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FACTORIZATION OF OPERATORS THROUGH SUBSPACES OF L^1-SPACES

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Abstract

We analyze domination properties and factorization of operators in Banach spaces through subspaces of L^1-spaces. Using vector measure integration and extending classical arguments based on scalar integral bounds, we provide characterizations of operators factoring through subspaces of L^1-spaces of finite measures. Some special cases involving positivity and compactness of the operators are considered.

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1. Introduction

Domination by scalar-valued integrals is the main tool for factoring operators through (subspaces of) L^p-spaces, providing fundamental results as Pietsch's theorem and the Maurey–Rosenthal factorization theorem. This kind of factorization is at the core of modern functional analysis. It is connected with operator ideal theory as there are several operator ideals that are characterized by factorizations through subspaces of L^p-spaces: summing, integral, nuclear, factorable, etc. (see, for example, [5, 8]). Another important source of factorization arguments comes from the works of Krivine, Kwapień, Maurey, Pisier and Rosenthal in the 1970s regarding geometric properties of operators acting in Banach lattices (convexity and concavity). This theory (nowadays called Maurey–Rosenthal factorization) is well developed and has many applications in other areas like harmonic analysis. For detailed information on the Maurey–Rosenthal factorization theory we refer to [12, 15, 18]; some recent contributions can be found in [17] and [4, 6, 7, 11, 13, 16]. A sample result follows (see, for example, [4, Section 4.3]).

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Theorem 1.1 (Maurey–Rosenthal). *Let X be an order continuous Banach function space over a finite measure μ, let Y be a Banach space and $T : X \to Y$ a 1-concave operator, that is, there is a constant $C > 0$ such that*

$$\sum_{j=1}^{n} \|T(x_j)\| \le C \Big\| \sum_{j=1}^{n} |x_j| \Big\| \tag{1.1}$$

for all $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$. Then T factors as

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ L^1(\mu) \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a positive (multiplication) operator and R is an operator.

In the usual proof of Theorem 1.1, the geometric inequality (1.1) gives rise to a domination inequality by means of a *scalar-valued* integral, which in turn yields the desired factorization. In this paper we are interested in showing how the natural extension of this type of domination inequality to the case of *vector-valued* integrals can provide a general understanding of the factorization problem, as well as new results and applications. The setting for such extension is provided by the spaces of scalar functions which are integrable with respect to vector measures. This class of Banach function spaces is a powerful tool for the analysis of operators on function spaces (see [17, Chs 4 and 6] and the references therein).

We shall focus on factorizations through *subspaces of L^1-spaces of finite measures*, that is, subspaces of $L^1(\mu)$ where μ is a finite measure. Typical examples of Banach spaces which are (isomorphic to) subspaces of L^1-spaces of a finite measure are ℓ^p, all L^p-spaces of a finite measure, and H^p for $1 \le p \le 2$. Subspaces of L^1-spaces have been thoroughly studied in the literature. Classical results of Rosenthal [19, 20] ensure that a subspace of $L^1(\mu)$ (μ a finite measure) not containing ℓ^1 is reflexive and isomorphic to a subspace of an L^p-space of a finite measure for some $1 < p \le 2$. We stress that factorizations through L^1-spaces of arbitrary nonnegative measures are related to the ideal of 1-factorable operators. In this direction, a result of Kwapień (see, for example, [8, Theorem 9.13]) states that an operator between Banach spaces $T : X \to Y$ factors through a subspace of an L^1-space of a nonnegative measure if and only if there is a constant $C > 0$ such that, whenever the finite sets $U, V \subseteq X$ satisfy

$$\sum_{x \in U} |\langle x^*, x\rangle| \le \sum_{x \in V} |\langle x^*, x\rangle| \quad \text{for all } x^* \in X^*,$$

we have

$$\sum_{x \in U} \|T(x)\| \le C \sum_{x \in V} \|x\|.$$

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We next summarize the content of this paper. In Section 2 we provide a characterization of operators factoring through subspaces of L^1-spaces of finite measures. Our general result (Theorem 2.1) involves some domination inequalities by means of integrals with respect to vector measures. In Section 3 we study such factorizations when the first factor is a positive and/or compact operator. In the positive case, the factorization through a subspace of an L^1-space is sometimes equivalent to the factorization through an L^1-space (Proposition 3.1). On the other hand, the compact case is related to a certain summability property of the operator (Theorem 3.5). From the technical point of view, our proof of Theorem 3.5 uses some recent results from [3] on compactness in L^1-spaces of vector measures. Finally, in Section 4 we deal with the particular case of operators acting in $C(K)$ spaces.

Terminology. All our linear spaces are real. If X is a Banach space, we will write B_X for its closed unit ball and X^* for its dual space. The evaluation of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$. The norm of X is denoted by $\|\cdot\|_X$ or simply $\|\cdot\|$. When X is a Banach lattice, the symbol X^+ stands for its positive cone, that is, the set of all nonnegative elements of X. An *operator* is a linear continuous map between Banach spaces. A *subspace* of a Banach space is a closed linear subspace. By a *nonnegative measure* we mean a $[0, \infty]$-valued countably additive measure defined on a measurable space. Nonnegative finite measures are simply called *finite measures*. By a *Banach function space* over a finite measure μ we mean an order ideal of $L^1(\mu)$ containing all simple functions which is equipped with a complete lattice norm.

A *vector measure* is a countably additive measure m defined on a measurable space (Ω, Σ) and taking values in a Banach space X. We will say that m is *positive* if X is a Banach lattice and $m(\Sigma) \subseteq X^+$. A *Rybakov control measure* of m is a finite measure of the form $\mu = |\langle m, x_0^*\rangle|$ for some $x_0^* \in B_{X^*}$ such that $m(A) = 0$ whenever $\mu(A) = 0$. Here $\langle m, x^*\rangle$ denotes the real-valued measure obtained by composing m with any $x^* \in X^*$. We refer the reader to [17] for the basic properties of the Banach space $L^1(m)$ of (equivalence classes of) real-valued functions on Ω which are integrable with respect to m. The space $L^1(m)$ is a Banach function space over any Rybakov control measure of m when equipped with the norm

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \quad f \in L^1(m).$$

Any order continuous Banach lattice with weak unit is order isometric to $L^1(m)$ for some vector measure m. We write $I_m : L^1(m) \to X$ for the *integration operator* given by

$$I_m(f) := \int_\Omega f\, dm, \quad f \in L^1(m).$$

The symbol 'Ω' will be omitted in formulas involving integrals when no confusion arises, so we write expressions like $\int f\, dm$ to denote the integral over the total set on which the measure is defined. We will also deal with the Banach space $L^\infty(m)$, which is defined as $L^\infty(\mu)$ for any Rybakov control measure μ of m.

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2. General factorization through a subspace of an L^1-space

Let X and Y be Banach spaces and let $T : X \to Y$ be an operator factoring through a subspace $E \subseteq L^1(\nu)$ for some finite measure ν, that is, there exist operators $S : X \to E$ and $R : E \to Y$ such that $T = R \circ S$. Then

$$\begin{aligned}\sum_{j=1}^{n} \|T(x_j)\| &\le \|R\| \sum_{j=1}^{n} \|S(x_j)\|_{L^1(\nu)} \\ &= \|R\| \sum_{j=1}^{n} \int |S(x_j)|\,d\nu = \|R\| \Big\| \sum_{j=1}^{n} |S(x_j)| \Big\|_{L^1(\nu)}\end{aligned}$$

for every $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$. This '1-concavity type' inequality is the starting point of the following characterization that relates domination by integrals with respect to vector measures and factorization through subspaces of L^1-spaces.

Theorem 2.1. *Let X and Y be Banach spaces. The following assertions are equivalent for an operator $T : X \to Y$.*

(i) *There exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S and R are operators.

(ii) *There exist a finite measure ν, an $L^1(\nu)$-valued vector measure m, and an operator $i : X \to L^1(m)$ such that*

$$\|T(x)\| \le \Big\| \int i(x)\,dm \Big\|_{L^1(\nu)} \tag{2.1}$$

for every $x \in X$.

(iii) *There exist an order continuous Banach lattice with weak unit L, an L-valued vector measure m, and an operator $i : X \to L^1(m)$ such that*

$$\sum_{j=1}^{n} \|T(x_j)\| \le \Big\| \sum_{j=1}^{n} \Big| \int i(x_j)\,dm \Big| \Big\|_{L} \tag{2.2}$$

for every $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$.

The following simple lemma will be used in the proof of Theorem 2.1; it gives a basic tool that relates domination and factorization of operators.

Lemma 2.2. *Let X, Y, and Z be Banach spaces and let $T : X \to Y$ and $S : X \to Z$ be operators such that*

$$\|T(x)\|_Y \le \|S(x)\|_Z \quad \text{for all } x \in X. \tag{2.3}$$

Then there is an operator $R : \overline{S(X)} \to Y$ such that $T = R \circ S$.

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PROOF. By (2.3), we can define a linear continuous mapping $r : S(X) \to Y$ by declaring $r(S(x)) := T(x)$ for all $x \in X$. Then r can be extended uniquely to an operator $R : \overline{S(X)} \to Y$ satisfying the required property. □

PROOF OF THEOREM 2.1. (i) $\Longrightarrow$ (ii) Let (Ω, Σ) be the measurable space on which ν is defined. Take the vector measure $m : \Sigma \to L^1(\nu)$ given by $m(A) := \chi_A$ (the characteristic function of A) for all $A \in \Sigma$. In this case, the integration operator $I_m : L^1(m) \to L^1(\nu)$ is an order isometry (see, for example, [17, Corollary 3.66]). Consider now the operator $i : X \to L^1(m)$ given by $i := I_m^{-1} \circ S$. We can assume without loss of generality that $\|R\| = 1$. Then for every $x \in X$ we have

$$\begin{aligned} \|T(x)\| = \|R(S(x))\| &\le \|R\| \, \|S(x)\|_{L^1(\nu)} \\ &= \|I_m(i(x))\|_{L^1(\nu)} = \Big\| \int i(x)\, dm \Big\|_{L^1(\nu)}. \end{aligned}$$

Therefore, inequality (2.1) holds.

(ii) $\Longrightarrow$ (i) We can apply Lemma 2.2 to $S := I_m \circ i : X \to L^1(\nu)$ in order to find an operator $R : \overline{S(X)} \to Y$ such that $R \circ S = T$.

(ii) $\Longrightarrow$ (iii) This is clear by taking $L := L^1(\nu)$.

(iii) $\Longrightarrow$ (ii) We begin by proving the following claim.

CLAIM 2.3. There is $\xi \in B_{L^*}$ such that

$$\|T(x)\| \le \Big\langle \Big| \int i(x)\, dm \Big|, \xi \Big\rangle \quad \text{for all } x \in X. \tag{2.4}$$

Indeed, for each $\bar{x} = (x_1, \ldots, x_n) \in X^n$, $n \in \mathbb{N}$, we define $\phi_{\bar{x}} : B_{L^*} \to \mathbb{R}$ as

$$\phi_{\bar{x}}(\xi) := \sum_{j=1}^{n} \|T(x_j)\| - \Big\langle \sum_{j=1}^{n} \Big| \int i(x_j)\, dm \Big|, \xi \Big\rangle.$$

Clearly, $\phi_{\bar{x}}$ is convex, w^*-continuous, and there exists $\xi_{\bar{x}} \in B_{L^*}$ such that $\phi_{\bar{x}}(\xi_{\bar{x}}) \le 0$ (by (2.2)). Note also that the collection of all functions of the form $\phi_{\bar{x}}$ is a convex cone of $\mathbb{R}^{B_{L^*}}$. An appeal to Ky Fan's lemma (see, for example, [8, Lemma 9.10]) ensures that there is $\xi \in B_{L^*}$ such that $\phi_{\bar{x}}(\xi) \le 0$ for all $\phi_{\bar{x}}$ as above. In particular, inequality (2.4) holds and the claim is proved.

Since L is an order continuous Banach lattice with weak unit, there exist a Banach function space Z over some probability space (Ω, Σ, μ), an order isometry $J : L \to Z$, and a function $h \in L^1(\mu)$ such that

$$\langle u, \xi \rangle = \int_\Omega J(u) h \, d\mu \quad \text{for all } u \in L \tag{2.5}$$

(see, for example, [15, Theorem 1.b.14]). Let ν be the finite measure defined by the formula $\nu(A) := \int_A |h| \, d\mu$ for all $A \in \Sigma$, so that the identity map $\alpha : Z \to L^1(\nu)$ is an operator. Clearly, the operator $j := \alpha \circ J : L \to L^1(\nu)$ satisfies

$$j(|u|) = |j(u)| \quad \text{for every } u \in L. \tag{2.6}$$

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Define an $L^1(\nu)$-valued vector measure by $\tilde{m} := j \circ m$. Then every m-integrable function is $\tilde{m}$-integrable and the identity map $\beta : L^1(m) \to L^1(\tilde{m})$ is an operator satisfying $I_{\tilde{m}} \circ \beta = j \circ I_m$ (see, for example, [17, Lemma 3.27]). Bearing in mind (2.4), for every $x \in X$ we have:

$$\begin{aligned}
\|T(x)\| &\leq \Big\langle \Big|\int i(x)\, dm\Big|, \xi \Big\rangle \overset{(2.5)}{=} \int_\Omega J\Big(\Big|\int i(x)\, dm\Big|\Big) h\, d\mu \\
&\leq \int_\Omega J\Big(\Big|\int i(x)\, dm\Big|\Big)|h|\, d\mu = \int_\Omega j\Big(\Big|\int i(x)\, dm\Big|\Big)\, d\nu \\
&\overset{(2.6)}{=} \int_\Omega \Big| j\Big(\int i(x)\, dm\Big)\Big|\, d\nu \\
&= \int_\Omega \Big|\int \beta(i(x))\, d\tilde{m}\Big|\, d\nu = \Big\|\int (\beta \circ i)(x)\, d\tilde{m}\Big\|_{L^1(\nu)}.
\end{aligned}$$

Therefore, inequality (2.1) holds for the $L^1(\nu)$-valued vector measure $\tilde{m}$ and the operator $\beta \circ i : X \to L^1(\tilde{m})$, hence T satisfies (ii). The proof is finished. □

Remark 2.4. When X is an order continuous Banach function space over a finite measure, we get an alternative proof for (i) $\Longrightarrow$ (ii) in Theorem 2.1, with a different choice of operators and vector measures.

Proof. Let (Ω, Σ, μ) be the finite measure space on which X is based. Suppose (i) in Theorem 2.1 holds. We can assume without loss of generality that $\|R\| = 1$. Define $m(A) := S(\chi_A)$ for every $A \in \Sigma$. Since S is an operator and X is order continuous, the following statements hold: $m : \Sigma \to L^1(\nu)$ is a vector measure; every element of X belongs to $L^1(m)$; the identity map $i : X \to L^1(m)$ is an operator; and $S = I_m \circ i$ (see, for example, [17, Proposition 4.4]). Then

$$\|T(x)\| = \|R(S(x))\| \leq \|S(x)\| = \|I_m(i(x))\|_{L^1(\nu)} = \Big\|\int_\Omega i(x)\, dm\Big\|_{L^1(\nu)}$$

for every $x \in X$. This proves that (ii) in Theorem 2.1 holds. □

Corollary 2.5. *Let X be a Banach space and L an order continuous Banach lattice with weak unit. If $T : X \to L$ is an operator satisfying*

$$\sum_{j=1}^n \|T(x_j)\| \leq \Big\|\sum_{j=1}^n |T(x_j)|\Big\|_L \quad \textit{for all } x_1, \ldots, x_n \in X, n \in \mathbb{N}, \tag{2.7}$$

then T factors through a subspace of $L^1(\nu)$ for some finite measure ν.

Proof. By [15, Theorem 1.b.14], we can assume that L is a Banach function space over some probability space (Ω, Σ, μ). The set function $m : \Sigma \to L$ defined by $m(A) := \chi_A$ is a vector measure such that the integration operator $I_m : L^1(m) \to L$ is an isomorphism (see, for example, [17, Corollary 3.66]). Inequality (2.7) means that (iii) in Theorem 2.1 holds by taking $i := I_m^{-1} \circ T$. Therefore, T factors through a subspace of $L^1(\nu)$ for some finite measure ν. □

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In general, an operator factoring through a subspace of an L^1-space need not factor through an L^1-space.

EXAMPLE 2.6. The identity operator $T : \ell^2 \to \ell^2$ factors through the subspace of $L^1[0, 1]$ generated by the Rademacher functions (which is isomorphic to ℓ^2). However, T cannot be factored through an L^1-space. Indeed, T is not 1-summing, while Grothendieck's theorem ensures that every operator from an L^1-space to ℓ^2 is 1-summing (see, for example, [8, Theorem 3.4]).

A Banach space Y is called *injective* (respectively, *separably injective*) if, for every Banach space (respectively, separable Banach space) Z and every subspace $Z_0 \subseteq Z$, any operator $R : Z_0 \to Y$ can be extended to an operator $\tilde{R} : Z \to Y$. A typical example of injective (respectively, separably injective) space is ℓ^∞ (respectively, c_0). We refer the reader to [1, 22] for basic information on injective and separably injective spaces.

REMARK 2.7. Let X and Y be Banach spaces and $T : X \to Y$ an operator satisfying the statements of Theorem 2.1. Then T factors through $L^1(\nu)$ if either Y is injective or Y is separably injective and X is separable.

PROOF. The injective case is obvious. On the other hand, if X is separable, then so is $Z_0 := \overline{S(X)} \subseteq E$. Since $L^1(\nu)$ is weakly compactly generated, it has the separable complementation property (see, for example, [10, Section 13.2]), hence there is a separable *complemented* subspace Z of $L^1(\nu)$ such that $Z_0 \subseteq Z$. If, in addition, Y is separably injective, then the restriction $R|_{Z_0}$ can be extended to an operator from Z to Y. Since Z is complemented in $L^1(\nu)$, such an operator can be obviously extended to an operator from $L^1(\nu)$ to Y. $\square$

3. Positive and compact factorizations

We begin this section by showing that, under some adequate requirements, factoring through a subspace of an L^1-space is equivalent to factoring through an L^1-space. To this end we shall apply the Maurey–Rosenthal theorem (Theorem 1.1).

PROPOSITION 3.1. *Let X be an order continuous Banach function space over a finite measure, Y a Banach space and $T : X \to Y$ an operator. The following statements are equivalent.*

(i) *There exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a positive *operator and R is an operator.*

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(ii) *There is a finite measure ν such that T factors as*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ L^1(\nu) \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a positive *operator and R is an operator.*

Proof. Only (i) $\Longrightarrow$ (ii) requires a proof. Let (Ω, Σ, μ) be the finite measure space on which X is based. Since S is positive and $L^1(\nu)$ is 1-concave, S is 1-concave (see, for example, [15, Proposition 1.d.9]). The Maurey–Rosenthal theorem ensures that S factors as

$$\begin{tikzcd} X \arrow[r, "S"] \arrow[d, "U"'] & E \\ L^1(\mu) \arrow[ur, "V"'] & \end{tikzcd}$$

where U is a positive (multiplication) operator and V is an operator. Then we have the factorization $T = (R \circ V) \circ U$ and so (ii) holds. □

The following remark should be compared with Corollary 2.5.

Remark 3.2. Let X be an order continuous Banach function space over a finite measure, Y a Banach lattice and $T : X \to Y$ a positive operator satisfying

$$\sum_{j=1}^{n} \|T(x_j)\| \leq \bigg\| \sum_{j=1}^{n} |T(x_j)| \bigg\| \quad \text{for all } x_1, \ldots, x_n \in X, n \in \mathbb{N}. \tag{3.1}$$

Then T factors through $L^1(\nu)$ for some finite measure ν.

Proof. Every positive operator between Banach lattices satisfying (3.1) is 1-concave (see, for example, [15, Proposition 1.d.9]). Hence the conclusion follows from the Maurey–Rosenthal theorem. □

Some straightforward verifications in the proof of Theorem 2.1 yield the following characterization.

Theorem 3.3. *Let X be a Banach lattice, Y a Banach space, and $T : X \to Y$ an operator. The following statements are equivalent.*

(i) *There exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a positive *operator and R is an operator.*

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(ii) *There exist a finite measure ν, an $L^1(\nu)$-valued* positive *vector measure m, and a* positive *operator $i : X \to L^1(m)$ such that*

$$\|T(x)\| \le \left\| \int i(x)\, dm \right\|_{L^1(\nu)}$$

for every $x \in X$.

(iii) *There exist an order continuous Banach lattice with weak unit L, an L-valued* positive *vector measure m, and a* positive *operator $i : X \to L^1(m)$ such that*

$$\sum_{j=1}^{n} \|T(x_j)\| \le \left\| \sum_{j=1}^{n} \left| \int i(x_j)\, dm \right| \right\|_L$$

for every $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$.

From now on we focus on factorizations for which the first factor is a *compact* operator. In this case, domination by L^∞-valued operators appears in a natural way and the key property for the vector measure is to have norm relatively compact range. This type of factorization is related to the following summability property.

Definition 3.4. Let X and Y be Banach spaces. An operator $T : X \to Y$ satisfies property (S_∞) if there exist a finite measure ν, an $L^1(\nu)$-valued vector measure m with norm relatively compact range, and an operator $i : X \to L^\infty(m)$ such that

$$\sum_{j=1}^{n} \|T(x_j)\| \le \sup_{h \in B_{L^\infty(m)}} \sum_{j=1}^{n} \left\| \int i(x_j) h\, dm \right\|_{L^1(\nu)} \tag{3.2}$$

for every $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$.

Theorem 3.5. *Let X and Y be Banach spaces. If an operator $T : X \to Y$ satisfies property (S_∞), then there exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a compact *operator and R is an operator.*

In the proof of Theorem 3.5 we will use an auxiliary locally convex Hausdorff topology τ_m on $L^1(m)$ (m a vector measure). A net (f_α) in $L^1(m)$ is said to be τ_m-convergent to $f \in L^1(m)$ if and only if for every $h \in L^\infty(m)$ we have

$$\int f_\alpha h\, dm \to \int f h\, dm \text{ in norm.}$$

The topology τ_m has been studied recently in [3, 21].

Proof of Theorem 3.5. Since m has norm relatively compact range, $K := B_{L^\infty(m)}$ is τ_m-compact as a subset of $L^1(m)$ (see [3, Proposition 3.5]). From now on K is equipped with the topology τ_m. For each $g \in L^\infty(m)$ we consider the operator

$$T_g : L^1(m) \to L^1(\nu), \quad T_g(f) = \int fg\,dm.$$

Since T_g is τ_m-norm continuous, the restriction $T_g|_K$ is Bochner integrable with respect to any regular Borel probability on K. Moreover, the function $\widetilde{g} : K \to \mathbb{R}$ given by

$$\widetilde{g}(h) := \|T_g(h)\|_{L^1(\nu)} \tag{3.3}$$

is continuous. We denote by $C(K, L^1(\nu))$ the Banach space of all $L^1(\nu)$-valued functions on K which are τ_m-norm continuous, with the supremum norm.

Claim A. *The map $j : X \to C(K, L^1(\nu))$ given by $j(x) := T_{i(x)}|_K$ is an operator.* Indeed, the linearity of j is clear. On the other hand, the norm of any $f \in L^1(m)$ can be computed as

$$\|f\|_{L^1(m)} = \sup_{h\in K}\Big\|\int hf\,dm\Big\|_{L^1(\nu)} \tag{3.4}$$

(see, for example, [17, Lemma 3.11]) and so for every $x \in X$ we have

$$\begin{aligned}\|j(x)\|_{C(K,L^1(\nu))} &= \sup_{h\in K}\Big\|\int hi(x)\,dm\Big\|_{L^1(\nu)}\\ &\overset{(3.4)}{=} \|i(x)\|_{L^1(m)} \le \|m\|\cdot\|i(x)\|_{L^\infty(m)} \le \|m\|\cdot\|i\|\cdot\|x\|,\end{aligned}$$

where $\|m\|$ denotes the total semivariation of m. Therefore, j is continuous.

Claim B. *Let η be a regular Borel probability on K. Then the map*

$$\Phi : K \to L^1(\eta, L^1(\nu)), \quad \Phi(g) := T_g|_K,$$

is τ_m-norm continuous. Here $L^1(\eta, L^1(\nu))$ denotes the Banach space of all (equivalence classes of) $L^1(\nu)$-valued functions on K which are Bochner integrable with respect to η. Since K is a compact subset of the angelic space $(L^1(m), \tau_m)$ (see [3, Proposition 2.2]), it suffices to prove that Φ is τ_m-norm sequentially continuous. Let (g_n) be a sequence in K which τ_m-converges to $g \in K$. For each $n \in \mathbb{N}$ we define $f_n := g_n - g \in 2B_{L^\infty(m)}$ and we consider the function $\widetilde{f_n} \in C(K)$ defined by (3.3). Note that the sequence $(\widetilde{f_n})$ in $C(K)$ satisfies $\widetilde{f_n} \to 0$ pointwise on K (because $f_n \to 0$ with respect to τ_m) and

$$\|\widetilde{f_n}\|_{C(K)} = \sup_{h\in K}\Big\|\int hf_n\,dm\Big\|_{L^1(\nu)} \overset{(3.4)}{=} \|f_n\|_{L^1(m)} \le 2\|m\| \quad \text{for all } n \in \mathbb{N}.$$

By Lebesgue's dominated convergence theorem we get

$$\|\Phi(g_n) - \Phi(g)\|_{L^1(\eta,L^1(\nu))} = \int_K \|T_{g_n}(h) - T_g(h)\|_{L^1(\nu)}\,d\eta(h) = \int_K \widetilde{f_n}\,d\eta \to 0.$$

This proves Claim B.

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Consider now the w^*-compact convex set $\mathcal{P}(K) \subseteq C(K)^*$ of all regular Borel probability measures on K. For each $\bar{x} = (x_1, \ldots, x_n) \in X^n$, $n \in \mathbb{N}$, we define the function $\phi_{\bar{x}} : \mathcal{P}(K) \to \mathbb{R}$ by

$$\phi_{\bar{x}}(\eta) := \sum_{k=1}^{n} \|T(x_k)\| - \sum_{k=1}^{n} \int_K \widetilde{i(x_k)}\, d\eta.$$

Clearly, $\phi_{\bar{x}}$ is convex and w^*-continuous. Moreover, since $\sum_{k=1}^{n} \widetilde{i(x_k)}$ is continuous on K, there is $h_{\bar{x}} \in K$ such that

$$\sum_{k=1}^{n} \|T(x_k)\| \overset{(3.2)}{\leq} \sup_{h \in K} \sum_{k=1}^{n} \widetilde{i(x_k)}(h) = \sum_{k=1}^{n} \widetilde{i(x_k)}(h_{\bar{x}}),$$

hence $\phi_{\bar{x}}(\delta_{h_{\bar{x}}}) \leq 0$, where $\delta_{h_{\bar{x}}} \in \mathcal{P}(K)$ denotes the evaluation functional at $h_{\bar{x}}$. It is also easy to check that the collection of all functions of the form $\phi_{\bar{x}}$ is a convex cone of $\mathbb{R}^{\mathcal{P}(K)}$. According to Ky Fan's lemma (see, for example, [8, Lemma 9.10]) there is $\eta \in \mathcal{P}(K)$ such that $\phi_{\bar{x}}(\eta) \leq 0$ for all functions of the form $\phi_{\bar{x}}$. Thus

$$\|T(x)\| \leq \int_K \widetilde{i(x)}\, d\eta = \|S(x)\|_{L^1(\eta, L^1(\nu))} \quad \text{for every } x \in X, \tag{3.5}$$

where S is the composition of the operator j (of Claim A) with the identity operator from $C(K, L^1(\nu))$ to $L^1(\eta, L^1(\nu))$. In view of (3.5), we can apply Lemma 2.2 to find an operator $R : \overline{S(X)} \to Y$ such that $T = R \circ S$. Taking into account that $L^1(\eta, L^1(\nu))$ is isometrically isomorphic to $L^1(\eta \otimes \nu)$, we have a factorization of T through a subspace of the L^1-space of the finite measure $\eta \otimes \nu$. It remains to check that S is compact. Let $\rho > 0$ such that $i(\rho B_X) \subseteq K$. Since $S(\rho B_X) = \Phi(i(\rho B_X)) \subseteq \Phi(K)$ and $\Phi(K)$ is norm compact in $L^1(\eta, L^1(\nu))$ (by Claim B), the set $S(\rho B_X)$ is norm relatively compact, hence so is $S(B_X)$. The proof is finished. □

The proof that (ii) $\Longrightarrow$ (i) in the following remark follows the steps of the proof of (iii) $\Longrightarrow$ (ii) in Theorem 2.1.

Remark 3.6. Let X and Y be Banach spaces and $T : X \to Y$ an operator. The following assertions are equivalent and imply that T satisfies property (S_∞).

(i) There exist a finite measure ν, an $L^1(\nu)$-valued vector measure m with norm relatively compact range and an operator $i : X \to L^\infty(m)$ such that

$$\|T(x)\| \leq \left\| \int i(x)\, dm \right\|_{L^1(\nu)}$$

for every $x \in X$.

(ii) There exist an order continuous Banach lattice with weak unit L, an L-valued vector measure m with norm relatively compact range and an operator $i : X \to L^\infty(m)$ such that

$$\sum_{j=1}^{n} \|T(x_j)\| \leq \left\| \sum_{j=1}^{n} \left| \int i(x_j)\, dm \right| \right\|_L$$

for every $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}$.

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In order to prove a version of Theorem 3.5 for positive operators and positive vector measures we need the following technical result.

Lemma 3.7. *Let m be a vector measure. Then:*

(i) *$L^1(m)^+$ is τ_m-sequentially closed.*

(ii) *$K \cap L^1(m)^+$ is τ_m-compact whenever $K \subseteq L^1(m)$ is τ_m-compact.*

Proof. The convex set $L^1(m)^+$ is norm closed, hence weakly closed, so (i) follows from [3, Proposition 3.3]. Bearing in mind that τ_m-compactness and τ_m-sequential compactness coincide (see [3, Corollary 2.3]), part (ii) follows from (i). $\square$

Theorem 3.8. *Let X be a Banach lattice, Y a Banach space, and $T : X \to Y$ an operator such that there exist a finite measure ν, an $L^1(\nu)$-valued* positive *vector measure m with norm relatively compact range, and a* positive *operator $i : X \to L^\infty(m)$ such that*

$$\sum_{j=1}^{n} \|T(x_j)\| \le \sup_{h \in B_{L^\infty(m)}} \sum_{j=1}^{n} \left\| \int i(x_j) h \, dm \right\|_{L^1(\nu)}$$

for every $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}$.

Then there exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "S"'] & Y \\
E \arrow[ur, "R"'] &
\end{tikzcd}$$

where S is a positive compact *operator and R is an operator.*

Proof. This can be proved as Theorem 3.5 with only a few changes:

(1) Take the τ_m-compact set $K := B_{L^\infty(m)} \cap L^1(m)^+$ (Lemma 3.7).

(2) $\|j(x)\|_{C(K, L^1(\nu))} \le \|i(x)\|_{L^1(m)}$.

(3) $\|\widetilde{f_n}\|_{C(K)} \le \|f_n\|_{L^1(m)}$.

(4) $\phi_{\bar{x}}(\eta) := \sum_{k=1}^{n} \|T(x_k)\| - 2 \sum_{k=1}^{n} \int_K \widetilde{i(x_k)} \, d\eta$.

(5) $\|T(x)\| \le 2 \int_K \widetilde{i(x)} \, d\eta = \|2S(x)\|_{L^1(\eta, L^1(\nu))}$.

(6) $i(\rho B_X \cap X^+) \subseteq K$.

Since m and i are positive, the operator j is positive and the same holds for S. $\square$

4. Factorization of operators acting in $C(K)$ spaces

Throughout this section we consider the case of operators acting in a $C(K)$ space (K a compact Hausdorff topological space), that is, the Banach space of all real-valued continuous functions on K. Such operators play an important role in the general theory of Banach spaces. A fundamental result of Bartle, Dunford, and Schwartz [2] (see, for example, [9, VI.2])states that if $T : C(K) \to Y$ is a weakly compact operator

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(Y a Banach space), then there is an Y-valued regular Borel vector measure m on K such that $T(f) = \int_K f\, dm$ for all $f \in C(K)$.

REMARK 4.1. Let K be a compact Hausdorff topological space, Y a Banach space, and $T : C(K) \to Y$ an operator. The following statements are equivalent.

(i) T is 1-*summing*.

(ii) There exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as

$$\begin{tikzcd} C(K) \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a *positive* operator and R is an operator.

(ii′) The same as (ii) with a (not necessarily finite) nonnegative measure ν.

(iii) There is a finite measure ν such that T factors as

$$\begin{tikzcd} C(K) \arrow[r, "T"] \arrow[d, "S"'] & Y \\ L^1(\nu) \arrow[ur, "R"'] & \end{tikzcd}$$

where S is a *positive* operator and R is an operator.

(iii′) The same as (iii) with a (not necessarily finite) nonnegative measure ν.

PROOF. The implication (i) $\Longrightarrow$ (iii) follows from Pietsch's factorization theorem (see, for example, [8, Corollary 2.15]). (ii′) $\Longrightarrow$ (i) is a consequence of the fact that every positive operator from $C(K)$ to a 1-concave Banach lattice (like $L^1(\nu)$) is 1-summing (see, for example, [15, Theorem 1.d.10]). The remaining implications are obvious. □

It turns out that an operator acting in a $C(K)$ space factors through a subspace of an L^1-space if and only if it is 2-*summing*. These operators can also be characterized in the spirit of Theorem 2.1, as follows.

THEOREM 4.2. *Let K be a compact Hausdorff topological space and Y a Banach space. The following assertions are equivalent for an operator $T : C(K) \to Y$.*

(i) *T is 2-summing.*

(ii) *There exist a finite measure ν and a subspace $E \subseteq L^1(\nu)$ such that T factors as*

$$\begin{tikzcd} C(K) \arrow[r, "T"] \arrow[d, "S"'] & Y \\ E \arrow[ur, "R"'] & \end{tikzcd}$$

where S and R are operators.

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(ii′) *The same as (ii) with a (not necessarily finite) nonnegative measure ν.*

(iii) *There exist a finite measure ν and an $L^1(\nu)$-valued regular Borel vector measure m on K such that*

$$\|T(f)\| \leq \left\| \int_K f\, dm \right\|_{L^1(\nu)}$$

for every $f \in C(K)$.

(iv) *There exist an order continuous Banach lattice with weak unit L and an L-valued regular Borel vector measure m on K such that*

$$\sum_{j=1}^{n} \|T(f_j)\| \leq \left\| \sum_{j=1}^{n} \left| \int_K f_j\, dm \right| \right\|_L$$

for every $f_1, \ldots, f_n \in C(K)$, $n \in \mathbb{N}$.

Proof. (i) $\Longrightarrow$ (ii) This implication holds for operators defined on arbitrary Banach spaces. Indeed, any 2-summing operator factors through a *Hilbert space* (see, for example, [8, Corollary 2.16]) and every Hilbert space embeds isomorphically into an L^1-space of a finite measure (see, for example, [14, p. 128, Theorem 12]).

(ii) $\Longrightarrow$ (ii′) Trivial.

(ii′) $\Longrightarrow$ (i) Since every operator from $C(K)$ to an L^1-space is 2-summing (see, for example, [8, Theorem 3.5]), S is 2-summing and so is $T = R \circ S$.

(ii) $\Longrightarrow$ (iii) We can assume without loss of generality that $\|R\| = 1$. Since $L^1(\nu)$ contains no subspace isomorphic to c_0, the operator S is weakly compact (see, for example, [9, p. 159, Theorem 15]). Therefore, there is an E-valued regular Borel vector measure m on K such that $S = I_m \circ i$, where $i : C(K) \to L^1(m)$ is the identity operator (see, for example, [9, VI.2]). Then

$$\|T(f)\| = \|R(S(f))\| \leq \|S(f)\| = \|I_m(f)\|_{L^1(\nu)} = \left\| \int_K f\, dm \right\|_{L^1(\nu)}$$

for every $f \in C(K)$. This proves that (iii) holds.

(iii) $\Longrightarrow$ (iv) This is clear by taking $L := L^1(\nu)$.

(iv) $\Longrightarrow$ (iii) This can be proved as in (iii) $\Longrightarrow$ (ii) of Theorem 2.1.

(iii) $\Longrightarrow$ (ii) Let $i : C(K) \to L^1(m)$ be the identity operator and consider the operator $S := I_m \circ i : C(K) \to L^1(\nu)$. Then Lemma 2.2 applied to T and S gives the desired factorization. The proof is complete. □

The following example shows that compact operators from a $C(K)$ space need not factor through subspaces of L^1-spaces.

Example 4.3. Consider the standard basis (e_n) of c_0, a sequence (I_n) of pairwise disjoint open subintervals of $[0, 1]$ and points $k_n \in I_n$ for all $n \in \mathbb{N}$. Define an operator $T : C[0, 1] \to c_0$ by

$$T(f) := \sum_{n=1}^{\infty} \frac{f(k_n)}{\sqrt{n}} e_n, \quad f \in C[0, 1].$$

Then T is compact but not 2-summing (hence T does not factor through a subspace of an L^1-space).

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PROOF. Clearly, T is the limit (in the operator norm) of the sequence (T_p) of finite-rank operators $T_p : C[0,1] \to c_0$ defined by

$$T_p(f) := \sum_{n=1}^{p} \frac{f(k_n)}{\sqrt{n}} e_n, \quad f \in C[0,1].$$

Hence T is compact. We prove that T is not 2-summing by contradiction. If T were 2-summing, then Pietsch's domination theorem (see, for example, [8, Theorem 2.12]) would ensure the existence of a finite Borel measure μ on $[0,1]$ such that

$$\|T(f)\|_{c_0} \le \|f\|_{L^2(\mu)} \quad \text{for all } f \in C(K).$$

Take a sequence (f_n) in $C[0,1]$ such that $0 \le f_n \le 1$, $\mathrm{supp}(f_n) \subseteq I_n$, and $f_n(k_n) = 1$ for all $n \in \mathbb{N}$. Then, for any $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} \frac{1}{n} = \sum_{n=1}^{N} \|T(f_n)\|_{c_0}^2 \le \sum_{n=1}^{N} \|f_n\|_{L^2(\mu)}^2 = \int_{[0,1]} \Big(\sum_{n=1}^{N} f_n^2\Big) d\mu \le \mu([0,1]) < \infty,$$

because $0 \le \sum_{n=1}^{N} f_n^2 \le 1$. This gives a contradiction. Therefore, T is not 2-summing and cannot be factored through a subspace of an L^1-space. □

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