqslant_e M$. Then $E(M) \overset{f}{\cong} E(x_1R) \oplus \cdots \oplus E(x_nR)$. Write $f(x) = y_1 + \cdots + y_n$ with $y_i \in E(x_iR)$. We have $M \cong f(x)R = (y_1 + \cdots + y_n)R \subseteq y_1R \oplus \cdots \oplus y_nR$. Therefore, to show (a) it suffices to show that every uniform cyclic non-singular right R-module is embeddable in a free module. So we may assume that $M = xR$ is uniform. Since xR is non-singular, $x^\perp$ is not essential in R. Thus, $x^\perp \cap I = 0$ for some $0 \neq I \subseteq R_R$, implying $I \hookrightarrow xR$. Then $E(I) \cong E(xR)$ since xR is uniform. Now it follows from (c) that xR is embeddable in R. ∎

Let R be a right Ore-domain but not a left Ore-domain and $T = M_2(R)$ be the 2×2 matrix ring over R. Then T is a semiprime right Goldie ring, T_T is not T-tight [**6**, Remark 3.7], and R_R is R-tight.

THEOREM 5. *The following are equivalent for a ring R:*

(a) *R is a right CC-ring;*
(b) *R is a right QI-ring, R_R is R-tight, and every singular cyclic R-module is compressible;*
(c) *R is semiprime right Goldie, R_R is R-tight, and every singular cyclic R-module is compressible.*

PROOF: $(a) \Rightarrow (b)$. By Boyle [**2**], right QI-rings can be characterised as being those right Noetherian rings for which every uniform cyclic module is strongly-prime, where a module M is called strongly-prime if M is contained in every nonzero quasi-injective submodule of $E(M)$. Note that every uniform cyclic module being compressible implies every uniform cyclic module being strongly-prime. Therefore, the implication follows from Theorem 3.

$(b) \Rightarrow (c)$. Obvious.

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$(c) \Rightarrow (a)$. Let $M = xR$ be a cyclic module and $N \leqslant_e M$. There exists a non-singular submodule K of N such that $Z(N) \oplus K \leqslant_e N \subseteq M$. Then $E(M) = E(Z(N)) \oplus E(K)$. Write $x = a+b$, where $a \in E(Z(N))$ and $b \in E(K)$. Since R is right non-singular, aR is singular. Then, by (c), we have $aR \hookrightarrow Z(N)$ since $Z(N) \cap aR \leqslant_e aR$. Note that bR is non-singular and hence is compressible by Proposition 4. It follows that $bR \hookrightarrow K$ since $K \cap bR \leqslant_e bR$. Therefore, we have $aR \oplus bR \hookrightarrow Z(N) \oplus K \subseteq N$, implying $xR \hookrightarrow N$. Therefore, M is compressible. ∎

3. Module analogues of right weakly semisimple rings and right CC-rings

DEFINITION 6: Let M and N be R-modules. N is said to be tight (respectively, R-tight) with respect to M, if every finitely generated (respectively, cyclic) submodule of $E_M(N)$ is embeddable in N. N is said to be weakly-injective (respectively, weakly R-injective) with respect to M, if for every finitely generated (respectively, cyclic) submodule Y of $E_M(N)$ there exists $X \subseteq E_M(N)$ such that $Y \subseteq X \cong N$.

REMARKS 7. (1) For any generator M in Mod$-R$, a module N is tight (or R-tight, or weakly-injective, or weakly R-injective, respectively) with respect to M if and only if N is tight (or R-tight, or weakly-injective, or weakly R-injective, respectively).

(2) A tight (respectively, R-tight) module N is tight (respectively, R-tight) with respect to M for any $M \in$ Mod-R. But if $R = \mathbf{Z}$ and $M = \mathbf{Z}_2$ then $E(M) = \mathbf{Z}_2(\infty)$ and $E_M(M) = \mathbf{Z}_2$. Therefore, M is tight with respect to M but not R-tight.

(3) A weakly-injective R-module may not be weakly R-injective with respect to some module M. For instance, consider $R = \mathbf{Z}_4$ and $M = 2R$. Then $E(R_R) = R$ and $E_M(R) = M$. Therefore R_R is weakly-injective. Obviously there does not exist $X \subseteq E_M(R)$ such that $M \subseteq X \cong R$. Thus, R_R is not weakly R-injective with respect to M.

THEOREM 8. *The following are equivalent for a module M:*

(a) *Every module in $\sigma[M]$ is weakly R-injective (or R-tight) with respect to M;*

(b) *M is locally Noetherian and every finitely generated module in $\sigma[M]$ is weakly R-injective (or R-tight) with respect to M;*

(c) *M is locally Noetherian and every cyclic module in $\sigma[M]$ is weakly R-injective (or R-tight) with respect to M;*

(d) *M is locally Noetherian and every uniform cyclic module in $\sigma[M]$ is weakly R-injective (or R-tight) with respect to M;*

(e) *M is locally Noetherian and every uniform cyclic module in $\sigma[M]$ is compressible;*

(f) *Every cyclic module in $\sigma[M]$ is compressible.*

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PROOF: $(a) \Rightarrow (b)$. Let N be any cyclic submodule of M. For any simple module $X \in \sigma[M]$, we have $X \leqslant_e E_M(X)$. By (a), every cyclic submodule of $E_M(X)$ is embeddable in X, and so $X = E_M(X)$ is M-injective. Therefore, X is N-injective. Note that any simple module which is not in $\sigma[M]$ is trivially N-injective. This shows that N is a V-module. By using the same argument in the proof that $(f) \Rightarrow (e)$ of Theorem 3, we can show that every factor module of N has finitely generated socle. By Lemma 2, N is Noetherian. Therefore, M is locally Noetherian.

$(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$. Clear.

$(e) \Rightarrow (f)$. Let N be a cyclic module in $\sigma[M]$. Note that N is embeddable in a factor module of a finite direct sum of cyclic submodules of M. Since M is locally Noetherian, N is Noetherian and hence has finite Goldie dimension. Now as in the proof that $(e) \Rightarrow (f)$ of Theorem 3, N is compressible.

$(f) \Rightarrow (a)$. First we note that every cyclic in $\sigma[M]$ is compressible if and only if every module in $\sigma[M]$ is R-tight with respect to M. In particular, by the implication $(a) \Rightarrow (b)$, M is locally Noetherian. Let N be a module in $\sigma[M]$ and Y a cyclic submodule of $E_M(N)$. Since $N \cap Y \leqslant_e Y$, we have an embedding $Y \overset{f}{\hookrightarrow} N \cap Y$ since Y is compressible. Note that $E_M(Y)$ is quasi-injective. There exists a homomorphism $g : E_M(Y) \longrightarrow E_M(Y)$ which extends f. Since M is locally Noetherian, Y is Noetherian and hence has finite Goldie dimension. It follows that g is an isomorphism. Note that $E_M(N) = E_M(Y) \oplus Z$ for some $Z \subseteq E_M(N)$. If we define $h : E_M(N) \longrightarrow E_M(N)$ by $h(a+b) = g(a) + b$ for all $a \in E_M(Y)$ and $b \in Z$, then h is an isomorphism which extends g. Let $X = h^{-1}(N)$. Then $Y \subseteq X \cong N$. Therefore, N is weakly R-injective. ∎

THEOREM 9. *The following are equivalent for a module M:*

(a) *Every module in $\sigma[M]$ is weakly-injective (or tight) with respect to M;*
(b) *M is locally Noetherian and every finitely generated module in $\sigma[M]$ is weakly-injective (or tight) with respect to M;*
(c) *M is locally Noetherian and every cyclic module in $\sigma[M]$ is weakly injective (or tight) with respect to M;*
(d) *M is locally Noetherian and every uniform cyclic module in $\sigma[M]$ is weakly injective (or tight) with respect to M;*
(e) *M is locally Noetherian and every uniform finitely generated module in $\sigma[M]$ is compressible;*
(f) *Every finitely generated module in $\sigma[M]$ is compressible.*

PROOF: $(a) \Rightarrow (b)$. By Theorem 8.

$(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)$. Clear.

$(e) \Rightarrow (f)$. Let N be a finitely generated module in $\sigma[M]$ and $P \leqslant_e N$. Since M is

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locally Noetherian, N is Noetherian and hence has finite Goldie dimension. Then there exist cyclic uniform submodules x_iR of N $(i = 1, \cdots, n)$ such that $x_1R + \cdots + x_nR = x_1R \oplus \cdots \oplus x_nR \leqslant_e P$. Then $E_M(N) \overset{f}{\cong} E_M(x_1R) \oplus \cdots \oplus E_M(x_nR)$. Since N is finitely generated, there exist finitely generated submodules Y_i of $E_M(x_iR)$ such that $f(N) \subseteq Y_1 \oplus \cdots \oplus Y_n$. Each Y_i is a finitely generated uniform module and hence compressible by (e). Therefore, we have an embedding $Y_i \overset{g_i}{\hookrightarrow} Y_i \cap x_iR$. Define $g : Y_1 \oplus \cdots \oplus Y_n \longrightarrow x_1R \oplus \cdots \oplus x_nR$ by $g(y_1 + \cdots + y_n) = g_1(y_1) + \cdots + g_n(y_n)$ for all $y_i \in Y_i$. Then g is one-to-one and we have $N \overset{g \circ f}{\hookrightarrow} P$, showing that N is compressible.

$(f) \Rightarrow (a)$. Similar to the proof that $(f) \Rightarrow (a)$ in Theorem 8. □

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