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# An uncountable Furstenberg–Zimmer structure theory

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*Abstract.* Furstenberg–Zimmer structure theory refers to the extension of the dichotomy between the compact and weakly mixing parts of a measure-preserving dynamical system and the algebraic and geometric descriptions of such parts to a conditional setting, where such dichotomy is established relative to a factor and conditional analogs of those algebraic and geometric descriptions are sought. Although the unconditional dichotomy and the characterizations are known for arbitrary systems, the relative situation is understood under certain countability and separability hypotheses on the underlying groups and spaces. The aim of this article is to remove these restrictions in the relative situation and establish a Furstenberg–Zimmer structure theory in full generality. As an independent byproduct, we establish a connection between the relative analysis of systems in ergodic theory and the internal logic in certain Boolean topoi.

Key words: structure theory, measure preserving systems, ergodic theory
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## 1. *Introduction*

1.1. *Motivation.* In the pioneering work [17], Furstenberg applied the countable Furstenberg–Zimmer structure theory, developed by him in the same article and by Zimmer in [47, 48] around the same time, to prove multiple recurrence for $\mathbb{Z}$-actions and show that this multiple recurrence theorem is equivalent to Szemerédi's theorem [43]. (Here countable refers to systems of countable measure-theoretic complexity such as measure-preserving $\mathbb{Z}$-actions on standard Borel probability spaces. The precise meaning of countable measure-theoretic complexity is given later in the introduction.) The Furstenberg–Zimmer structure theory is significantly refined in the Host–Kra–Ziegler structure theory [26, 46] which establishes a description of the characteristic factors governing multiple recurrence for $\mathbb{Z}$-actions as inverse limits of rotations on nilmanifolds. The finitary counterpart of the Host–Kra–Ziegler structure theorem is known as the inverse theorem for the Gowers uniformity norms by Green, Tao, and Ziegler [23]—a foundational result in the area of higher-order Fourier analysis initiated by Gower's seminal work [21, 22] on Szemerédi's theorem.

Beyond the natural (intrinsic) motivation of generalizing ergodic theory to uncountable systems, we hope that an uncountable Host–Kra–Ziegler structure theory (such as for the action of hyperfinite abelian groups on Loeb probability spaces) may help to clarify the relationship between the ergodic-theoretical and analytic (higher-order Fourier analysis) approaches to Szemerédi's theorem. See [29, 30] for some recent progress.

By proving an uncountable Moore–Schmidt theorem [33] and an uncountable Mackey–Zimmer theorem [32] (and clarifying in [31] some foundational aspects of uncountable measure theory), first basic results in ergodic structure theory were successfully extended to uncountable settings. The present paper aims to establish another part by developing an uncountable Furstenberg–Zimmer structure theory.

The uncountable Furstenberg–Zimmer structure theory of this paper was applied in the joint work [10] by the author and others to obtain an uncountable and uniform extension of the double recurrence theorem for amenable groups by Bergelson, McCutcheon, and Zhang [4].

1.2. *Main results.* Let $\Gamma$ be a group and $X$ be a $\sigma$-complete Boolean algebra equipped with a countably additive measure $\mu$ of total mass 1. Suppose that $\Gamma$ acts on the probability algebra $(X, \mu)$ by a group homomorphism (this definition is equivalent to that given in Definition 2.1, where by reversing the arrows in the category of probability algebras, we can work with an action of $\Gamma$ instead of its opposite) $T : \Gamma^{\mathrm{op}} \to \mathrm{Aut}(X, \mu)$, where $\mathrm{Aut}(X, \mu)$ denotes the automorphism group of $(X, \mu)$ and $\Gamma^{\mathrm{op}}$ the opposite group. The automorphisms of $(X, \mu)$ are measure-preserving bijective Boolean homomorphisms of $X$. (In the ergodic theory literature, it is common to call the pair $(X, \mu)$, where $X$ is a $\sigma$-complete Boolean algebra and $\mu$ is a countably additive probability measure on $X$, a *measure algebra*, e.g., see [19, 36]. We follow the conventions of Fremlin [16] to call $(X, \mu)$ a *probability algebra* and reserve the name measure algebra for abstract measures of arbitrary total mass.) We call $\mathcal{X} = (X, \mu, T)$ a *probability algebra* $\Gamma$*-dynamical system*. The category of such dynamical systems and some basic tools to study them are collected in the preliminaries in §2. A main goal of this paper is to establish the following.

THEOREM 1.1. (Uncountable Furstenberg–Zimmer structure theorem) *Let $\Gamma$ be a group. For any probability algebra $\Gamma$-dynamical system $\mathcal{X} = (X, \mu, T)$, there exists an ordinal $\beta$ and for each ordinal $\alpha \leq \beta$, a factor $\mathcal{X} \to \mathcal{Y}_\alpha = (Y_\alpha, \nu_\alpha, S_\alpha)$ satisfying the following properties:*

(i) *$\mathcal{Y}_0$ is trivial;*
(ii) *$\mathcal{Y}_{\alpha+1} \to \mathcal{Y}_\alpha$ is a relatively compact extension for every successor ordinal $\alpha + 1 \leq \beta$;*
(iii) *$\mathcal{Y}_\alpha$ is the inverse limit of the systems $\mathcal{Y}_\eta$, $\eta < \alpha$ for every limit ordinal $\alpha \leq \beta$, in the sense that $Y_\alpha$ is generated by $\bigcup_{\eta<\alpha} Y_\eta$ as a $\sigma$-complete Boolean algebra;*
(iv) *$\mathcal{X} \to \mathcal{Y}_\beta$ is a relatively weakly mixing extension.*

Theorem 1.1 follows from the following relative dichotomy between relatively weakly mixing and relatively compact extensions of probability algebra $\Gamma$-dynamical systems by transfinite induction.

THEOREM 1.2. (Uncountable relative dichotomy) *Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be probability algebra $\Gamma$-dynamical systems and $\pi : \mathcal{X} \to \mathcal{Y}$ an extension. Exactly one of the following statements is true:*

(i) *$\pi : \mathcal{X} \to \mathcal{Y}$ is a relatively weakly mixing extension;*
(ii) *there exist a probability algebra $\Gamma$-dynamical system $\mathcal{Z} = (Z, \lambda, R)$ and extensions $\phi : \mathcal{X} \to \mathcal{Z}$ and $\psi : \mathcal{Z} \to \mathcal{Y}$ such that $\psi$ is a non-trivial relatively compact extension.*

We prove Theorems 1.1 and 1.2 in §6. Compact and weakly mixing extensions of probability algebra systems are defined in §§4 and 6 respectively. Moreover, we establish various characterizations of relatively compact extensions in terms of (a) certain invariant finitely generated modules, (b) non-trivial invariant conditional Hilbert–Schmidt operators, and (c) conditionally almost periodic orbits. We establish the equivalence of (a)–(c) in smaller conditional $L^\infty$-modules similarly to the classical theory and also in larger conditional $L^0$-modules (where $L^0$ denotes the ring of equivalence classes of complex measurable functions), and we show that these descriptions in the larger and the smaller modules are equivalent. In the case of ergodic systems, we prove in §5 that relatively compact extensions are isomorphic to isometric extensions (that is, probability algebra homogeneous skew-product systems as introduced in [**32**]).

These results are well understood for systems of countable measure-theoretic complexity, see [**17**, **47**, **48**] for the original papers and [**12**, **18**, **19**, **36**, **44**] for some textbook expositions. The aim of this paper is to extend them to systems of uncountable measure-theoretic complexity. We say that a probability algebra $\Gamma$-dynamical system $(X, \mu, T)$ has *countable measure-theoretic complexity* if $\Gamma$ is countable and $(X, \mu)$ is separable (we have a metric structure on $X$ defined by $d(E, F) = \mu(E \Delta F)$), and has *uncountable measure-theoretic complexity* otherwise. In Table 1, we compare some features of systems of countable and uncountable complexity.

We establish a Furstenberg–Zimmer structure theory for not necessarily countable groups and not necessarily separable probability algebras. To achieve such generality, we have to introduce novel tools to study the structure of probability algebra systems relative to some factor since many classical tools, such as disintegration of measures, measurable selection lemmas, and the theory of measurable Hilbert bundles, rely on such countability and separability hypotheses. Instead, we apply tools from topos theory as suggested in [**45**] by Tao. We will not use topos theory directly nor define a sheaf anywhere in this paper. Instead, we use the closely related conditional analysis as developed in [**6**, **9**, **15**, **28**]. A main advantage of conditional analysis is that it can be setup without much cost and understood with basic knowledge in measure theory and functional analysis. The ergodic theoretic results of this paper are organized in a self-contained presentation and no familiarity with topos theory is required for the reader purely interested in the ergodic theoretic content. In fact, the use of conditional analysis and the greater generality lead to

TABLE 1. The group-theoretic, measure-theoretic, topological, and methodological differences of systems of countable and uncountable complexity (with a view towards their Furstenberg–Zimmer structure theory).

| Countable complexity | Uncountable complexity |
|---|---|
| Countable group actions | Uncountable group actions |
| Standard Borel probability spaces | Inseparable probability algebras |
| Pointwise action by measurable functions | Pointfree action by Boolean homomorphisms |
| Cantor models | Canonical Stonean models |
| Borel measurability | Baire measurability |
| Homogeneous skew-product extensions by metrizable compact groups | Homogeneous skew-product extensions by compact Hausdorff groups |
| von Neumann mean ergodic theorem | Alaoglu–Birkhoff ergodic theorem |
| Countable Furstenberg towers | Uncountable Furstenberg towers |
| Classical disintegration of measures | Canonical disintegration of measures |
| Measurable Hilbert bundles | Conditional Hilbert spaces |
| Measurable selection theory | Conditional analysis |

significant simplifications of the proofs of several results in Furstenberg–Zimmer structure theory.

For the readers familiar with topos theory and interested in the connection between ergodic theory and topos theory, we include several remarks relating the conditional analysis of this paper to the internal discourse in Boolean Grothendieck sheaf topoi. To the best knowledge of the author, these remarks comprise novel insight into the semantics of such topoi.

We would like to point out that in recent work [11], a slightly different proof of Theorem 1.1 is provided which is based on a functional analytic decomposition theorem for group representations on Kaplansky–Hilbert modules.

1.3. *Technical innovations.* *A priori* probability algebras are not $\sigma$-algebras of subsets of an underlying set. We view probability algebras as abstract pointfree probability spaces as opposed to concrete point-set probability spaces defined on $\sigma$-algebras of sets. In countable ergodic theory, the starting points are concrete countably generated probability spaces on which a measure-preserving action by a countable or second-countable group on the underlying set is defined. In the countable theory, a distinction is made between an action which is defined everywhere on the underlying set and a near-action which is only defined almost everywhere, e.g., see [48]. This distinction is non-trivial, since there are natural examples of systems of countable complexity for Polish group near-actions that cannot be realized by a pointwise action, e.g., see [20, 39].

In uncountable ergodic theory, the starting points are arbitrary (not necessarily countably generated or separable) probability algebras on which an arbitrary (not necessarily countable or second-countable) group acts by measure-preserving Boolean isomorphisms. A major advantage of working with pointfree probability algebras is that one systematically avoids to have to deal with null set issues. In this regard, the abstract pointfree perspective, which has been already implemented in our previous works [32, 33] and systematically

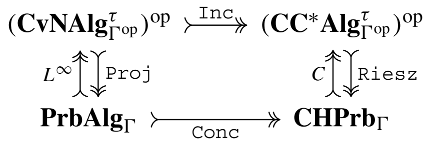


FIGURE 1. Adjacent auxiliary dynamical categories. Tailed arrows are faithful functors, while arrows with two heads are full functors. The diagram commutes up to natural isomorphisms.

developed in the foundational paper [**31**], seems to be more natural in uncountable ergodic theory.

Furthermore, the category of probability algebra dynamical systems is dually equivalent to the dynamical category of commutative tracial von Neumann algebras (see Figure 1) such that all structural results established in this paper for probability algebra dynamical systems have natural analogs in the dynamical category of commutative tracial von Neumann algebras, see [**36**, §3.2] for this correspondence for systems of countable complexity. See [**41**, §2] for a proof of a non-commutative Furstenberg–Zimmer-type dichotomy of relatively weakly mixing and relatively compact extensions for countable group actions on separable von Neumann algebras. In fact, the analogies extend also to the other two dynamical categories of commutative tracial $C^*$-algebras and topological dynamical systems preserving a Baire–Radon probability measure in Figure 1 by restricting to the image of the `Conc` and `Inc` functors, see §2 for the definition of these functors.

However, by quotienting out the null ideal, to any concrete countable complexity system we can associate an abstract probability algebra system. In this regard, the abstract perspective is implicitly present in the countable theory, e.g., see [**18**, §5], [**19**, §2], the preliminaries of [**47**, **48**], and the expositions in [**36**, **44**].

To study the Furstenberg–Zimmer structure theory of abstract pointfree systems, analogs of some basic and more advanced tools from measure theory and functional analysis are required. These are the basic Hilbert theory of $L^2$-spaces, disintegration of measures, and the theory of measurable Hilbert bundles. However, classical disintegration of measures and the theory of measurable Hilbert bundles are, in general, only applicable for systems of countable measure-theoretic complexity. Therefore, the main technical difficulty in extending the Furstenberg–Zimmer structure theory is to find viable alternatives which work for abstract systems of uncountable measure-theoretic complexity as well. We divide the above tools into two sets. The first set are basic tools from measure theory and the second one consists of tools needed for the analysis relative to some factor.

For the first group, we have two options. We can either use the abstract measure theory for measure algebras, as systematically developed in [**16**], or we can use concrete measure theory after passing to a *canonical model*. The canonical model, which is introduced in §2 (see also [**13**] and the references given in Remark 2.5), is a functor associating to any probability algebra dynamical system a unique compact Hausdorff probability measure-preserving dynamical system where the underlying group acts by homeomorphisms. The canonical model comes with some favorable properties, for example, a canonical disintegration of measures, see §2.4.

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We replace the second set of tools by conditional analysis which permits performing analysis in relative situations without any countability or separability assumptions. More precisely, we introduce a conditional Hilbert space and its conditional tensor product representing conditional Hilbert–Schmidt operators, and we study a conditional Gram–Schmidt process and a conditional version of the spectral theorem for these conditional Hilbert–Schmidt operators in §3 and Appendix A.

We found it convenient to explicitly use the language of category theory in this paper. It helps to clearly separate the concrete from the abstract probability spaces, and clarifies the relationships between these categories by way of *functorial relations*. These functors work particularly well with dynamics. This allows to safely pass back and forth between these categories, and thus take advantage of tools available in both the concrete and abstract settings. Functors establishing dualities of categories, such as the useful Gelfand duality, permit to efficiently translate into functional analytic categories to a similar effect. Practically, these categorical relations help with a basic housekeeping, avoiding making unnecessary mistakes. An illustrating example may be the clarification of category-dependent notions of measurability and various product constructions whose relations to each other can be worked out by functorial relations, cf. [**31**, **32**]. In fact, once these categorical relations are explicitly established and well understood, certain frequently used constructions and arguments automatize and simplify.

## 2. *Preliminaries*

2.1. *The categories.* Throughout this paper, we fix an arbitrary group $\Gamma$. We define the category $\mathbf{PrbAlg}_\Gamma$ of probability algebra $\Gamma$-dynamical systems to be the opposite of a category of Boolean algebras equipped with a probability measure-preserving dynamical structure.

A thorough discussion of some categorical aspects of ergodic theory and adjacent areas in measure theory and functional analysis, such as the auxiliary categories depicted in Figure 1, together with useful references to basic category theory can be found in the accompanying paper [**31**], to the relevant parts of which we will provide exact referencing as we proceed. The interested reader is also referred to [**31**] for our categorical notation and conventions.

*Definition 2.1.* (The category $\mathbf{PrbAlg}_\Gamma$)

(i) A $\mathbf{Bool}_\sigma$-object is a *$\sigma$-complete Boolean algebra* $X = (X, \vee, \wedge, \bar{\cdot}, 0, 1)$ and a $\mathbf{Bool}_\sigma$-morphism from a $\mathbf{Bool}_\sigma$-algebra $X$ to another $\mathbf{Bool}_\sigma$-algebra $Y$ is a *$\sigma$-complete Boolean homomorphism* $f : X \to Y$.

(ii) The category **AbsMbl** of *abstract measurable spaces* is the opposite category of $\mathbf{Bool}_\sigma$.

(iii) The objects of **PrbAlg** are *probability algebras* $(X, \mu)$, where $X$ is a $\mathbf{Bool}_\sigma$-algebra and $\mu : X \to [0, 1]$ is a countably additive probability measure, that is (a) $\mu(\bigvee_n E_n) = \sum_n \mu(E_n)$ for any sequence $(E_n)$ of pairwise disjoint elements in $X$, (b) $\mu(X) = 1$, and (c) $\mu(E) = 0$ if and only if $E = 0$. A **PrbAlg**-*morphism* from a **PrbAlg**-algebra $(X, \mu)$ to another **PrbAlg**-algebra $(Y, \nu)$ is a Boolean algebra homomorphism $\pi : Y \to X$ such that $\mu_*\pi(E) = \nu(E)$ for all $E \in Y$ (notice the

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opposite direction of the arrow. We implicitly work with an opposite category here to keep certain functors covariant), where we denote by $\mu_*\pi$ the pullback probability measure on $Y$ defined by $\mu_*\pi(E) := \mu(\pi(E))$. The $\mathbf{Bool}_\sigma$-algebra $X$ of a **PrbAlg**-object automatically upgrades to a complete Boolean algebra (that is, a Boolean algebra in which arbitrary joins and meets exist) and the underlying Boolean algebra homomorphism of a **PrbAlg**-morphism automatically upgrades to a complete Boolean algebra homomorphism (that is, one that preserves arbitrary joins and meets), see Remark 2.2. Let $\mathrm{Aut}(X, \mu)$ denote the automorphism group of a **PrbAlg**-algebra $(X, \mu)$ in the category **PrbAlg**.

(iv) A $\mathbf{PrbAlg}_\Gamma$-*system* is a tuple $(X, \mu, T)$ where $(X, \mu)$ is a **PrbAlg**-algebra and $T : \Gamma \to \mathrm{Aut}(X, \mu)$ is a group homomorphism $\gamma \mapsto T_\gamma$. A $\mathbf{PrbAlg}_\Gamma$-*morphism* from a $\mathbf{PrbAlg}_\Gamma$-system $(X, \mu, T)$ to another $\mathbf{PrbAlg}_\Gamma$-system $(Y, \nu, S)$ is a **PrbAlg**-morphism $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi \circ T_\gamma = S_\gamma \circ \pi$ for all $\gamma \in \Gamma$. We call $\mathbf{PrbAlg}_\Gamma$ the category of *probability algebra* $\Gamma$*-dynamical systems*. We denote a $\mathbf{PrbAlg}_\Gamma$-system by $\mathcal{X} = (X, \mu, T)$. If $\pi : \mathcal{X} \to \mathcal{Y}$ is a $\mathbf{PrbAlg}_\Gamma$-morphism, we call $\mathcal{X}$ a $\mathbf{PrbAlg}_\Gamma$-*extension* of $\mathcal{Y}$ and $\mathcal{Y}$ a $\mathbf{PrbAlg}_\Gamma$-*factor* of $\mathcal{X}$. We sometimes also refer to the $\mathbf{PrbAlg}_\Gamma$-morphism $\pi$ itself as a $\mathbf{PrbAlg}_\Gamma$-extension or a $\mathbf{PrbAlg}_\Gamma$-factor.

An extensive discussion of the previously introduced categories can be found in [**31**, §6].

*Remark 2.2.* It is well known that the $\mathbf{Bool}_\sigma$-algebra $X$ of a **PrbAlg**-algebra $(X, \mu)$ is automatically a complete Boolean algebra. We provide the short proof for the convenience of the reader. Let $(E_i)_{i \in I}$ be an arbitrary family in $X$. Well order $I$ by $<$ and let $F_i = E_i \wedge \overline{\bigvee_{j<i} E_j}$ for each $i \in I$. By construction, $(F_i)_{i \in I}$ is a family of pairwise disjoint elements of $X$. Since $\mu$ is a finite measure, only countably many of the $F_i$ are non-zero (this property is also called the *countable chain condition*). Thus $\bigvee_{i \in I} E_i = \bigvee_{i \in I, F_i > 0} F_i$, and by $\sigma$-completeness of $X$, $\bigvee_{i \in I, F_i \neq 0} F_i \in X$. Moreover, if $\pi : (Y, \nu) \to (X, \mu)$ is a **PrbAlg**-morphism (the underlying $\mathbf{Bool}_\sigma$-morphism of which does not necessarily preserve countable joins and meets by definition), then $\pi$ automatically upgrades to a complete Boolean homomorphism due to the automatic injectivity of $\pi$ and the countable chain condition, cf. [**32**, Remark 1.2].

Next we introduce some basic tools to study the structure of $\mathbf{PrbAlg}_\Gamma$-systems. We obtain these tools by relating the category $\mathbf{PrbAlg}_\Gamma$ to some adjacent auxiliary dynamical categories of algebras and spaces, as illustrated in Figure 1. The symbol op on a category indicates the use of opposite category, while $\Gamma^{\mathrm{op}}$ for a group $\Gamma$ means the use of the opposite group.

*Definition 2.3.* (Adjacent dynamical categories) First we introduce the remaining categories in Figure 1 without a dynamical structure. Then we apply a universal procedure to turn these categories into corresponding dynamical categories.

(i) A **CHPrb**-space is a tuple $(X, \mathcal{B}a(X), \mu)$, where $X$ is a compact Hausdorff space, $\mathcal{B}a(X)$ is the Baire $\sigma$-algebra of $X$ (that is, the smallest $\sigma$-algebra on $X$ rendering continuous functions measurable), and $\mu$ is a Baire-Radon probability measure on the measurable space $(X, \mathcal{B}a(X))$. A **CHPrb**-morphism

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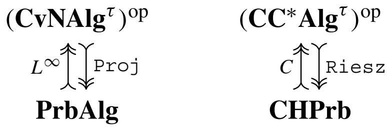


FIGURE 2. Gelfand-type dualities between opposite probability algebras and tracial commutative von Neumann algebra and between compact Hausdorff probability spaces and tracial commutative unital $C^*$-algebras.

$\pi : (X, \mathcal{B}a(X), \mu) \to (Y, \mathcal{B}a(Y), \nu)$ between compact Hausdorff probability spaces is a continuous function $\pi : X \to Y$ such that $\pi_*\mu = \nu$, where $\pi_*\mu$ denotes pushforward measure (a continuous function between compact Hausdorff spaces is always Baire-measurable since its pullback preservers compact $G_\delta$ sets).

(ii) A $\mathbf{CC^*Alg}^\tau$*-algebra* $(\mathcal{A}, \tau)$ is a unital commutative $C^*$-algebra equipped with a state $\tau : \mathcal{A} \to \mathbb{C}$ that is a bounded linear functional which is non-negative (it maps non-negative elements to non-negative reals) and is of operator norm 1. A $\mathbf{CC^*Alg}^\tau$*-morphism* $\Phi : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \tau_{\mathcal{B}})$ between $\mathbf{CC^*Alg}^\tau$-algebras is a unital $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\tau_{\mathcal{B}} \circ \Phi = \tau_{\mathcal{A}}$.

(iii) A $\mathbf{CvNAlg}^\tau$-algebra $(\mathcal{A}, \tau_{\mathcal{A}})$ is a commutative von Neumann algebra $\mathcal{A}$ equipped with a faithful trace $\tau_{\mathcal{A}}$, that is to say a $*$-linear functional $\tau_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$ with $\tau_{\mathcal{A}}(1) = 1$, and $\tau_{\mathcal{A}}(aa^*) \geq 0$ for any $a \in \mathcal{A}$, with equality if and only if $a = 0$. A $\mathbf{CvNAlg}^\tau$-morphism $\Phi : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \tau_{\mathcal{B}})$ between $\mathbf{CvNAlg}^\tau$-algebras is a von Neumann algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\tau_{\mathcal{A}} = \tau_{\mathcal{B}} \circ \Phi$.

Let $C$ be one of the categories **CHPrb**, $\mathbf{CC^*Alg}^\tau$, or $\mathbf{CvNAlg}^\tau$. For a $C$-object $X$, let $\mathrm{Aut}_C(X)$ be the automorphism group of $X$ in $C$. We define a $C_\Gamma$-object (respectively a $C_{\Gamma^{\mathrm{op}}}$-object) to be a tuple $(X, T)$, where $X$ is a $C$-object and $T : \Gamma \to \mathrm{Aut}_C(X)$ is a homomorphism $\gamma \mapsto T^\gamma$ of groups (respectively $T : \Gamma^{\mathrm{op}} \to \mathrm{Aut}_C(X)$ is a homomorphism $\gamma^{\mathrm{op}} \mapsto T^{\gamma^{\mathrm{op}}}$ of groups). A $C_\Gamma$-morphism $\pi : (X, T) \to (Y, S)$ (respectively $C_{\Gamma^{\mathrm{op}}}$-morphism $\pi : (Y, S) \to (X, T)$) between $C_\Gamma$-objects (respectively $C_{\Gamma^{\mathrm{op}}}$-objects) is a $C$-morphism $\pi : X \to Y$ (respectively a $C$-morphism $\pi : Y \to X$) obeying the intertwining property $\pi \circ T^\gamma = S^\gamma \circ \pi$ for all $\gamma \in \Gamma$ (respectively $\pi \circ S^{\gamma^{\mathrm{op}}} = T^{\gamma^{\mathrm{op}}} \circ \pi$ for all $\gamma^{\mathrm{op}} \in \Gamma^{\mathrm{op}}$).

2.2. *The functors.* The functors in Figure 1, in particular $L^\infty$, $C$, `Proj`, `Riesz`, and `Conc`, are the means by which we represent the objects and morphisms in the dynamical category $\mathbf{PrbAlg}_\Gamma$ by objects and morphisms in the adjacent dynamical categories $\mathbf{CHPrb}_\Gamma$, $\mathbf{CvNAlg}^\tau_{\Gamma^{\mathrm{op}}}$, $\mathbf{CC^*Alg}^\tau_{\Gamma^{\mathrm{op}}}$ to use the tools available in the latter categories to study structural properties of systems in the former, while the canonical model functor `Conc` is our main player. We start by describing the two Gelfand-type dualities in Figure 2 which yield the dualities of the dynamical categories in Figure 1 from which we then derive the canonical model functor `Conc` in the same figure.

Gelfand duality (cf. Figure 3) establishes a well-known equivalence between the category **CH** of compact Hausdorff spaces and the category $\mathbf{CC^*Alg}$ of unital commutative $C^*$-algebras. The functor `Spec` associates to each unital commutative $C^*$-algebra $\mathcal{A}$ its

$$\begin{array}{c} (\mathbf{CC^*Alg})^{\mathrm{op}} \\ C \big\uparrow \big\downarrow \mathtt{Spec} \\ \mathbf{CH} \end{array}$$

FIGURE 3. Gelfand duality between the category of compact Hausdorff spaces and commutative unital $C^*$-algebras.

spectrum $\mathtt{Spec}(\mathcal{A})$ (the set of $\mathbf{CC^*Alg}$-morphisms from $\mathcal{A}$ to $\mathbb{C}$) which is a compact Hausdorff space by the Banach–Alaoglu theorem and to each unital $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ the continuous function $\mathtt{Spec}(\Phi) : \mathtt{Spec}(\mathcal{B}) \to \mathtt{Spec}(\mathcal{A})$, $\mathtt{Spec}(\Phi)(f) := f \circ \Phi$. The functor $C$ associates with each compact Hausdorff space $X$ the unital commutative $C^*$-algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ and with each continuous function $\pi : X \to Y$ between $\mathbf{CH}$-spaces the Koopmann operator $C(\pi) : C(Y) \to C(X)$, $C(\pi)(f) := f \circ \pi$. See [31, §2] for a more detailed exposition.

The Gelfand duality in Figure 3 extends to a 'Riesz duality' between $\mathbf{CHPrb}$ and $\mathbf{CC^*Alg}^\tau$ as follows. A $\mathbf{CHPrb}$-space $(X, \mathcal{B}a(X), \mu)$ is mapped via the $C$-functor to the $\mathbf{CC^*Alg}^\tau$-algebra $(C(X), \tau_\mu)$, where the state $\tau_\mu : C(X) \to \mathbb{C}$ is defined by integration $\tau_\mu(f) := \int_X f\, d\mu$. If $\pi : (X, \mathcal{B}a(X), \mu) \to (Y, \mathcal{B}a(Y), \nu)$ is a $\mathbf{CHPrb}$-morphism, then the Koopman operator $C(\pi) : C(Y) \to C(X)$ satisfies $\tau_\nu = \tau_\mu \circ C(\pi)$. Conversely, using Gelfand duality and the Riesz representation theorem, we can associate to each $\mathbf{CC^*Alg}^\tau$-algebra $(\mathcal{A}, \tau_{\mathcal{A}})$ a unique $\mathbf{CHPrb}$-space $\mathtt{Riesz}(\mathcal{A}, \tau_{\mathcal{A}}) := (\mathtt{Spec}(\mathcal{A}), \mathcal{B}a(\mathtt{Spec}(\mathcal{A})), \mu_{\tau_{\mathcal{A}}})$. Uniqueness in the Riesz representation theorem also yields that if $\Phi : (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \tau_{\mathcal{B}})$ is a $\mathbf{CC^*Alg}^\tau$-morphism, then the continuous function $\mathtt{Spec}(\Phi) : \mathtt{Spec}(\mathcal{B}) \to \mathtt{Spec}(\mathcal{A})$ pushes forward $\nu_{\tau_{\mathcal{B}}}$ to $\mu_{\tau_{\mathcal{A}}}$. This 'Riesz duality' between $\mathbf{CHPrb}$ and $\mathbf{CC^*Alg}^\tau$ can be promoted in the obvious way to a duality between the corresponding dynamical categories $\mathbf{CHPrb}_\Gamma$ and $\mathbf{CC^*Alg}^\tau_{\Gamma^{\mathrm{op}}}$. This establishes the duality on the right-hand side of Figure 2. See [31, §5] for a more detailed exposition.

As for the duality on the left-hand side of Figure 2, let $(X, \mu)$ be a $\mathbf{PrbAlg}$-space. We define $L^\infty(X)$ to be the set of $\mathbf{AbsMbl}$-morphisms $f : X \to \mathcal{B}o(\mathbb{C})$, where $\mathcal{B}o(\mathbb{C})$ denotes the Borel $\sigma$-algebra of the complex numbers $\mathbb{C}$, which are bounded in the sense that there exists a real number $M \geq 0$ such that $f(\{z \in \mathbb{C} : |z| \leq M\}) = 1$, with $\|f\|_{L^\infty(X)}$ defined to equal the infimum of all such $M$. We can equip $L^\infty(X)$ with the structure of a commutative $*$-algebra by lifting all $*$-algebra operations from $\mathbb{C}$ to the set of all $\mathbf{AbsMbl}$-morphisms $f : X \to \mathcal{B}o(\mathbb{C})$. Every element $E$ of $X$ generates an idempotent element $1_E$ of $L^\infty(X)$. The set of finite linear combinations of idempotents form a dense subspace of $L^\infty(X)$. One can then define a trace $\tau$ on this algebra by defining $\tau(\sum_{n=1}^N c_n 1_{E_n}) := \sum_{n=1}^N c_n \mu(E_n)$ for any finite sequence of complex numbers $c_n$ and $E_n \in X$, and then extending by density. Thus, $L^\infty(X)$ can be viewed as an element of $\mathbf{CvNAlg}^\tau$. If $\pi : X \to Y$ is a $\mathbf{PrbAlg}$-morphism, one can define the $\mathbf{CvNAlg}^\tau$-morphism $L^\infty(\pi) : L^\infty(Y) \to L^\infty(X)$ by the Koopman operator $L^\infty(\pi)(f) := f \circ \pi$. See [31, §§6 and 7] for details of these constructions.

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Conversely, suppose that $(\mathcal{A}, \tau_{\mathcal{A}})$ is a $\mathbf{CvNAlg}^{\tau}$-algebra. We can form the collection $\mathcal{P}_{\mathcal{A}}$ of real projections in $\mathcal{A}$. As is well-known, these projections have the structure of a complete Boolean algebra. The trace $\tau_{\mathcal{A}}$ then becomes a countably additive probability measure on $\mathcal{P}_{\mathcal{A}}$, and we write $\texttt{Proj}(\mathcal{A}, \tau_{\mathcal{A}})$ for the opposite probability algebra of $(\mathcal{P}_{\mathcal{A}}, \tau_{\mathcal{A}})$. If $\Phi\colon (\mathcal{A}, \tau_{\mathcal{A}}) \to (\mathcal{B}, \tau_{\mathcal{B}})$ is a $\mathbf{CvNAlg}^{\tau}$-morphism, we observe that the associated von Neumann algebra homomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ maps projections in $\mathcal{P}_{\mathcal{A}}$ to projections in $\mathcal{P}_{\mathcal{B}}$, in a manner that preserves the trace as well as being a $\mathbf{Bool}_{\sigma}$-morphism. We then define $\texttt{Proj}(\Phi)\colon \texttt{Proj}(\mathcal{B}, \tau_{\mathcal{B}}) \to \texttt{Proj}(\mathcal{A}, \tau_{\mathcal{A}})$ to be the **PrbAlg**-morphism associated with this $\mathbf{Bool}_{\sigma}$-morphism. See [31, §7] for a proof that $\texttt{Proj}$ and $L^{\infty}$ establish a duality of categories between $\mathbf{CvNAlg}^{\tau}$ and **PrbAlg**. This duality then extends naturally to the dynamical categories $\mathbf{CvNAlg}^{\tau}_{\Gamma^{\mathrm{op}}}$ and $\mathbf{PrbAlg}_{\Gamma}$.

Every von Neumann algebra is also a unital $C^*$-algebra, and a faithful trace on a commutative von Neumann algebra becomes a state on the corresponding $C^*$-algebra. From this, it is easy to see that there is a forgetful inclusion functor $\texttt{Inc}$ from $\mathbf{CvNAlg}^{\tau}_{\Gamma^{\mathrm{op}}}$ to $\mathbf{CC}^*\mathbf{Alg}^{\tau}_{\Gamma^{\mathrm{op}}}$.

Finally, we define the *canonical model functor* by

$$\texttt{Conc} := \texttt{Riesz} \circ \texttt{Inc} \circ L^{\infty}.$$

*Remark 2.4.* There is a more direct way to define the canonical model functor using the Stone and Loomis–Sikorski representation theorems where the underlying compact Hausdorff space is the Stonean space associated to a complete Boolean algebra by Stone's representation theorem, e.g., see [31, §9].

*Remark 2.5.* The canonical model has been implicitly introduced in the literature at several occasions, e.g., see [7, 8, 13, 14, 16, 24, 34, 42], where such models are also referred to as 'Stone' or 'Kakutani models'.

For a countable group $\Gamma$, let us denote by $\mathbf{PrbAlg}^{\sigma}_{\Gamma}$ the subcategory of $\mathbf{PrbAlg}_{\Gamma}$ whose systems consist of separable probability algebras. In [19, §2], a 'Cantor model' is associated to each $\mathbf{PrbAlg}^{\sigma}_{\Gamma}$-system which satisfies some functoriality properties, (such Cantor models are formed on closed subspaces of the classical Cantor set). These Cantor models are not canonical, but amenable to classical disintegration of measures. Furstenberg [18, §§5–6] works with 'separable systems' which are concrete probability spaces on which a countable group acts measure-preservingly (modulo null sets) such that the associated $\mathbf{PrbAlg}_{\Gamma}$-systems (with the help of the $\texttt{AlgAbs}$-functor which is described below) are in the subcategory $\mathbf{PrbAlg}^{\sigma}_{\Gamma}$.

The categories $\mathbf{PrbAlg}_{\Gamma}$ and $\mathbf{CHPrb}_{\Gamma}$ are not dual to each other (similarly, as the categories $\mathbf{CvNAlg}^{\tau}_{\Gamma^{\mathrm{op}}}$ and $\mathbf{CC}^*\mathbf{Alg}^{\tau}_{\Gamma^{\mathrm{op}}}$ are not so). However, we have a functor $\texttt{AlgAbs}$ (cf. Figure 4) that associates with every $\mathbf{CHPrb}_{\Gamma}$-system a $\mathbf{PrbAlg}_{\Gamma}$-system such that $\texttt{AlgAbs} \circ \texttt{Conc}$ is naturally isomorphic to the identity functor on $\mathbf{PrbAlg}_{\Gamma}$.

Next we define the functor $\texttt{AlgAbs}$. Let $(X, \mathcal{B}a(X), \mu)$ be a **CHPrb**-space. Denote by $\mathcal{N}_{\mu}$ the $\mu$-null ideal of $\mathcal{B}a(X)$, that is, the collection of $E \in \mathcal{B}a(X)$ such that $\mu(E) = 0$. We can form the quotient Boolean $\sigma$-algebra $X_{\mu} = \mathcal{B}a(X)/\mathcal{N}_{\mu}$ by identifying $E, F \in \mathcal{B}a(X)$ if $E \Delta F \in \mathcal{N}_{\mu}$. Let $\pi_X : \mathcal{B}a(X) \to X_{\mu}$ be the canonical

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$$\mathbf{PrbAlg}_\Gamma \xleftarrow{\texttt{AlgAbs}} \mathbf{CHPrb}_\Gamma$$

FIGURE 4. The `AlgAbs` functor first abstracts away from the set structure of a $\mathbf{CHPrb}_\Gamma$-system and then deletes the null ideal of this abstract system to obtain a $\mathbf{PrbAlg}_\Gamma$-system.

$\mathbf{Bool}_\sigma$-epimorphism. We define a probability measure $\hat{\mu} : X_\mu \to [0, 1]$ by $\hat{\mu}(\pi(E)) := \mu(E)$. We obtain a **PrbAlg**-algebra $(X_\mu, \hat{\mu})$, applying the opposite functor to $(X_\mu, \hat{\mu})$ associates with $(X, \mathcal{B}a(X), \mu)$ a **PrbAlg**-space. If $f : (X, \mathcal{B}a(X), \mu) \to (Y, \mathcal{B}a(Y), \nu)$ is a **CHPrb**-morphism, then the pullback map $f^* : \mathcal{B}a(Y) \to \mathcal{B}a(X)$ defined by $E \mapsto f^*(E) := f^{-1}(E)$ is a $\mathbf{Bool}_\sigma$-morphism. We define the **PrbAlg**-morphism $\texttt{AlgAbs}(f) : (X_\mu, \hat{\mu}) \to (Y_\nu, \hat{\nu})$ by $\pi_Y(E) \mapsto \texttt{AlgAbs}(f)(\pi_Y(E)) := \pi_X(f^*(E))$. The functor `AlgAbs` promotes to a functor from $\mathbf{CHPrb}_\Gamma$ to $\mathbf{PrbAlg}_\Gamma$. The functor `AlgAbs` is not injective on objects since a **CHPrb**-space and its measure-theoretic completion are mapped to the same **PrbAlg**-space. Moreover, $\texttt{Conc}(\texttt{AlgAbs}(X, \mathcal{B}a(X), \mu))$ is typically much larger than $(X, \mathcal{B}a(X), \mu)$.

2.3. *Abstract Lebesgue spaces.* The following notation will be used throughout the remainder of this paper.

Let $\mathcal{X} = (X, \mu, T)$ be a $\mathbf{PrbAlg}_\Gamma$-system. We denote the canonical model $\texttt{Conc}(\mathcal{X})$ by $\tilde{\mathcal{X}}$, its underlying Stonean space by $\tilde{X}$ (cf. Remark 2.4), its Baire–Radon measure by $\tilde{\mu}$, and its $\Gamma$-action by $\tilde{T}$, so that

$$\texttt{Conc}(\mathcal{X}) = \tilde{\mathcal{X}} = (\tilde{X}, \mathcal{B}a(\tilde{X}), \tilde{\mu}, \tilde{T}).$$

If $\pi : \mathcal{X} \to \mathcal{Y}$ is a $\mathbf{PrbAlg}_\Gamma$-factor, then we denote its canonical representation $\texttt{Conc}(\pi)$ by $\tilde{\pi}$.

We can use the canonical model to define integration and Lebesgue spaces for probability algebras (in [16, §§363–366], a direct construction for integration and Lebesgue spaces for measure algebras is provided which is equivalent to our construction via canonical models). For any $0 \leq p \leq \infty$, we define the complex vector space

$$L^p(X) = L^p(X, \mu) := L^p(\tilde{X}, \mathcal{B}a(\tilde{X}), \tilde{\mu}).$$

For any $f \in L^1(X)$, we define

$$\int_X f \, d\mu := \int_{\tilde{X}} f \, d\tilde{\mu}.$$

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a $\mathbf{PrbAlg}_\Gamma$-factor with canonical representation $\tilde{\pi}$. Then $\tilde{\pi}$ gives rise to a Koopman operator $\tilde{\pi}^* : L^p(Y) \to L^p(X)$ defined by pullback as

$$\tilde{\pi}^* f := f \circ \tilde{\pi}.$$

In particular, we can use the Koopman operator to identify $L^p(Y)$ with a (closed) subspace of $L^p(X)$. Henceforth, we will make use of these identifications without further notice.

2.4. *Canonical disintegration and relatively independent products.* A main advantage of the canonical model is that it yields a canonical disintegration of measures from

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which, in turn, several key basic measure-theoretic concepts can be defined. Canonical disintegration of measures relies on an important property which we coined the *strong Lusin property* in [31] (a property well known in the literature, though under a different guise). It implies that we have a $\mathbf{CC^*Alg}$-isomorphism

$$L^\infty(X) \equiv C(\tilde{X}) \tag{1}$$

for any $\mathbf{PrbAlg}$-algebra $(X, \mu)$ with Stone space $\tilde{X}$. The strong Lusin property implies the following disintegration result.

THEOREM 2.6. (Canonical disintegration) [31, §8] *Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be $\mathbf{PrbAlg}_\Gamma$-systems and $\pi : \mathcal{X} \to \mathcal{Y}$ be a $\mathbf{PrbAlg}_\Gamma$-factor. Then there is a unique Radon probability measure $\mu_y$ on $\tilde{X}$ for each $y \in \tilde{Y}$ which depends continuously on $y$ in the vague topology in the sense that $y \mapsto \int_{\tilde{X}} f \, d\mu_y$ is continuous for every $f \in C(\tilde{X})$, and such that*

$$\int_{\tilde{X}} f \tilde{\pi}^* g \, d\tilde{\mu} = \int_{\tilde{Y}} \bigg( \int_{\tilde{X}} f \, d\mu_y \bigg) g \, d\tilde{\nu} \tag{2}$$

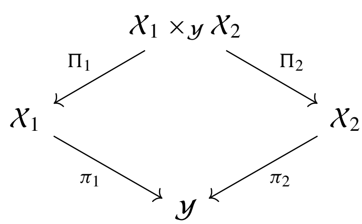
*for all $f \in C(\tilde{X})$, $g \in C(\tilde{Y})$. Furthermore, for each $y \in \tilde{Y}$, $\mu_y$ is supported on the compact set $\tilde{\pi}^{-1}(\{y\})$, in the sense that $\mu_y(E) = 0$ whenever $E$ is a measurable set disjoint from $\tilde{\pi}^{-1}(\{y\})$ (this conclusion does not require the fibers $\tilde{\pi}^{-1}(\{y\})$ to be measurable). Moreover, we have*

$$\mu_{\tilde{S}^\gamma(y)} = \tilde{T}^\gamma_* \mu_y \tag{3}$$

*for all $y \in \tilde{Y}$ and $\gamma \in \Gamma$.*

We can apply the canonical disintegration to construct relatively independent products in the dynamical category $\mathbf{PrbAlg}_\Gamma$ (cf. [14, §2]).

THEOREM 2.7. (Relative products in $\mathbf{PrbAlg}_\Gamma$) [31, §8] *Let $\mathcal{X}_1 = (X_1, \mu_1, T_1)$, $\mathcal{X}_2 = (X_2, \mu_2, T_2)$, and $\mathcal{Y} = (Y, \nu, S)$ be $\mathbf{PrbAlg}_\Gamma$-systems. Suppose that one has $\mathbf{PrbAlg}_\Gamma$-factors $\pi_1 : \mathcal{X}_1 \to \mathcal{Y}$, $\pi_2 : \mathcal{X}_2 \to \mathcal{Y}$. Then there exists a $\mathbf{PrbAlg}_\Gamma$-commutative diagram*




*for some $\mathbf{PrbAlg}_\Gamma$-system $\mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 = (X_1 \times_Y X_2, \mu \times_Y \mu, T \times T)$ and $\mathbf{PrbAlg}_\Gamma$-morphisms $\Pi_1 : \mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 \to \mathcal{X}_1$, $\Pi_2 : \mathcal{X}_1 \times_{\mathcal{Y}} \mathcal{X}_2 \to \mathcal{X}_2$ such that one has*

$$\int_{X_1 \times_Y X_2} f_1 f_2 \, d\mu \times_Y \mu = \int_Y \mathbb{E}(f_1|Y)\mathbb{E}(f_2|Y) \, d\nu \tag{4}$$

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*for all* $f_1 \in L^\infty(X_1)$, $f_2 \in L^\infty(X_2)$, *where we embed* $L^\infty(Y)$ *into* $L^\infty(X_1)$, $L^\infty(X_2)$, *and embed these algebras in turn into* $L^\infty(X_1 \times_Y X_2)$. *Furthermore, the* $\mathbf{Bool}_\sigma$*-algebra of* $X_1 \times_Y X_2$ *is generated by the* $\mathbf{Bool}_\sigma$*-algebras* $X_1$, $X_2$ *(where we identify the latter with* $\sigma$*-complete subalgebras of the former in the obvious way).*

2.5. *The invariant factor functor and the Alaoglu–Birkhoff ergodic theorem.* Since we assume no structure on $\Gamma$ other than it being a group, the only ergodic theorem available to us is the Alaoglu–Birkhoff ergodic theorem. We first introduce the functor $\mathtt{Inv}_\Gamma$ from $\mathbf{PrbAlg}_\Gamma$ into itself.

*Definition 2.8.* (Invariant factor functor)

(i) If $\mathcal{X} = (X, \mu, T)$ is a $\mathbf{PrbAlg}_\Gamma$-system, we define

$$\mathtt{Inv}_\Gamma(\mathcal{X}) = (\mathtt{Inv}_\Gamma(X), \mu_{\mathtt{Inv}_\Gamma(X)}, T_{\mathtt{Inv}_\Gamma(X)})$$

to be the $\mathbf{PrbAlg}_\Gamma$-system with $\mathbf{AbsMbl}$-space

$$\mathtt{Inv}_\Gamma(X) := \{E \in X : T^\gamma(E) = E \text{ for all } \gamma \in \Gamma\},$$

measure

$$\mu_{\mathtt{Inv}_\Gamma(X)}(E) := \mu(E)$$

for all $E \in X$, and action defined by

$$T^\gamma_{\mathtt{Inv}_\Gamma(X)}(E) := T^\gamma(E)$$

for all $E \in \mathtt{Inv}_\Gamma(X)$ and $\gamma \in \Gamma$.

(ii) If $f : \mathcal{X} \to \mathcal{Y}$ is a $\mathbf{PrbAlg}_\Gamma$-factor, we define $\mathtt{Inv}_\Gamma(f) : \mathtt{Inv}_\Gamma(\mathcal{X}) \to \mathtt{Inv}_\Gamma(\mathcal{Y})$ by

$$\mathtt{Inv}_\Gamma(f)(E) := f(E)$$

whenever $E \in \mathtt{Inv}_\Gamma(Y)$.

(iii) A $\mathbf{PrbAlg}_\Gamma$-system $\mathcal{X}$ is said to be *ergodic* if $\mathtt{Inv}_\Gamma(X)$ is the trivial algebra $\{0, 1\}$.

There is a natural epimorphism from the identity functor on $\mathbf{PrbAlg}_\Gamma$ to $\mathtt{Inv}_\Gamma$ that gives a $\mathbf{PrbAlg}_\Gamma$-factor $\pi : \mathcal{X} \to \mathtt{Inv}_\Gamma(\mathcal{X})$ defined by inclusion. Using this factor, one can view $L^2(\mathtt{Inv}_\Gamma(X))$ as a subspace of $L^2(X)$. Each shift $T^\gamma : X \to X$ induces a unitary Koopman operator $(T^\gamma)^* : L^2(X) \to L^2(X)$. For any $f \in L^0(X)$, we denote by $\mathtt{Orb}_\Gamma(f)$ the orbit $\{(T^\gamma)^* f : \gamma \in \Gamma\}$ of $f$.

THEOREM 2.9. ((Alaoglu–Birkhoff abstract ergodic theorem) [1]) *Let* $\mathcal{X} = (X, \mu, T)$ *be a* $\mathbf{PrbAlg}_\Gamma$*-system and* $\mathbb{E}(\cdot|\mathtt{Inv}_\Gamma(X)) : L^2(X) \to L^2(\mathtt{Inv}_\Gamma(X))$ *be the orthogonal projection. Then for any* $f \in L^2(X)$, $\mathbb{E}(f|\mathtt{Inv}_\Gamma(X))$ *is the unique element of minimal norm in the closed convex hull of* $\mathtt{Orb}_\Gamma(f)$.

## 3. *Conditional Hilbert spaces and Hilbert–Schmidt operators*

In this section, we develop some conditional analysis which is used in §4 to study relatively compact extensions. Specifically, we introduce a conditional Hilbert space and its conditional tensor product and establish some useful properties of them. We do not aim to give the most general and systematic treatment of conditional Hilbert space theory, but present, in a self-contained manner, the minimal amount of that theory needed, directed towards its application to the study of extensions of $\mathbf{PrbAlg}_\Gamma$-systems. In several remarks, we discuss and provide references on how to interpret the results in Boolean sheaf topoi, which are addressed to the reader with familiarity to topos theory and categorical logic. The ergodic-theoretically inclined reader can view the theory of conditional Hilbert spaces as a natural extension of the theory of measurable Hilbert bundles (e.g., see [19, §9] and [18, §6]) in situations where separability and countability hypotheses are not satisfied.

We do not assume any dynamical structure in this section and fix a **PrbAlg**-morphism $\pi : (X, \mu) \to (Y, \nu)$ throughout. Recall that we have corresponding embeddings $\pi^* : L^0(Y) \to L^0(X)$ and $\pi^* : L^2(Y) \to L^2(X)$ defined by $\pi^*(f) = f \circ \tilde{\pi}$, where $\tilde{\pi} : \tilde{X} \to \tilde{Y}$ is the canonical representation. We identify $L^0(Y)$ with the closed subspace $\pi^*(L^0(Y))$ of $L^0(X)$ (with respect to convergence in probability) and $L^2(Y)$ with the closed subspace $\pi^*(L^2(Y))$ of $L^2(X)$. We also identify $\mathcal{B}a(\tilde{Y})$ with the sub-$\sigma$-algebra $\tilde{\pi}^*(\mathcal{B}a(\tilde{Y}))$ of $\mathcal{B}a(\tilde{X})$. Throughout, all equalities, inequalities, and inclusions between measurable functions and measurable sets are understood in the almost sure sense. We have the canonical disintegration $(\mu_y)_{y\in\tilde{Y}}$ of $\mu$ over $Y$. This leads to the relatively independent product $(X \times_Y X, \mu \times_Y \mu)$. We can use the canonical **PrbAlg**-morphism $\psi : (X \times_Y X, \mu \times_Y \mu) \to (Y, \nu)$ to identify $L^0(Y)$ with a subspace of $L^0(X \times_Y X)$, and similarly for the respective $L^2$ spaces.

A partition $\mathcal{P}$ of $\tilde{X}$ is called a *Y-partition* if its elements are measurable sets in $\mathcal{B}a(\tilde{Y})$. For technical reasons, we allow the empty set to be an element of a partition. Let $\mathcal{P}$ be a $Y$-partition and $f_E \in L^0(X)$ for each $E \in \mathcal{P}$. Then there is a unique $f \in L^0(X)$ such that $f 1_E = f_E 1_E$ for all $E \in \mathcal{P}$. We denote this unique element $f$ by $\sum_{E\in\mathcal{P}} f_E 1_E$ and call it a *Y-countable concatenation* in $L^0(X)$. Similarly, we define $Y$-countable concatenations in $L^0(X \times_Y X)$. We will be interested in subspaces of $L^0(X)$ and $L^0(X \times_Y X)$ which are closed under $Y$-countable concatenations. More specifically, these subspaces are a conditional version of the classical $L^2$ space and its conditional Hilbert space tensor product which are defined as follows.

*Definition 3.1.* We define the *conditional Hilbert space*

$$\mathtt{L}^2(\mathtt{X|Y}) := \left\{ f \in L^0(X) : \mathbb{E}(|f|^2|Y) = \int_{\tilde{X}} |f|^2 \, d\mu_y < \infty \right\}$$

and the *conditional Hilbert-Schmidt space*

$$\mathtt{HS}(\mathtt{X|Y}) := \left\{ K \in L^0(X \times_Y X) : \mathbb{E}(|K|^2|Y) = \int_{\tilde{X}\times\tilde{X}} |K|^2 \, d\mu_y \times \mu_y < \infty \right\}.$$

One easily verifies that

$$\mathtt{L}^2(\mathtt{X|Y}) = \{ f \in L^0(X) : \text{there exist } g \in L^0(Y), h \in L^2(X) \text{ such that } f = gh \},$$
$$\mathtt{HS}(\mathtt{X|Y}) = \{ f \in L^0(X \times_Y X) : \text{there exist } g \in L^0(Y), h \in L^2(X \times_Y X) \text{ such that } f = gh \}.$$

In particular, both $\mathtt{L}^2(\mathtt{X|Y})$ and $\mathtt{HS}(\mathtt{X|Y})$ are modules over the ring $L^0(Y)$ closed under $Y$-countable concatenations.

*Remark 3.2.* In [44, §2.13], the same notation $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is used for the smaller $L^\infty(Y)$-module

$$\{f \in L^2(X) : \|\mathbb{E}(|f|^2|Y)\|_{L^\infty(Y)} < \infty\}.$$

The crucial difference between the smaller conditional Hilbert space in [44] and our larger conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is that the latter is closed under $Y$-countable concatenations. It is exactly this property that makes $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ amenable to topos-theoretic tools, see the following Remark 3.5.

We define a conditional version of inner product and norm.

*Definition 3.3.* For $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, we define their *conditional inner product* as

$$\langle f, g\rangle_{L^2(X|Y)} := \mathbb{E}(f\bar{g}|Y). \tag{5}$$

For $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, we define its *conditional norm* as

$$\|f\|_{L^2(X|Y)} := \sqrt{\langle f, f\rangle_{L^2(X|Y)}}. \tag{6}$$

The conditional norm $\|\cdot\|_{L^2(X|Y)}$ induces a probabilistic metric on $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ by

$$\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}(f, g) := \int_Y \min(1, \|f - g\|_{L^2(X|Y)})\, d\nu.$$

Similarly, we define $\langle\cdot,\cdot\rangle_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$, $\|\cdot\|_{\mathtt{HS}(X|Y)}$, and $\mathtt{d}_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$.

We verify some basic properties.

PROPOSITION 3.4.

(i) *The conditional inner product $\langle\cdot,\cdot\rangle_{L^2(X|Y)}$ satisfies the following properties.*

(a) *For all $Y$-partitions $\mathcal{P}, \mathcal{P}'$ and families $(f_E)_{E\in\mathcal{P}}, (f_{E'})_{E'\in\mathcal{P}'}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, we have*

$$\left\langle \sum_{E\in\mathcal{P}} f_E 1_E, \sum_{E'\in\mathcal{P}'} f_{E'} 1_{E'}\right\rangle_{L^2(X|Y)} = \sum_{E\cap E'\in\mathcal{P}\cap\mathcal{P}'} \langle f_E, f_{E'}\rangle_{L^2(X|Y)} 1_{E\cap E'},$$

*where $\mathcal{P}\cap\mathcal{P}' := \{E\cap E' : E\in\mathcal{P},\ E'\in\mathcal{P}'\}$.*

(b) *For all $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, it holds that $\langle f, g\rangle_{L^2(X|Y)} = \overline{\langle g, f\rangle_{L^2(X|Y)}}$.*

(c) *For all $f, g, h \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ and $a \in L^0(Y)$, it holds that*

$$\langle af + g, h\rangle_{L^2(X|Y)} = a\langle f, h\rangle_{L^2(X|Y)} + \langle g, h\rangle_{L^2(X|Y)}.$$

(d) *For all $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, it holds that $\langle f, f\rangle_{L^2(X|Y)} \geq 0$.*

*Similarly, for $\langle\cdot,\cdot\rangle_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$.*

(ii) *The conditional norm $\|\cdot\|_{L^2(X|Y)}$ satisfies the following properties.*

(a) *For all $Y$-partitions and families $(f_E)_{E\in\mathcal{P}}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, we have*

$$\left\| \sum_{E\in\mathcal{P}} f_E 1_E \right\|_{L^2(X|Y)} = \sum_{E\in\mathcal{P}} \|f_E\|_{L^2(X|Y)} 1_E.$$

(b) $\|f\|_{L^2(X|Y)} = 0$ *if and only if* $f = 0$.

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(c) *For every* $a \in L^0(Y)$ *and* $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*, we have* $\|af\|_{L^2(X|Y)} = |a|\|f\|_{L^2(X|Y)}$.

(d) *For all* $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*, we have* $\|f+g\|_{L^2(X|Y)} \leq \|f\|_{L^2(X|Y)} + \|g\|_{L^2(X|Y)}$.

*Similarly, for* $\|\cdot\|_{\mathtt{HS}(X|Y)}$.

(iii) *(Conditional Cauchy–Schwarz inequality) For all* $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*, it holds that*

$$|\langle f, g\rangle_{L^2(X|Y)}| \leq \|f\|_{L^2(X|Y)}\|g\|_{L^2(X|Y)}. \tag{7}$$

*Similarly, for* $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$.

(iv) *The space*

$$\mathcal{D} = \left\{ \sum_{i=1}^{n} c_i f_i \otimes g_i : n \in \mathbb{N}, c_i \in \mathbb{C}, f_i, g_i \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y}) \right\},$$

*where* $f \otimes g(x, x') := f(x)g(x')$*, is dense in* $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$ *with respect to the metric* $\mathtt{d}_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$.

(v) *For any* $K \in \mathtt{HS}(\mathtt{X}|\mathtt{Y})$ *and* $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*, define*

$$K *_Y f(x) := \langle K(x, \cdot), \bar{f}\rangle_{L^2(X|Y)}(\tilde{\pi}(x)).$$

*The mapping* $f \mapsto K *_Y f$ *is well defined from* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ *to* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*. Moreover, it holds*

$$\|K *_Y f\|_{L^2(X|Y)} \leq \|K\|_{\mathtt{HS}(X|Y)}\|f\|_{L^2(X|Y)} \tag{8}$$

*for all* $K \in \mathtt{HS}(\mathtt{X}|\mathtt{Y})$ *and* $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*. In particular,* $f \mapsto K *_Y f$ *from* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ *to* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ *is continuous with respect to the metric* $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$.

(vi) *Convergence with respect to* $L^2$*-metric is equivalent to convergence with respect to the metric* $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$ *on* $L^2(X)$*. Similarly for* $\mathtt{d}_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$ *on* $L^2(X \times_Y X)$.

(vii) *The space* $L^\infty(X)$ *(respectively* $L^\infty(X \times_Y X)$*) is dense in* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ *(respectively* $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$*) with respect to the topology induced by the metric* $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$ *(respectively* $\mathtt{d}_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$*).*

*Proof.* The assertions in (i) and (ii) follow by standard properties of conditional expectations.

As for (iii), we follow the proof of the classical Cauchy–Schwarz inequality with some necessary modifications. First suppose that $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ satisfy $\|f\|_{L^2(X|Y)}$, $\|g\|_{L^2(X|Y)}$, $\|f - ag\|_{L^2(X|Y)} > 0$ for all $a \in L^0(Y)$. Then we have

$$0 < \|f - ag\|_{L^2(X|Y)} = \|f\|^2_{L^2(X|Y)} - a\langle f, g\rangle_{L^2(X|Y)} - \bar{a}\langle f, g\rangle_{L^2(X|Y)} + a\bar{a}\|g\|^2_{L^2(X|Y)}$$

for all $a \in L^0(Y)$. Setting

$$a = \frac{|\langle f, g\rangle_{L^2(X|Y)}|^2}{\|f'\|^2_{L^2(X|Y)}\langle f, f'\rangle_{L^2(X|Y)}}$$

and after some elementary algebraic manipulations, we obtain equation (7) in this case.

Second suppose that $f, g \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ satisfy only $\|f\|_{L^2(X|Y)}, \|g\|_{L^2(X|Y)} > 0$. Let

$$\mathcal{E} = \{E \in \mathcal{B}a(\tilde{Y}) : \text{ there exists } a \in L^0(Y), f = ag \text{ on } E\}.$$

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By completeness of $Y$ (see Remark 2.2), there exists a least upper bound $E_{\max}$ for $\mathcal{E}$ (with respect to almost sure inclusion). By the countable chain condition, there is $\mathcal{E}' \subset \mathcal{E}$ countable such that $E_{\max} = \bigcup \mathcal{E}'$. We can assume that $\mathcal{E}'$ is a $Y$-partition of $E_{\max}$. Choose $a \in L^0(Y)$ such that $a = a_E$ on $E$ with $a_E$ such that $f = a_E g$ for all $E \in \mathcal{E}'$. Then $f = ag$ on $E_{\max}$. It holds that $\|f\|_{L^2(X|Y)}, \|g\|_{L^2(X|Y)}, \|f - ag\|_{L^2(X|Y)} > 0$ on $E^c_{\max}$ for all $a \in L^0(Y)$. Thus, equation (7) is satisfied on $E^c_{\max}$ by repeating the first step on $E^c_{\max}$. Since equation (7) is also trivially satisfied on $E_{\max}$, we obtain equation (7) also in this case.

Finally, suppose that $\|f\|_{L^2(X|Y)}$ or $\|g\|_{L^2(X|Y)}$ are equal to 0 on a set of positive measure. Let $E = \{\|f\|_{L^2(X|Y)} = 0\} \cup \{\|g\|_{L^2(X|Y)} = 0\}$. Then equation (7) is satisfied on $E^c$ by repeating the second step on $E^c$. Since equation (7) is also trivially satisfied on $E$, we obtain equation (7) also in this case, and this completes the proof of (iii).

It is enough to prove (iv) for $K \in \mathtt{HS(X|Y)}$ with $K \geq 0$. By the Stone–Weierstraß theorem, the algebra of finite disjoint unions of rectangles $E \times F$ with $E, F \in \mathcal{B}a(\tilde{X})$ generates $\mathcal{B}a(\tilde{X} \times \tilde{X})$. Hence there is a sequence $(K_n)$ in $\mathcal{D}$ such that $K_n \uparrow K\, \tilde{\mu} \times_{\tilde{Y}} \tilde{\mu}$-almost surely as $n \to \infty$. By monotone convergence, $\|K_n - K\|_{\mathtt{HS}(X|Y)} \to 0$ $\tilde{\nu}$-almost surely. Hence $\|K_n - K\|_{\mathtt{HS}(X|Y)} \to 0$ in convergence in $\tilde{\nu}$-measure, and this proves (iv).

As for (v), let $K \in \mathtt{HS(X|Y)}$ and $f \in \mathtt{L}^2\mathtt{(X|Y)}$. By the conditional Cauchy–Schwarz inequality and the Fubini–Tonelli theorem, one obtains

$$\begin{aligned}\int_{\tilde{X}} |K *_Y f|^2(x)\, d\mu_y(x) &= \int_{\tilde{X}} |\langle K(x,\cdot), \bar{f}\rangle_{L^2(X|Y)}(\tilde{\pi}(x))|^2\, d\mu_y(x) \\ &\leq \|f\|^2_{L^2(X|Y)} \int_{\tilde{X}} \|K(x,\cdot)\|^2_{L^2(X|Y)}(\tilde{\pi}(x))\, d\mu_y(x) \\ &= \|K\|^2_{\mathtt{HS}(X|Y)} \|f\|^2_{L^2(X|Y)} < \infty.\end{aligned}$$

This shows that $K *_Y f \in \mathtt{L}^2\mathtt{(X|Y)}$, equation (8), and continuity.

As for (vi), we only prove the claim for $\mathtt{L}^2\mathtt{(X|Y)}$ (the arguments for $\mathtt{HS(X|Y)}$ are identical). For $f \in L^2(X)$, it holds

$$\int_{\tilde{X}} f\, d\tilde{\mu} = \int_{\tilde{Y}} \int_{\tilde{X}} f\, d\mu_y\, d\tilde{\nu}. \tag{9}$$

In particular, we have

$$\|f\|^2_{L^2(X)} = \int_Y \|f\|^2_{L^2(X|Y)}\, d\nu. \tag{10}$$

The assertion follows from the fact that convergence in $L^2$ is equivalent to convergence in measure.

Finally, as for (vii), let $f \in \mathtt{L}^2\mathtt{(X|Y)}$. Define $f_m = f 1_{\|f\|_{L^2(X|Y)} \leq m}$ for $m \geq 1$. Then $\mathtt{d}_{\mathtt{L}^2\mathtt{(X|Y)}}(f, f_m) \to 0$ as $m \to \infty$. By (ii) and equation (9), $f_m \in L^2(X)$. As $L^\infty(X)$ is dense in $L^2(X)$ in $L^2$-topology, we can approximate each $f_m$ by a sequence $(f_{m,n})$ in $L^\infty(X)$ in $L^2$-metric. The claim now follows from equation (9), (vi), and triangle inequality. □

*Remark 3.5.* Let $\mathtt{Sh}(Y)$ be the Grothendieck topos of sheaves on the site $(Y, J)$, where $J$ is the Grothendieck sup-topology on the complete Boolean algebra $Y$. We refer the interested

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reader to [38] for an introduction to Grothendieck sheaf topoi. By the countable chain condition (see Remark 2.2), a Grothendieck basis of the sup-topology $J$ is given by the sets of countable partitions of elements of $Y$. Since the spaces $L^0(Y)$, $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$ are closed under countable $Y$-concatenations, they can be given the structure of a sheaf on the site $(Y, J)$. Since $Y$ is a complete Boolean algebra, the topos $\mathtt{Sh}(Y)$ is Boolean and satisfies the axiom of choice. Thus its internal logic is strong enough to permit an internal mathematical discourse sufficient for the bulk of classical mathematics, see [38, Ch. IV]. In particular, we can interpret the external space $L^0(Y)$ as the internal complex numbers of $\mathtt{Sh}(Y)$. Now Proposition 3.4 shows that the external space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ can be interpreted as an internal complex Hilbert space, and the external space $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$ as its internal Hilbert space tensor product whose elements represent the internal Hilbert–Schmidt operators. Externally, $L^0(Y)$ is a commutative ring and $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ and $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$ are topological modules over $L^0(Y)$ (given the topology induced by $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$, $\mathtt{d}_{\mathtt{HS}(\mathtt{X}|\mathtt{Y})}$, and convergence in probability).

The conditional analysis of this section is an external interpretation and significant simplification of some results in the internal Hilbert space theory of $\mathtt{Sh}(Y)$. For example, Proposition 3.4(iv) and (v) essentially describe the $\mathtt{Sh}(Y)$-spectral theorem of $\mathtt{Sh}(Y)$-Hilbert–Schmidt operators. This interpretation is enabled by conditional set theory (the connection of conditional set theory to Boolean Grothendieck topoi is established in [27], see also [5] for a relation to the akin theory of Boolean-valued models) and conditional analysis as developed in [6, 9, 15], which, initially, were developed independently of topos theory and can be viewed in retrospect as an exploration of the semantics of the internal discourse of Boolean Grothendieck topoi.

We continue to develop some basic conditional analysis of the conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$.

*Definition 3.6.* An *$L^0(Y)$-linear combination* in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is an expression of the form $\sum_{f\in\mathcal{F}} a_f f$ for some finite set $\mathcal{F}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ and $a_f \in L^0(Y)$, $f \in \mathcal{F}$. A subset $\mathcal{M} \subset \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is called a *finitely generated $L^0(Y)$-submodule* if there is a finite collection $\mathcal{F}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ such that each $f \in \mathcal{M}$ can be expressed as $\sum_{f\in\mathcal{F}} a_f f$ for some $(a_f)_{f\in\mathcal{F}}$ in $L^0(Y)$.

PROPOSITION 3.7. (Conditional Gram–Schmidt process) *Let $\mathcal{M}$ be a finitely generated $L^0(Y)$-submodule of $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$. Then there is a finite $Y$-partition $\mathcal{P}$ and for each $E \in \mathcal{P}$, there is a finite set $\mathcal{F}_E$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ satisfying the following properties.*

(i) *There is $E_0 \in \mathcal{P}$ such that $\mathcal{F}_{E_0} = \{0\}$.*
(ii) *$\|f\|_{L^2(X|Y)} = 1$ on $E$ for all $f \in \mathcal{F}_E$ and $E \in \mathcal{P}\setminus\{E_0\}$.*
(iii) *$\langle f, g\rangle_{L^2(X|Y)} = 0$ on $E$ for all distinct $f, g \in \mathcal{F}_E$ and $E \in \mathcal{P}$.*
(iv) *It holds that*

$$\mathcal{M} = \left\{ \sum_{E\in\mathcal{P}} f_E 1_E : \text{for all } E \in \mathcal{P} \text{ there exists } (a_f^E)_{f\in\mathcal{F}_E} \subset L^0(Y) \text{ such that } f_E = \sum_{f\in\mathcal{F}_E} a_f^E f \right\}.$$

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*We call* $\mathtt{cdim}(\mathcal{M}) = \sum_{E\in\mathcal{P}} \#(\mathcal{F}_E)|E$ *the conditional dimension of* $\mathcal{M}$*, where* # *denotes cardinality (note that* $\mathtt{cdim}(\mathcal{M})$ *is a* $\mathcal{B}a(\tilde{Y})$*-measurable random variable).*

*Proof.* Let $f_1, \ldots, f_n \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ be a collection of generators of $\mathcal{M}$. Define

$$h_1 = \frac{f_1}{\|f_1\|_{L^2(X|Y)}} 1_{\|f_1\|_{L^2(X|Y)}>0}.$$

Next suppose we defined $h_1, \ldots, h_k$ with $k \le n-1$. Set

$$g_{k+1} = f_{k+1} - \sum_{i=1}^{k} \langle f_{k+1}, h_i\rangle_{L^2(X|Y)} h_i$$

and define

$$h_{k+1} = \frac{g_{k+1}}{\|g_{k+1}\|_{L^2(X|Y)}} 1_{\|g_{k+1}\|_{L^2(X|Y)}>0}.$$

Denote by $F_i = \{\|g_i\|_{L^2(X|Y)} > 0\}$ and $F_{i+n} = F_i^c$ for all $i = 1, \ldots, n$. Form all finite intersections $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le 2n$ for some $1 \le k \le 2n$. Let $\mathcal{P}_0$ denote the set of all such finite intersections whose measure is positive. Then $\mathcal{P}_0$ forms a $Y$-partition. For $E = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k} \in \mathcal{P}_0$, let $\mathcal{F}_E = \{h_{i_j} : i_j \le n, j = 1, \ldots, k\}$. Let $\mathcal{P}$ consist of those $E \in \mathcal{P}_0$ such that $\mathcal{F}_E$ is not empty. Set $E_0 = (\bigcup_{E\in\mathcal{P}} E)^c$. By construction, $\mathcal{P}$, $(\mathcal{F}_E)_{E\in\mathcal{P}}$, and $E_0$ satisfy properties (i)–(iv). □

LEMMA 3.8. *Let* $\mathcal{M}$ *be a finitely generated* $L^0(Y)$*-submodule of* $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$*. Then* $\mathcal{M}$ *is closed with respect to the metric* $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$.

*Proof.* Let $\mathcal{P}$ and $(\mathcal{F}_E)_{E\in\mathcal{P}}$ satisfy properties (i)–(iv) in Lemma 3.7 for $\mathcal{M}$. Let $(f_n)$ be a sequence in $\mathcal{M}$ such that $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}(f_n, f) \to 0$ as $n \to \infty$ for some $f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})$. Each $f_n$ is of the form

$$f_n = \sum_{E\in\mathcal{P}} \bigg( \sum_{f_E\in\mathcal{F}_E} a^n_{f_E} f_E \bigg)|E$$

for some $(a^n_{f_E})_{f\in\mathcal{F}_E}$ for each $E \in \mathcal{P}$. For arbitrary $n, m$, by Proposition 3.4(ii) and Proposition 3.7(ii) and (iii),

$$\|f_n - f_m\|_{L^2(X|Y)} = \sum_{E\in\mathcal{P}} \bigg( \sum_{f_E\in\mathcal{F}_E} |a^n_{f_E} - a^m_{f_E}| \bigg)|E. \tag{11}$$

Since $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}(f_n, f_m) \to 0$ as $n, m \to \infty$ by hypothesis, the claim follows by applying [35, Lemma 4.6] to (11). □

## 4. *Relatively compact extensions*

In this section, we establish various characterizations of relatively compact extensions of $\mathbf{PrbAlg}_\Gamma$-systems which are collected in the following.

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THEOREM 4.1. *Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be* $\mathbf{PrbAlg}_\Gamma$*-systems, and $\pi : \mathcal{X} \to \mathcal{Y}$ be a* $\mathbf{PrbAlg}_\Gamma$*-extension. Then the following are equivalent.*

(i) *The conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is the $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$-closure of*

$$\{K *_Y f : K \in \mathtt{HS}(\mathtt{X}|\mathtt{Y})\ \Gamma\text{-invariant},\ f \in \mathtt{L}^2(\mathtt{X}|\mathtt{Y})\}.$$

(ii) *The conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is the $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$-closure of the union of all its finitely generated and $\Gamma$-invariant $L^0(Y)$-submodules.*

(iii) *There exists a dense set $\mathcal{G}$ with respect to the metric $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$ such that for all $f \in \mathcal{G}$ and every $\varepsilon > 0$, there is a finite set $\mathcal{F}$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ such that for all $\gamma \in \Gamma$,*

$$\min_{g\in\mathcal{F}} \|(T^\gamma)^* f - g\|_{L^2(X|Y)} < \varepsilon.$$

(i)′ *The classical Hilbert space $L^2(X)$ is the $L^2$-closure of*

$$\{K *_Y f : K \in L^\infty(X \times_Y X)\ \Gamma\text{-invariant},\ f \in L^2(X)\}.$$

(ii)′ *The classical Hilbert space $L^2(X)$ is the $L^2$-closure of the union of all its closed, $\Gamma$-invariant, and finitely generated $L^\infty(Y)$-submodules.*

(iii)′ *There exists a dense set $\mathcal{H}$ in $L^2(X)$ such that for all $f \in \mathcal{H}$ and every $\varepsilon > 0$, there is a finite set $\mathcal{F}$ in $L^2(X)$ such that for all $\gamma \in \Gamma$,*

$$\min_{g\in\mathcal{F}} \|(T^\gamma)^* f - g\|_{L^2(X|Y)} < \varepsilon.$$

*A* $\mathbf{PrbAlg}_\Gamma$*-morphism $\pi$ fulfilling one (and therefore all) of the above six properties is called a* relatively compact $\mathbf{PrbAlg}_\Gamma$-extension.

*Remark 4.2.* Property (ii)′ is used by Zimmer in [**47**, **48**] under the name of ‘relatively discrete spectrum’. Ellis [**14**] used a variant of property (ii)′ to generalize some parts of Zimmer’s work, see §5. Properties (i)′ and (iii)′ appear in Furstenberg [**18**, §6]. We prove that (i)′, (ii)′, and (iii)′ are equivalent and relate them to their analogs in the larger conditional Hilbert spaces $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ and $\mathtt{HS}(\mathtt{X}|\mathtt{Y})$. Our proof of Theorem 4.1 combines ideas from the countable theory as in [**18**, §6] with the conditional analysis developed in §3 and Appendix A.

*Remark 4.3.* Using conditional integration theory as developed in [**28**] (in particular, a conditional version of the Carathéodory extension theorem and the results in [**28**, §4]), one can construct, in a slightly tedious process, an internal $\mathtt{Sh}(Y)$-system from an external extension $\pi : \mathcal{X} \to \mathcal{Y}$. Then the properties (i)–(iii) in Theorem 4.1 characterizing relatively compact extensions or systems of relative discrete spectrum can be viewed as an external interpretation of characterizations of internal compact systems or internal systems of discrete spectrum.

LEMMA 4.4. *Assertion (i)′ implies assertion (ii)′ in Theorem 4.1.*

*Proof.* To prove (ii)′ from (i)′, it suffices to show that the union of the closed and finitely generated $\Gamma$-invariant $L^\infty(Y)$-submodules of the range of $K *_Y : L^2(X) \to L^2(X)$

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is dense in the range of $K*_Y$ for any $\Gamma$-invariant $K \in L^\infty(X \times_Y X)$. (We thank the anonymous referee for pointing out a gap in a previous version of this argument. The current arguments are borrowed from a forthcoming work joint with Pieter Spaas, where we establish a non-commutative analog of several results in (uncountable) Furstenberg–Zimmer structure theory.) By splitting into real and imaginary parts and then further into positive and negative parts, we can assume that $K \in L^\infty(X \times_Y X)$ is non-negative. Note that $K*_Y : L^2(X) \to L^2(X)$ is then a bounded and positive operator (cf. Lemma A.5) and thus its spectrum is compact and contained in $[0, +\infty)$. For $\varepsilon > 0$, consider the spectral projection $P_\varepsilon := 1_{[\varepsilon, \|K\|_{L^\infty(X\times_Y X)}]}(K*_Y)$, where we apply Borel functional calculus to $K*_Y$ as an operator from $L^2(X)$ to $L^2(X)$ to define $P_\varepsilon$. Since $P_\varepsilon$ arises as a limit of polynomials in $K*_Y$ in the strong operator topology, $P_\varepsilon$ is $\Gamma$-invariant. Thus its range $\mathcal{H}_\varepsilon := P_\varepsilon(L^2(X))$ is a $\Gamma$-invariant $L^\infty(Y)$-submodule of $L^2(X)$ (this follows from that $L^\infty(Y)$-linearity is preserved when passing to strong operator limits as well). By standard properties of the Borel functional calculus, $P_\varepsilon \circ K = K \circ P_\varepsilon = K|_{\mathcal{H}_\varepsilon} \geq \varepsilon \mathrm{Id}_{\mathcal{H}_\varepsilon}$ (where the inequality is understood in the sense of operators on the Hilbert space $\mathcal{H}_\varepsilon$). Now we can apply Proposition A.6 and obtain that $\mathcal{H}_\varepsilon$ is necessarily finitely generated as an $L^\infty(Y)$-module. Letting $\varepsilon \to 0$, we get that the increasing union of the finitely generated $\Gamma$-invariant $L^\infty(Y)$-submodules of the range of $K$ is dense in the range of $K$. This concludes the proof of (i)′ ⇒ (ii)′. □

*Remark 4.5.* An alternative proof of the implication (i)′ ⇒ (ii)′, or better to say of (i) ⇒ (ii) (from which then (i)′ ⇒ (ii)′ could be easily deduced), would be to invoke a conditional spectral theorem for the conditional Hilbert–Schmidt operator $K*_Y : \mathtt{L}^2(\mathtt{X|Y}) \to \mathtt{L}^2(\mathtt{X|Y})$ for any $K \in \mathtt{HS}(\mathtt{X|Y})$. The internal logic of $\mathtt{Sh}(Y)$ admits such a $\mathtt{Sh}(Y)$-spectral theorem, whose proof one could then interpret externally by hand for the conditional Hilbert–Schmidt operator $K*_Y : \mathtt{L}^2(\mathtt{X|Y}) \to \mathtt{L}^2(\mathtt{X|Y})$ (in some sense, this is carried out in a topological setting in [**11**, Part I, §4], see the comments at the end of this part). We can also compare with the relative spectral theorem for measurable Hilbert bundles proved in [**19**, Ch. 9, §3] in the countable complexity setting. We manage to avoid using explicitly a relative/conditional spectral theorem by combining Borel functional calculus with a mix of conditional analysis and classical arguments as needed in the proof of Proposition A.6 in Appendix A.

LEMMA 4.6. *Assertion (ii) implies assertion (iii) in Theorem 4.1.*

*Proof.* Let $\mathfrak{A}$ be the union of finitely generated and $\Gamma$-invariant $L^0(Y)$-submodules and let $\mathcal{G} = \{f \in \mathfrak{A} : \text{there exists } M > 0 \text{such that} \|f\|_{L^2(X|Y)} < M\}$. By assumption and Lemma 3.8, $\mathcal{G}$ is dense in $\mathtt{L}^2(\mathtt{X|Y})$ with respect to the metric $\mathtt{d}_{\mathtt{L}^2(\mathtt{X|Y})}$.

Fix $f \in \mathcal{G}$ and $\varepsilon > 0$. Then there is a finitely generated and $\Gamma$-invariant $L^0(Y)$-submodule $\mathcal{M}$ such that $f \in \mathcal{M}$. By Proposition 3.7, there are a finite $Y$-partition $\mathcal{P}$ and for each $E \in \mathcal{P}$, a finite family $\mathcal{F}_E$ such that for all $f \in \mathcal{M}$, we have the representation $f = \sum_{E\in\mathcal{P}} f_E 1_E$, where $f_E = \sum_{g\in\mathcal{F}_E} a_g g$ is an $L^0(Y)$-linear combination for each $E \in \mathcal{P}$. Since $\|(T^\gamma)^* f\|_{L^2(X|Y)} = (T^\gamma)^* \|f\|_{L^2(X|Y)} < M$ for all $\gamma \in \Gamma$, the orbit $\mathtt{Orb}_\Gamma(f)$ is bounded with respect to the conditional norm $\|\cdot\|_{L^2(X|Y)}$. It follows from

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Proposition 3.4(ii) and Proposition 3.7 that for all $h \in \mathtt{Orb}_\Gamma(f)$,

$$\|h\|_{L^2(X|Y)} = \Big\| \sum_{E\in\mathcal{P}} h_E 1_E \Big\|_{L^2(X|Y)} = \sum_{E\in\mathcal{P}} \|h_E\|_{L^2(X|Y)} 1_E$$
$$= \sum_{E\in\mathcal{P}} \bigg( \sum_{g\in\mathcal{F}_E} |a_g^h|^2 \bigg)^{1/2} 1_E < M.$$

For each $E \in \mathcal{P}$, choose $c_1^E, \ldots, c_{j_E}^E \in \mathbb{C}^{\#(\mathcal{F}_E)}$ such that the closed ball in $\mathbb{C}^{\#(\mathcal{F}_E)}$ with radius $M$ and center the origin is covered by the union of the open balls of radius $\varepsilon$ and center $c_k^E$, $k = 1, \ldots, j_E$. Denote by $C_k^E$ the constant measurable vector in $L^0(Y)^{\#(\mathcal{F}_E)}$ with value $c_k^E$. For $f \in \mathcal{F}_E$, let $C_k^E(f)$ denote a component of the vector $C_k^E$. Put

$$\mathcal{L}_E = \bigg\{ \sum_{f\in\mathcal{F}_E} C_k^E(f) f : k = 1, \ldots, j_E \bigg\}$$

and define

$$\mathcal{L} = \bigg\{ \sum_{E\in\mathcal{P}} g_E|E : g_E \in \mathcal{L}_E \text{ for each } E \in \mathcal{P} \bigg\}.$$

Clearly, $\mathcal{L}$ is finite and by construction, it holds that

$$\min_{g\in\mathcal{L}} \|g - f\|_{L^2(X|Y)} < \varepsilon.$$

□

*Remark 4.7.* The idea of the previous proof is a conditional version of the Heine–Borel theorem (see [9, Theorem 4.6]) or the internal Heine–Borel theorem of $\mathtt{Sh}(Y)$.

LEMMA 4.8. *Assertion (i) is equivalent to assertion (i)′ in Theorem 4.1.*

*Proof.* The equivalence is a consequence of Proposition 3.4(v)–(vii). □

LEMMA 4.9. *Assertion (ii) is equivalent to assertion (ii)′ in Theorem 4.1.*

*Proof.* We first show that (ii) implies (ii)′. Let $\mathcal{M}$ be an finitely generated and $\Gamma$-invariant $L^0(Y)$-submodule of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$. For $\mathcal{M}$, choose a finite $Y$-partition $\mathcal{P}$ and for each $E \in \mathcal{P}$, a finite set $\mathcal{F}_E$ satisfying the properties in Proposition 3.7. Let

$$\mathcal{L} = \bigg\{ \sum_{E\in\mathcal{P}} g_E|E : g_E \in \mathcal{F}_E \text{ each } E \in \mathcal{P} \bigg\}.$$

By property (ii) in Proposition 3.7, $\|g\|_{L^2(X|Y)} \le 1$ for each $g \in \mathcal{L}$. Thus by equation (10), $\|g\|_{L^2(X)} \le 1$ for all $g \in \mathcal{L}$. Let $\mathcal{M}'$ be the $L^\infty(Y)$-submodule of $L^2(X)$ generated by $\mathcal{L}$. Then $\mathcal{M}'$ is $\Gamma$-invariant since $\mathcal{M}$ is $\Gamma$-invariant, and by Proposition 3.4(vi), $\mathcal{M}'$ is also $L^2$-closed. It remains to show that the union of all $\mathcal{M}'$, where $\mathcal{M}$ is a finitely generated and $\Gamma$-invariant $L^0(Y)$-submodule of $\mathrm{L}^2(\mathrm{X}|\mathrm{Y})$, is dense in $L^2(X)$. However, this follows from Proposition 3.4(v) and (vii).

Let us show that (ii)′ implies (ii). Let $\mathcal{M}'$ be an $L^\infty(Y)$-finitely generated, $\Gamma$-invariant, and closed submodule of $L^2(X)$. Let $\mathcal{L}$ be a finite set of generators of $\mathcal{M}'$. Let

$\mathcal{M} = \{\sum_{g\in\mathcal{L}} a_g g : a_g \in L^0(Y) \text{ for all } g \in \mathcal{L}\}$. By construction and Proposition 3.4(ii), $\mathcal{M}$ is a finitely generated and $\Gamma$-invariant $L^0(Y)$-submodule of $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$. By Proposition 3.4(vii), the union of all such $\mathcal{M}$ is dense in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ with respect to $\mathtt{d}_{\mathtt{L}^2(\mathtt{X}|\mathtt{Y})}$. □

LEMMA 4.10. *Assertion (iii) is equivalent to assertion (iii)′ in Theorem 4.1.*

*Proof.* Let $\mathcal{G}$ be a dense set in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ satisfying the property in (iii). Let $f \in \mathcal{G}$ and $M > 0$. By Proposition 3.4(ii), $f 1_{\|f\|_{L^2(X|Y)}\le M}$ is also an element of $\mathcal{G}$. Then $\mathcal{R} = \{f 1_{\|f\|_{L^2(X|Y)}\le M} : f \in \mathcal{S}, M > 0\}$ is a subset of $L^2(X)$ by Proposition 3.4(v) and dense in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$. Now (iii)′ follows from Proposition 3.4(vi). The converse direction is an immediate consequence of Proposition 3.4(vii). □

We conclude the proof of Theorem 4.1 by establishing the following.

LEMMA 4.11. *Assertion (iii)′ implies assertion (i)′ in Theorem 4.1.*

*Proof.* We show that if $f \in \mathcal{H}$ is orthogonal to all functions of the form $K *_Y g$, where $K$ ranges over $\Gamma$-invariant functions in $L^\infty(X \times_Y X)$ and $g$ ranges over $L^2(X)$, then $f = 0$. Let $K$ be the unique element of minimal norm in the closed convex hull of $\mathtt{Orb}_\Gamma(f \otimes \bar{f})$ in $L^2(X \times_Y X)$, which is $\Gamma$-invariant by Theorem 2.9. Set $K_M = K 1_{|K|\le M}$ for $M \ge 0$. By hypothesis, $f$ is orthogonal to $K_M *_Y \bar{f}$. We can rewrite this, using the disintegration $(\mu_y)_{y\in\tilde{Y}}$ of $\tilde{\mu}$, as

$$\begin{aligned} 0 &= \int_{\tilde{X}} f(x)\bigg(\int_{\tilde{X}} K_M(x,x')\bar{f}(x')\,d\mu_{\tilde{\pi}(x)}(x')\bigg)\,d\tilde{\mu}(x) \\ &= \int_{\tilde{Y}}\int_{\tilde{X}} f(x)\bigg(\int_{\tilde{X}} K_M(x,x')\bar{f}(x')\,d\mu_y(x')\bigg)\,d\mu_y(x)\,d\tilde{\nu}(y) \\ &= \int_{\tilde{X}\times\tilde{X}} f\otimes\bar{f} K_M\,d\tilde{\mu}\times_{\tilde{Y}}\tilde{\mu}. \end{aligned}$$

Thus $f \otimes \bar{f}$ is orthogonal to $K_M$ in $L^2(X \times_Y X)$. As $K_M$ is $\Gamma$-invariant, $(T^\gamma \times T^\gamma)^* f \otimes \bar{f}$ is orthogonal to $K_M$ for all $\gamma \in \Gamma$ as well. Thus $K$ must be orthogonal to all $K_M$ which implies that $K_M = 0$ for all $M \ge 0$, and therefore also $K = 0$.

Now choose a finite set $\mathcal{F}$ such that the property in (iii)′ is satisfied for $f$ and an arbitrary $\varepsilon > 0$. Let $(K_n)$ be a sequence in the convex hull of $\mathtt{Orb}_\Gamma(f \otimes \bar{f})$ such that $\lim \|K_n\|_{L^2(X\times_Y X)} \to 0$. By Cauchy–Schwarz, we also have $\langle K_n, g \otimes \bar{g}\rangle_{L^2(X\times_Y X)} \to 0$ for all $g \in \mathcal{F}$. Expanding the inner product and choosing $n$ large enough, we can always obtain a $\gamma \in \Gamma$ such that

$$\sum_{g\in\mathcal{F}} \|\langle (T^\gamma)^* f, g\rangle_{L^2(X|Y)}\|_{L^2(Y)}$$

is arbitrarily small. Thus for any $\delta > 0$, we can find a set $E \in \mathcal{B}a(\tilde{Y})$ with $\tilde{\nu}(E^c) < \delta$ such that

$$|\langle (T^\gamma)^* f, g\rangle_{L^2(X|Y)}| < \varepsilon \quad \text{on } E$$

for all $g \in \mathcal{F}$ for some $\gamma$. By (iii)$'$, we also have for all $\gamma \in \Gamma$,

$$\min_{g \in \mathcal{F}} \|(T^{\gamma})^* f - g\|_{L^2(X|Y)} < \varepsilon.$$

By triangle inequality,

$$\|f\|_{L^2(X|Y)} 1_{T^{-\gamma}(E)} = \|(T^{\gamma})^* f\|_{L^2(X|Y)} 1_E < 3\varepsilon$$

for some $\gamma$. Since $\varepsilon$, $\delta$ were chosen arbitrarily, it follows that $f = 0$, and this finishes the proof of Theorem 4.1. □

## 5. *The uncountable Mackey–Zimmer theorem and isometric extensions*

Under the hypothesis of ergodicity, relatively compact extensions are isomorphic to homogeneous skew-product extensions. This characterization is a relative version of the Halmos–von Neumann theorem (proved for $\mathbb{Z}$-actions by Halmos and von Neumann [**25**], then generalized by Mackey [**37**] to second-countable locally compact group actions, see [**13**, Theorem 17.5] for an uncountable Halmos–von Neumann theorem for ergodic Markov semi-group actions). The relative version of the Halmos–von Neumann theorem was obtained by Zimmer [**47**] for actions of second-countable locally compact groups on standard Borel spaces. In this section, we extend Zimmer's theorem to systems of uncountable complexity. The key ingredient in the proof is the Mackey–Zimmer theorem classifying ergodic homogeneous extensions. We use the uncountable Mackey–Zimmer theorem [**32**] established by Tao and the author. We point out that similar results to those in this section were obtained by Ellis [**14**, §5], though with different methods. In short, the methods of [**14**] rely on topological dynamics and the theory of Banach bundles of spaces of continuous functions which is akin to the methods used in [**11**], whereas our methods are ergodic-theoretic and based off measure theory and topos theory.

We start by recalling the uncountable Mackey–Zimmer theorem [**32**]. Let $K$ be a compact Hausdorff group and $(Y, \nu)$ a **PrbAlg**-space. We denote by $\mathtt{Cond}_Y(K)$ the set of all **AbsMbl**-morphisms from $Y$ to $\mathcal{B}a(K)$. Let $\widetilde{\mathtt{Cond}}_Y(K)$ denote the set of continuous functions from $\tilde{Y}$ to $K$. It is a remarkable property of the canonical model that we have the isomorphism

$$\mathtt{Cond}_Y(K) \equiv \widetilde{\mathtt{Cond}}_Y(K), \tag{12}$$

see [**31**, Proposition 7.9] for a proof (here, isomorphism is understood in the sense of sets, that is, there is a bijection between the sets $\mathtt{Cond}_Y(K)$ and $\widetilde{\mathtt{Cond}}_Y(K)$). The set $\widetilde{\mathtt{Cond}}_T(K)$ has the structure of a group defining the group law in a pointwise way. Therefore, also $\mathtt{Cond}_Y(K)$ has a group structure. We introduce $\mathbf{PrbAlg}_\Gamma$-cocycles, homogeneous skew-products, and extensions.

*Definition 5.1.* Let $\mathcal{Y} = (Y, \nu, S)$ be a $\mathbf{PrbAlg}_\Gamma$-system, $K$ a compact Hausdorff group, and $L \leq K$ a closed subgroup.

(i) A $K$-valued $\mathbf{PrbAlg}_\Gamma$-*cocycle* is a family $\rho = (\rho_\gamma)_{\gamma \in \Gamma}$ of elements $\rho_\gamma \in \mathtt{Cond}_Y(K)$ satisfying the $\mathbf{PrbAlg}_\Gamma$-cocycle property

$$\rho_{\gamma\gamma'} = (\rho_\gamma \circ S^{\gamma'})\rho_{\gamma'}$$

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for all $\gamma, \gamma' \in \Gamma$. Any $K$-valued $\mathbf{PrbAlg}_\Gamma$-cocycle $\rho = (\rho_\gamma)_{\gamma\in\Gamma}$ has a canonical representation $\tilde{\rho} = (\tilde{\rho}_\gamma)_{\gamma\in\Gamma}$, where $\tilde{\rho}_\gamma \in \widetilde{\mathtt{Cond}_Y}(K)$ is defined by the isomorphism in equation (12) such that $\tilde{\rho} = (\tilde{\rho}_\gamma)_{\gamma\in\Gamma}$ is a $\mathbf{CHPrb}_\Gamma$-cocycle on $\tilde{Y}$.

(ii) Let $\rho = (\rho_\gamma)_{\gamma\in\Gamma}$ be a $\mathbf{PrbAlg}_\Gamma$-cocycle with canonical representation $\tilde{\rho} = (\tilde{\rho}_\gamma)_{\gamma\in\Gamma}$. We denote by $\tilde{Y} \rtimes_{\tilde{\rho}} K/L$ the $\mathbf{CHPrb}_\Gamma$-*homogeneous skew-product* defined by the following data:
- the homogeneous space $K/L$ is equipped with Baire $\sigma$-algebra and Haar measure;
- we equip $\tilde{Y} \times K/L$ with the product Baire probability measure;
- the $\mathbf{CHPrb}_\Gamma$-action $T : \Gamma \to \mathrm{Aut}(\tilde{Y} \times K/L)$ is defined by

$$T^\gamma(y, kL) = (S^\gamma y, \tilde{\rho}_\gamma(y)kL)$$

for all $(y, kL) \in \tilde{Y} \times K/L$ and $\gamma \in \Gamma$.

We define the $\mathbf{PrbAlg}_\Gamma$-*homogeneous skew-product* $\mathcal{Y} \rtimes_\rho K/L$ to be the $\mathbf{PrbAlg}_\Gamma$-system

$$\mathtt{AlgAbs}(\tilde{Y} \rtimes_{\tilde{\rho}} K/L).$$

(iii) A $\mathbf{PrbAlg}_\Gamma$-*homogeneous extension* of $\mathcal{Y}$ by $K/L$ is a tuple $(\mathcal{X}, \pi, \theta, \rho)$, where $\mathcal{X} = (X, \mu, T)$ is a $\mathbf{PrbAlg}_\Gamma$-system, $\pi : (X, \mu, T) \to (Y, \nu, S)$ is a $\mathbf{PrbAlg}_\Gamma$-extension, $\theta \in \mathtt{Cond}_X(K/L)$ is a vertical coordinate such that $\pi, \theta$ jointly generate the $\mathbf{Bool}_\sigma$-algebra $X$, and $\rho = (\rho_\gamma)_{\gamma\in\Gamma}$ is a $\mathbf{PrbAlg}_\Gamma$-cocycle such that

$$\theta \circ T^\gamma = (\rho_\gamma \circ \pi)\theta$$

for all $\gamma \in \Gamma$ using the natural action of $\mathtt{Cond}_X(K)$ on $\mathtt{Cond}_X(K/L)$.

THEOREM 5.2. (Uncountable Mackey–Zimmer [32]) *Let $\mathcal{Y} = (Y, \nu, S)$ be an ergodic $\mathbf{PrbAlg}_\Gamma$-system and $K$ be a compact Hausdorff group. Every ergodic $\mathbf{PrbAlg}_\Gamma$- homogeneous extension $\mathcal{X}$ of $\mathcal{Y}$ by $K/L$ for some compact subgroup $L$ of $K$ is isomorphic in $\mathbf{PrbAlg}_\Gamma$ to a $\mathbf{PrbAlg}_\Gamma$-homogeneous skew-product $\mathcal{Y} \rtimes_\rho H/M$ for some compact subgroup $H$ of $K$, some compact subgroup $M$ of $H$, and some $H$-valued $\mathbf{PrbAlg}_\Gamma$-cocycle $\rho$.*

We apply the uncountable Mackey–Zimmer theorem to establish a geometric characterization of relatively compact extensions.

THEOREM 5.3. *If $\pi : (X, \mu, T) \to (Y, \nu, S)$ is an ergodic relatively compact $\mathbf{PrbAlg}_\Gamma$-extension, then $(X, \mu, T)$ is $\mathbf{PrbAlg}_\Gamma$-isomorphic to a homogeneous skew-product extension $\mathcal{Y} \rtimes_\rho H/M$ for a compact Hausdorff group $H$, a closed subgroup $M$ of $H$, and an $H$-valued $\mathbf{PrbAlg}_\Gamma$-cocycle $\rho$.*

In the proof, we need some properties about cocycles with values in quotient and product spaces which are recorded in the following lemma.

LEMMA 5.4. *Let $A$ be an index set and for every $\alpha \in A$, let $K_\alpha$ be a compact Hausdorff group and $L_\alpha$ a closed subgroup of $K_\alpha$. Let $(Y, \nu)$ be a $\mathbf{PrbAlg}$-space. Then the following* *identities hold:*

$$\mathtt{Cond}_Y\bigg(\prod_{\alpha\in A} K_\alpha\bigg) = \prod_{\alpha\in A} \mathtt{Cond}_Y(K_\alpha)$$

$$\mathtt{Cond}_Y\bigg(\prod_{\alpha\in A}(K_\alpha/L_\alpha)\bigg) = \mathtt{Cond}_Y\bigg(\prod_{\alpha\in A} K_\alpha/\prod_{\alpha\in A} L_\alpha\bigg).$$

*Proof.* The first identity is proved in [33, Corollary 3.5]. Routine verification shows that

$$\prod_{\alpha\in A}(K_\alpha/L_\alpha) \equiv \prod_{\alpha\in A} K_\alpha/\prod_{\alpha\in A} L_\alpha$$

is a **CH**-isomorphism. This proves the second identity. □

*Proof of Theorem 5.3.* Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a relatively compact $\mathbf{PrbAlg}_\Gamma$-extension of ergodic systems. Let $(\mathcal{M}_\alpha)_{\alpha\in A}$ be the family of all $\Gamma$-invariant closed finitely generated $L^\infty(Y)$-submodules of $L^2(X)$. By Theorem 4.1(ii)$'$, $L^2(X)$ is the $L^2$ closure of $\bigcup_{\alpha\in A} \mathcal{M}_\alpha$. Fix $\alpha \in A$. Since $\mathcal{M}_\alpha$ is $\Gamma$-invariant, the map $\mathtt{cdim}(\mathcal{M}_\alpha)$ is also $\Gamma$-invariant and by ergodicity, $\mathtt{cdim}(\mathcal{M}_\alpha) \equiv d_\alpha$ for some integer $d_\alpha \geq 1$. By Proposition 3.7, we find $f_1^\alpha, \ldots, f_{d_\alpha}^\alpha \in \mathcal{M}_\alpha$ such that $\|f_i^\alpha\|_{L^2(X|Y)} = 1$ for all $i = 1, \ldots, d_\alpha$ and $\langle f_i^\alpha, f_j^\alpha\rangle_{L^2(X|Y)} = 0$ whenever $i \neq j$. In fact, we can choose $f_1^\alpha, \ldots, f_{d_\alpha}^\alpha$ such that all $f_i^\alpha$ take values in the unit circle $\mathbb{S}$. For any $\gamma \in \Gamma$, then $(\tilde{T}^\gamma)^* f_1^\alpha, \ldots, (\tilde{T}^\gamma)^* f_{d_\alpha}^\alpha$ also satisfies $\|(\tilde{T}^\gamma)^* f_i^\alpha\|_{L^2(X|Y)} = 1$ for all $i = 1, \ldots, d_\alpha$ and $\langle(\tilde{T}^\gamma)^* f_i^\alpha, (\tilde{T}^\gamma)^* f_j^\alpha\rangle_{L^2(X|Y)} = 0$ whenever $i \neq j$. Let

$$\Lambda_\gamma^\alpha = \begin{bmatrix} a_{11}^\gamma & \cdots & a_{1d_\alpha}^\gamma \\ \vdots & \ddots & \vdots \\ a_{d_\alpha 1}^\gamma & \cdots & a_{d_\alpha d_\alpha}^\gamma \end{bmatrix}$$

where $a_{ij}^\gamma \in L^\infty(Y)$ is such that $f_i^\alpha = \sum_{j=1}^{d_\alpha} a_{ij}^\gamma (\tilde{T}^\gamma)^* f_j^\alpha$ for all $i = 1, \ldots, d_\alpha$. We can identify $\Lambda_\gamma^\alpha$ with an element of the function space $L^0(Y \to \mathbb{U}(d_\alpha))$ of equivalence classes of $\mathbb{U}(d_\alpha)$-valued random variables on $\tilde{Y}$, where $\mathbb{U}(d_\alpha)$ is the group of $d_\alpha \times d_\alpha$ unitary matrices. We obtain the identity $F^\alpha = \Lambda_\gamma^\alpha F_\gamma^\alpha$, where $F^\alpha = (f_1^\alpha, \ldots, f_{d_\alpha}^\alpha)$ and $F_\gamma^\alpha = ((\tilde{T}^\gamma)^* f_1^\alpha, \ldots, (\tilde{T}^\gamma)^* f_{d_\alpha}^\alpha)$. Let $\rho_\gamma^\alpha := ((\Lambda_\gamma^\alpha)^*)^{\mathrm{op}}$, which is an element of $\mathtt{Cond}_Y(\mathbb{U}(d_\alpha))$. We thus obtain a $\mathbb{U}(d_\alpha)$-valued $\mathbf{PrbAlg}_\Gamma$-cocycle $\rho^\alpha = (\rho_\gamma^\alpha)_{\gamma\in\Gamma}$ on $(Y, \nu, S)$. Define the **AbsMbl**-morphism $\theta_\alpha : Y \to \mathcal{B}a(\mathbb{S}^{2d_\alpha - 1})$ by $\theta_\alpha := (((1/\sqrt{d_\alpha})F^\alpha)^*)^{\mathrm{op}}$.

Now define $\rho = (\rho_\gamma)_{\gamma\in\Gamma}$ by $\rho_\gamma = (\rho_\gamma^\alpha)_{\alpha\in A}, \theta = (\theta_\alpha)_{\alpha\in A}$, $K = \prod_{\alpha\in A} \mathbb{U}(d_\alpha)$, and $L = \prod_{\alpha\in A} \mathbb{U}(d_\alpha - 1)$. By Lemma 5.4, $\theta \in \mathtt{Cond}_Y(K/L)$ is a vertical coordinate such that $\pi, \theta$ jointly generate the $\mathbf{Bool}_\sigma$-algebra $X$, and $\rho = (\rho_\gamma)_{\gamma\in\Gamma}$ is a $\mathbf{PrbAlg}_\Gamma$-cocycle such that $\theta \circ T^\gamma = (\rho_\gamma \circ \pi)\theta$. The claim follows from Theorem 5.2. □

*Remark 5.5.* Conversely, every (not necessarily ergodic) $\mathbf{PrbAlg}_\Gamma$-homogeneous skew-product $\mathcal{Y} \rtimes_\rho K/L$ is a relatively compact $\mathbf{PrbAlg}_\Gamma$-extension of $\mathcal{Y}$. It is sufficient to show this for $\mathbf{PrbAlg}_\Gamma$-group skew-products $\mathcal{Y} \rtimes_\rho K$. Indeed, if $\mathcal{Y} \rtimes_\rho K \to \mathcal{Y}$ is a relatively compact $\mathbf{PrbAlg}_\Gamma$-extension, then so is $\mathcal{Y} \rtimes_\rho K/L \to \mathcal{Y}$ since we have a $\mathbf{PrbAlg}_\Gamma$-extension $\mathcal{Y} \rtimes_\rho K \to \mathcal{Y} \rtimes_\rho K/L$.

Now $L^2(Y \rtimes_\rho K)$ is the Hilbert space tensor product of $L^2(Y)$ and $L^2(K)$. By the Peter–Weyl theorem, $L^2(K)$ is the closure of the union of finite-dimensional $K$-invariant subspaces. For any $g$ in a finite-dimensional $K$-invariant subspace of $L^2(K)$ and $f \in L^\infty(Y)$, the tensor $f \otimes g$ lies in a $\Gamma$-invariant finitely generated $L^\infty(Y)$-submodule of $L^2(Y \rtimes_\rho K)$. The claim follows from Theorem 4.1.

*Remark 5.6.* We call $(\mathcal{Y}_\alpha)_{\alpha \le \beta}$ in Theorem 1.1 the *Furstenberg tower* associated to the $\mathbf{PrbAlg}_\Gamma$-system $\mathcal{X}$ and call $\beta$ the *length* of this tower. If $\mathcal{X}$ is a $\mathbf{PrbAlg}_\Gamma$-system of countable complexity, then the length of the Furstenberg tower of $\mathcal{X}$ must be a countable ordinal since $L^2(X)$ is separable. Beleznay and Foreman [3] proved that any countable ordinal can be realized as the length of the Furstenberg tower of a $\mathbf{PrbAlg}_\Gamma$-system of countable complexity. By a transfinite recursion, we can build out of $\mathbf{PrbAlg}_\Gamma$-group skew-products $\mathbf{PrbAlg}_\Gamma$-systems of Furstenberg tower of arbitrary length (cf. Remark 5.5). We leave the details to the interested reader.

*Remark 5.7.* Austin establishes a relatively ergodic version of Theorem 5.3 for systems of countable complexity in [2, §4]. We think that a significantly heavier application of the topos-theoretic machinery will prove a generalization of Austin's result to systems of arbitrary (not necessarily countable) complexity. We hope to work out the details of this in future work.

## 6. *Dichotomy*

Given any $\mathbf{PrbAlg}_\Gamma$-system $X$, it is well known that $L^2(X)$ is the orthogonal sum of the compact and weakly mixing factors of $X$, e.g., see Ch. 1.7 of the lecture notes [40] by Peterson. This is an ergodic theoretic manifestation of the dichotomy between structure and randomness. In this section, we establish a relative or conditional version of this dichotomy for $\mathbf{PrbAlg}_\Gamma$-extensions. Let us first define relatively weakly mixing functions and extensions.

*Definition 6.1.* Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be $\mathbf{PrbAlg}_\Gamma$-systems. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a $\mathbf{PrbAlg}_\Gamma$-extension. A function $f \in L^2(X)$ with $\mathbb{E}(f|Y) = 0$ is said to be *relatively weakly mixing* if for all $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\|\langle (T^\gamma)^* f, f\rangle_{L^2(X|Y)}\|_{L^2(Y)} < \varepsilon.$$

Moreover, we say that $\mathcal{X}$ is a *relatively weakly mixing* $\mathbf{PrbAlg}_\Gamma$*-extension* of $\mathcal{Y}$ if all $f \in L^2(X)$ with $\mathbb{E}(f|Y) = 0$ are relatively weakly mixing. Finally, we denote by $\mathtt{WM}_{X|Y}$ the subspace of all relatively mixing functions in $L^2(X)$.

We denote by $\mathtt{AP}_{X|Y}$ the $L^2$ closure of all $\Gamma$-invariant finitely generated $L^\infty(Y)$-submodules of $L^2(X)$. The relative or conditional version of the classical dichotomy between the compact and weakly mixing parts of a system is

$$L^2(X) = \mathtt{AP}_{X|Y} \oplus \mathtt{WM}_{X|Y}, \tag{13}$$

where the sum is in the sense of the Hilbert space $L^2(X)$. This relative dichotomy is well understood for systems of countable measure-theoretic complexity, e.g., see [44, §2.14]

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for an exposition in the case of $\mathbb{Z}$-actions. Theorem 1.2 extends this relative dichotomy to systems of uncountable complexity which we restate next for the convenience of the reader.

THEOREM 6.2. (Uncountable relative dichotomy) *Let $\mathcal{X} = (X, \mu, T)$ and $\mathcal{Y} = (Y, \nu, S)$ be* $\mathbf{PrbAlg}_\Gamma$*-systems and $\pi : \mathcal{X} \to \mathcal{Y}$ be a* $\mathbf{PrbAlg}_\Gamma$*-extension. Exactly one of the following statements is true.*

(i) *$\pi : \mathcal{X} \to \mathcal{Y}$ is a relatively weakly mixing* $\mathbf{PrbAlg}_\Gamma$*-extension.*

(ii) *There exist a* $\mathbf{PrbAlg}_\Gamma$*-system $\mathcal{Z} = (Z, \lambda, R)$ and* $\mathbf{PrbAlg}_\Gamma$*-extensions $\phi : \mathcal{X} \to \mathcal{Z}$ and $\psi : \mathcal{Z} \to \mathcal{Y}$ such that $\psi$ is a non-trivial relatively compact* $\mathbf{PrbAlg}_\Gamma$*-extension.*

*Proof.* If (i) is false, there are $f \in L^2(X)$ with $\mathbb{E}(f|Y) = 0$ and $\varepsilon > 0$ such that for all $\gamma \in \Gamma$,

$$\langle (T^\gamma)^* f \otimes \bar{f}, f \otimes \bar{f} \rangle_{L^2(X \times_Y X)} = \| \langle (T^\gamma)^* f, f \rangle_{L^2(X|Y)} \|^2_{L^2(Y)} \geq \varepsilon. \tag{14}$$

Let $K$ be the unique element of minimal norm in the closed convex hull of $\mathtt{Orb}_\Gamma(f \otimes \bar{f})$. By Theorem 2.9, $K$ is $\Gamma$-invariant, and by equation (14), $K$ is non-trivial. Since $\mathbb{E}(f|Y) = 0$, there must exist $g \in L^2(X)$ such that $K *_Y g \in L^2(X) \backslash L^2(Y)$. Let $\mathcal{A} = \{K *_Y g \in L^\infty(X) : g \in L^2(X)\}$. By Theorem 4.1, every $K *_Y g \in \mathcal{A}$ is contained in a closed $\Gamma$-invariant finitely generated $L^\infty(Y)$-submodule of $L^2(X)$. It is not difficult to check that $\mathcal{A}$ is a $\Gamma$-invariant vector subspace of $L^\infty(X)$ closed under complex conjugation and multiplication. Thus $\mathcal{A}$ has the structure of a $\mathbf{CvNAlg}^\tau_{\Gamma^{\mathrm{op}}}$-system such that $\Phi : \mathcal{A} \to L^\infty(X, \mu, T)$ and $\Psi : L^\infty(Y, \nu, S) \to \mathcal{A}$ are $\mathbf{CvNAlg}^\tau_{\Gamma^{\mathrm{op}}}$-factors with $\Psi$ non-trivial. Then $(Z, \lambda, R) := \mathtt{Proj}(\mathcal{A})$, $\phi := \mathtt{Proj}(\Phi)$, and $\psi := \mathtt{Proj}(\Psi)$ satisfy (ii).

Now let $f$ be relatively weakly mixing and $g \in \mathtt{AP}_{X|Y}$. Without loss of generality, we may assume that $f$ is bounded and $g$ is an element of an invariant finitely generated $L^\infty(Y)$-submodule $\mathcal{M}$ of $L^2(X)$. For any $\gamma \in \Gamma$ and $h \in \mathcal{M}$, we have

$$\begin{aligned}
&\| \langle f, g \rangle_{L^2(X|Y)} \|_{L^2(Y)} \\
&\quad = \| \langle (T^\gamma)^* f, (T^\gamma)^* g \rangle_{L^2(X|Y)} \|_{L^2(Y)} \\
&\quad \leq \| \langle (T^\gamma)^* f, (T^\gamma)^* g - h \rangle_{L^2(X|Y)} \|_{L^2(Y)} + \| \langle (T^\gamma)^* f, h \rangle_{L^2(X|Y)} \|_{L^2(Y)} \\
&\quad \leq \| \| (T^\gamma)^* f \|_{L^2(X|Y)} \| (T^\gamma)^* g - h \|_{L^2(X|Y)} \|_{L^2(Y)} + \| \langle (T^\gamma)^* f, h \rangle_{L^2(X|Y)} \|_{L^2(Y)} \\
&\quad \leq \| f \|_{L^\infty(X)} \| \| (T^\gamma)^* g - h \|_{L^2(X|Y)} \|_{L^2(Y)} + \| \langle (T^\gamma)^* f, h \rangle_{L^2(X|Y)} \|_{L^2(Y)},
\end{aligned}$$

where the second inequality follows from the conditional Cauchy–Schwarz inequality. For any $\delta > 0$, we can first choose $h$ and then $\gamma$ such that the last term is $< \delta$ uniformly in $\gamma$. Hence $\| \langle f, g \rangle_{L^2(X|Y)} \|_{L^2(Y)} = 0$, and thus $\langle f, g \rangle_{L^2(X)} = 0$. It remains to show that if $f \notin \mathtt{WM}_{X|Y}$, then $f$ must correlate with some $g \in \mathtt{AP}_{X|Y}$, but this follows by a similar argument as in the first part of the proof. □

*Remark 6.3.* A $\mathbf{PrbAlg}_\Gamma$-extension $\pi : \mathcal{X} \to \mathcal{Y}$ is said to be relatively ergodic if $\mathtt{Inv}_\Gamma(\mathcal{X}) = \mathtt{Inv}_\Gamma(\mathcal{Y})$ (after identifying $\mathtt{Inv}_\Gamma(\mathcal{Y})$ with a subalgebra of $\mathcal{X}$). Using the relative dichotomy in equation (13), it is not difficult to show that a $\mathbf{PrbAlg}_\Gamma$-extension $\pi : \mathcal{X} \to \mathcal{Y}$ is relatively weakly mixing if and only if $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a relatively ergodic extension of $\mathcal{Y}$.

*Remark 6.4.* The relative dichotomy in equation (13) can be extended to a conditional Halmos–von Neumann-type decomposition of the conditional Hilbert space $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$, which is the external interpretation of the internal Halmos–von Neumann theorem of the topos $\mathtt{Sh}(Y)$. In fact, this provides an alternative route to the proof of Theorem 6.2. To be more precise, construct from a $\mathbf{PrbAlg}_\Gamma$-extension $\pi : \mathcal{X} \to \mathcal{Y}$ an internal measure-preserving dynamical system in $\mathtt{Sh}(Y)$ (see Remark 4.3), apply to that internal system the internal Halmos–von Neumann theorem, and then externally interpret the resulting internal orthogonal decomposition. This reasoning, which basically reduces to a translation process between the external and internal universes, is enabled by a logical Boolean transfer principle powerful enough to yield a Halmos–von Neumann theorem in the internal logic of the topos $\mathtt{Sh}(Y)$. In fact, the above proof of Theorem 6.2 is a reflection and validation of this transfer principle.

We can use Theorem 6.2 to prove our main result, Theorem 1.1, which we now restate.

THEOREM 6.5. *Let $\mathcal{X} = (X, \mu, T)$ be a $\mathbf{PrbAlg}_\Gamma$-system. Then there is an ordinal number $\beta$ such that for each ordinal number $\alpha \leq \beta$, there are a $\mathbf{PrbAlg}_\Gamma$-system $\mathcal{Y}_\alpha = (Y_\alpha, \nu_\alpha, S_\alpha)$ and a $\mathbf{PrbAlg}_\Gamma$-factor $\pi_\alpha : \mathcal{X} \to \mathcal{Y}_\alpha$, and for every successor ordinal number $\alpha + 1 \leq \beta$, there is a $\mathbf{PrbAlg}_\Gamma$-factor $\pi_{\alpha+1,\alpha} : \mathcal{Y}_{\alpha+1} \to \mathcal{Y}_\alpha$ with the following properties:*

(i) *$Y_0$ is the trivial algebra;*
(ii) *$\pi_{\alpha+1,\alpha} : \mathcal{Y}_{\alpha+1} \to \mathcal{Y}_\alpha$ is a non-trivial relatively compact $\mathbf{PrbAlg}_\Gamma$-extension;*
(iii) *$\mathcal{Y}_\alpha$ is the inverse limit of the systems $\mathcal{Y}_\eta$, $\eta < \alpha$ for every limit ordinal number $\alpha \leq \beta$, in the sense that $Y_\alpha$ is generated by $\bigcup_{\eta<\alpha} Y_\eta$ as a $\sigma$-complete Boolean algebra;*
(iv) *$\pi_\beta : \mathcal{X} \to \mathcal{Y}_\beta$ is a relatively weakly mixing $\mathbf{PrbAlg}_\Gamma$-extension.*

*Proof.* Let $\mathcal{Y}_0$ be the trivial $\mathbf{PrbAlg}_\Gamma$-system, and let $\pi_0 : \mathcal{X} \to \mathcal{Y}_0$ be the corresponding $\mathbf{PrbAlg}_\Gamma$-factor map. By Theorem 6.2, $\pi_0 : \mathcal{X} \to \mathcal{Y}_0$ is either a relatively weakly mixing $\mathbf{PrbAlg}_\Gamma$-extension, in which case we set $\beta = 0$, or there are a $\mathbf{PrbAlg}_\Gamma$-system $\mathcal{Z}$ and $\mathbf{PrbAlg}_\Gamma$-extensions $\phi : \mathcal{X} \to \mathcal{Z}$ and $\psi : \mathcal{Z} \to \mathcal{Y}_0$ such that $\psi : \mathcal{Z} \to \mathcal{Y}_0$ is a non-trivial relatively compact $\mathbf{PrbAlg}_\Gamma$-extension of $\mathcal{Y}_0$. Then set $\mathcal{Y}_1 := \mathtt{AP}_{\mathtt{X}|\mathtt{Y}_0}$, let $\pi_1 : \mathcal{X} \to \mathcal{Y}_1$ and $\pi_{1,0} : \mathcal{Y}_1 \to \mathcal{Y}_0$ be the corresponding $\mathbf{PrbAlg}_\Gamma$-factors, and repeat the previous step with $\mathcal{X}$ and $\mathcal{Y}_1$ instead of $\mathcal{Y}_0$ using Theorem 6.2. This process may end here if $\pi_1 : \mathcal{X} \to \mathcal{Y}_1$ is a relatively weakly mixing $\mathbf{PrbAlg}_\Gamma$-extension and we set $\beta = 1$, or we may need to continue and repeat the previous step again. Now repeating the previous steps in a transfinite recursion, while passing to inverse limits at limit ordinals, this process will eventually terminate at an ordinal number $\beta$. □

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## A. *Appendix. A conditional spectral analysis*

In this appendix, we collect some auxiliary results needed in the proof of Lemma 4.4. The notation is the same as in §3.

*Definition A.1.* (Conditional orthonormal sets) A set $M$ in $\mathtt{L}^2(\mathtt{X}|\mathtt{Y})$ is said to be *conditional orthonormal* if and only if $\langle f, g\rangle_{L^2(X|Y)} = 0$ for all $f, g \in M$ and $\langle f, f\rangle_{L^2(X|Y)} = 1_E$ for some $E \in Y$ for all $f \in M$ (where $E$ may depend on $f$).

LEMMA A.2. *Let $K \in L^\infty(X \times_Y X)$, and let $M$ be a countable conditional orthonormal set. Then $\sum_{f\in M} \|K *_Y f\|^2_{L^2(X)} < \infty$.*

*Proof.* Applying Bessel's inequality pointwise almost everywhere, we have

$$\sum_{f\in M} |(K *_Y f)(x)|^2 \le \|K(x,\cdot)\|^2_{L^2(X|Y)}(\tilde{\pi}(x)) < C,$$

where $|K|^2$ is bounded by the constant $C > 0$. The claim follows from the monotone convergence theorem. □

*Remark A.3.* The assumption that $M$ has countable cardinality, which guarantees measurability when applying a pointwise argument, can be removed from Lemma A.2 as follows. Instead of using Bessel's inequality pointwise, we could apply a conditional version of Bessel's inequality for arbitrary conditional orthonormal sets. This conditional version of Bessel's inequality is an interpretation of the internal $\mathtt{Sh}(Y)$-Bessel's inequality. However, we do not need this stronger version of the conditional Bessel's inequality here.

We need a slight modification of some results in [36, Appendix C].

LEMMA A.4. (Cf. [36, Lemma C.10]) *Let $f \in L^2(X)$ be non-zero. Then for every $\varepsilon > 0$, there is $g \in L^\infty(Y)f$ such that $\langle g, g\rangle_{L^2(X|Y)} = 1_E$ for some non-zero $E \in Y$ and $\|f - \langle f, g\rangle_{L^2(X|Y)} g\|_{L^2(X)} < \varepsilon$.*

*Proof.* Let $E_N = \{y \in \tilde{Y} : 0 < \|f\|_{L^2(X|Y)}(y) \le N\}$ for an integer $N \ge 1$. Next, choose $N$ large enough such that $\|f 1_{E_N^c}\|_{L^2(X)} = \|\|f\|_{L^2(X|Y)} 1_{E_N^c}\|_{L^2(Y)} < \varepsilon$. Put $E = E_N$ and define $h \in L^\infty(Y)$ by $1_E h^2 \langle f, f\rangle_{L^2(X|Y)} = 1_E$. Then $g := 1_E h f$ has the desired properties. □

LEMMA A.5. (Cf. [36, Lemma C.14]) *Let $K \in L^\infty(X \times_Y X)$, and let $\mathcal{H}$ be a closed $L^\infty(Y)$-submodule of $L^2(X)$. Then $K *_Y : \mathcal{H} \to \mathcal{H}$ is a bounded $L^\infty(Y)$-linear operator. Moreover, its classical operator norm can be computed as*

$$\begin{aligned}\|K *_Y\| &:= \sup\{\|K *_Y f\|_{L^2(X)} \mid f \in \mathcal{H},\ \|f\|_{L^2(X)} \le 1\}\\ &= \sup\{\|K *_Y f\|_{L^2(X)} \mid f \in \mathcal{H},\ \langle f, f\rangle_{L^2(X|Y)} = 1_E \text{ for some } E \in Y\}.\end{aligned}$$

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*Proof.* By definition, $K*_Y$ is $L^\infty(Y)$-linear (meaning that $K *_Y (gf) = gK_*(f)$ for all $f \in \mathcal{H}$, $g \in L^\infty(Y)$). By equation (8), the operator norm $\|K *_Y\| \le \|K\|_{L^\infty(X\times_Y X)}$, and thus $K*_Y : \mathcal{H} \to \mathcal{H}$ is a bounded operator.

Now fix $\varepsilon > 0$. Then there is a non-zero $f \in \mathcal{H}$ with $\|f\|_{L^2(X)} \le 1$ such that $\|K *_Y f\|_{L^2(X)} \ge \|K *_Y\| - \varepsilon/2$. For any $\delta > 0$, using Lemma A.2, we can find $g \in L^\infty(Y)f$ such that $\langle g, g\rangle_{L^2(X|Y)} = 1_E$ for some non-zero $E \in Y$ and $\|f - \langle f, g\rangle_{L^2(X|Y)} g\|_{L^2(X)} < \delta$. We obtain

$$\|K *_Y f - K *_Y (\langle f, g\rangle_{L^2(X|Y)} g)\|_{L^2(X)} \le \|K *_Y\| \|f - \langle f, g\rangle_{L^2(X|Y)} g\|_{L^2(X)} < \delta \|K *_Y\|.$$

Upon taking $\delta$ small enough, we can thus ensure that

$$\|K *_Y (\langle f, g\rangle_{L^2(X|Y)} g)\|_{L^2(X)} \ge \|K *_Y f\|_{L^2(X)} - \frac{\varepsilon}{2}.$$

Meanwhile, we obtain from the conditional Cauchy–Schwarz inequality (7):

$$\begin{aligned}\|\langle f, g\rangle_{L^2(X|Y)}\|^2_{L^2(Y)} &= \int_Y |\langle f, g\rangle_{L^2(X|Y)}|^2 \, d\nu \\ &\le \int_Y \|f\|^2_{L^2(X|Y)} \|g\|^2_{L^2(X|Y)} \, d\nu \\ &\le 1.\end{aligned}$$

Putting things together, we get

$$\begin{aligned}\|K *_Y g\|_{L^2(X)} &\ge \|\langle f, g\rangle_{L^2(X|Y)}\|_{L^2(Y)} \|K *_Y g\|_{L^2(X)} \\ &\ge \|K *_Y (\langle f, g\rangle_{L^2(X|Y)} g)\|_{L^2(X)} \ge \|K *_Y\| - \varepsilon,\end{aligned}$$

which finishes the proof of the lemma. □

The following proposition will serve us as a substitute of a conditional version of the spectral theorem (when combined with spectral projections).

PROPOSITION A.6. (Cf. [36, Proposition C.15]) *Let $K \in L^\infty(X \times_Y X)$, let $\mathcal{H}_0$ be a closed $L^\infty(Y)$-submodule of $L^2(X)$, and let $\varepsilon > 0$ be arbitrary. Then there is a finite conditional orthonormal set $M \subset \mathcal{H}_0$ such that $\|K *_Y f\|_{L^2(X)} \le \varepsilon \|f\|_{L^2(X)}$ for every $f \in \mathcal{H}_0$ which is conditionally orthogonal to M.*

*Proof.* For notational convenience, we denote $U := K*_Y$. Note that $U : \mathcal{H}_0 \to \mathcal{H}_0$ is a bounded and $L^\infty(Y)$-linear operator. Fix $\varepsilon > 0$. By the previous Lemma A.5, we can find $f_1 \in \mathcal{H}_0$ such that $\|U(f_1)\|_{L^2(X)} \ge \|U\| - \varepsilon/2$ and $\langle f_1, f_1\rangle_{L^2(X|Y)} = 1_{E_1}$ for some non-zero $E_1 \in Y$. Let $\mathcal{H}_1 = \{g \in \mathcal{H}_0 : \langle f_1, g\rangle_{L^2(X|Y)} = 0\}$, and note that $\mathcal{H}_1$ is a closed $L^\infty(Y)$-submodule of $\mathcal{H}_0$. We consider the restriction $U_1$ of $U$ to $\mathcal{H}_1$. We can apply again Lemma A.5 now to $U_1 : \mathcal{H}_1 \to \mathcal{H}_1$ to find $f_2 \in \mathcal{H}_1$ such that $\|U_1(f_2)\|_{L^2(X)} \ge \|U_1\| - \varepsilon/2$ and $\langle f_2, f_2\rangle_{L^2(X|Y)} = 1_{E_2}$ for some non-zero $E_2 \in Y$. Continuing in this way, we can construct a conditional orthonormal sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $\|U_n(f_{n+1})\|_{L^2(X)} \ge \|U_n\| - \varepsilon/2$, where $\mathcal{H}_n = \{g \in \mathcal{H}_0 : \langle f_k, g\rangle_{L^2(X|Y)} = 0$, for all $k = 1, \ldots, n\}$ and $f_{n+1} \in \mathcal{H}_n$. By Lemma A.2, we can take $n$ large enough so that $\|U(f_n)\|_{L^2(X)} \le \varepsilon/2$, and thus $\|U_n\| \le \varepsilon$. Then $M = \{f_1, \ldots, f_n\}$ satisfies the desired proof. □

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## REFERENCES


**[1]** L. Alaoglu and G. Birkhoff. General ergodic theorems. *Ann. of Math. (2)* **41** (1940), 293–309.

**[2]** T. Austin. Extensions of probability-preserving systems by measurably-varying homogeneous spaces and applications. *Fund. Math.* **210**(2) (2010), 133–206.

**[3]** F. Beleznay and M. Foreman. The complexity of the collection of measure-distal transformations. *Ergod. Th. & Dynam. Sys.* **16**(5) (1996), 929–962.

**[4]** V. Bergelson, R. McCutcheon and Q. Zhang. A Roth theorem for amenable groups. *Amer. J. Math.* **119** (1997), 1173–1211.

**[5]** M. Carl and A. Jamneshan. A transfer principle for second-order arithmetic, and applications. *J. Log. Anal.* **10**(8) (2018), 1–32.

**[6]** P. Cheridito, M. Kupper and N. Vogelpoth. Conditional analysis on $\mathbb{R}^d$. *Set Optimization and Applications (Springer Proceedings in Mathematics & Statistics, 151)*. Eds. A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff and C. Schrage. Springer, Berlin–Heidelberg, 2015, pp. 179–211.

**[7]** R. Derndinger, G. Palm and R. Nagel. Ergodic theory in the perspective of functional analysis. unpublished.

**[8]** J. Dieudonné. Sur le théorème de Lebesgue–Nikodym. III. *Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.)* **23** (1948), 25–53.

**[9]** S. Drapeau, A. Jamneshan, M. Karliczek and M. Kupper. The algebra of conditional sets and the concepts of conditional topology and compactness. *J. Math. Anal. Appl.* **437**(1) (2016), 561–589.

**[10]** P. Durcik, R. Greenfeld, A. Iseli, A. Jamneshan and J. Madrid. An uncountable ergodic Roth theorem and applications. *Preprint*, 2022, arXiv:2101.00685.

**[11]** N. Edeko, M. Haase and H. Kreidler. A decomposition theorem for unitary group representations on Kaplansky–Hilbert modules and the Furstenberg–Zimmer structure theorem. *Preprint*, 2021, arXiv:2104.04865.

**[12]** M. Einsiedler and T. Ward. *Ergodic Theory: With a View Towards Number Theory (Graduate Texts in Mathematics, 259)*. Springer, London, 2010.

**[13]** T. Eisner, B. Farkas, M. Haase and R. Nagel. *Operator Theoretic Aspects of Ergodic Theory (Graduate Texts in Mathematics, 272)*. Springer, Cham, 2015.

**[14]** R. Ellis. Topological dynamics and ergodic theory. *Ergod. Th. & Dynam. Sys.* **7**(1) (1987), 25–47.

**[15]** D. Filipović, M. Kupper and N. Vogelpoth. Separation and duality in locally $L^0$-convex modules. *J. Funct. Anal.* **256** (2009), 3996–4029.

**[16]** D. H. Fremlin. *Measure Theory*. Vol. 3. Torres Fremlin, Colchester, 2004. Measure algebras, Corrected second printing of the 2002 original.

**[17]** H. Furstenberg. Ergodic behaviour of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Anal. Math.* **31** (1977), 204–256.

**[18]** H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory (Princeton Legacy Library, 119)*. Princeton University Press, Princeton, NJ, 2014.

**[19]** E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2015.

**[20]** E. Glasner, B. Tsirelson and B. Weiss. The automorphism group of the Gaussian measure cannot act pointwise. *Israel J. Math.* **148** (2005), 305–329.

**[21]** W. T. Gowers. A new proof of Szemerédi's theorem for arithmetic progressions of length four. *Geom. Funct. Anal.* **8**(3) (1998), 529–551.

**[22]** W. T. Gowers. A new proof of Szemerédi's theorem. *Geom. Funct. Anal.* **11**(3) (2001), 465–588.

**[23]** B. Green, T. Tao and T. Ziegler. An inverse theorem for the Gowers $U^{s+1}[N]$-norm. *Ann. of Math. (2)* **176**(2) (2012), 1231–1372.

**[24]** P. R. Halmos. On a theorem of Dieudonné. *Proc. Natl. Acad. Sci. USA* **35** (1949), 38–42.

**[25]** P. R. Halmos and J. von Neumann. Operator methods in classical mechanics. II. *Ann. of Math. (2)* **43** (1942), 332–350.

**[26]** B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. of Math. (2)* **161**(1) (2005), 397–488.

**[27]** A. Jamneshan. Sheaves and conditional sets. *Preprint*, 2019, arXiv:1403.5405.

**[28]** A. Jamneshan, M. Kupper and M. Streckfuß. Measures and integrals in conditional set theory. *Set-Valued Var. Anal.* **26**(4) (2018), 947–973.

**[29]** A. Jamneshan, O. Shalom and T. Tao. The structure of arbitrary Conze–Lesigne systems. *Preprint*, 2021, arXiv:2112.02056.

**[30]** A. Jamneshan and T. Tao. The inverse theorem for the $U^3$ Gowers uniformity norm on arbitrary finite abelian groups: Fourier-analytic and ergodic approaches. *Preprint*, 2021, arXiv:2112.13759.


https://doi.org/10.1017/etds.2022.43 Published online by Cambridge University Press

**[31]** A. Jamneshan and T. Tao. Foundational aspects of uncountable measure theory: Gelfand duality, Riesz representation, canonical models, and canonical disintegration. *Preprint*, 2022, arXiv:2010.00681.
**[32]** A. Jamneshan and T. Tao. An uncountable Mackey–Zimmer theorem. *Studia Math.*, to appear.
**[33]** A. Jamneshan and T. Tao. An uncountable Moore–Schmidt theorem. *Ergod. Th. & Dynam. Sys.* doi:10.1017/etds.2022.36. Published online 11 May 2022.
**[34]** S. Kakutani. Concrete representation of abstract $(L)$-spaces and the mean ergodic theorem. *Ann. of Math. (2)* **42** (1941), 523–537.
**[35]** O. Kallenberg. *Foundations of Modern Probability (Probability and Its Applications)*, 2nd edn. Springer-Verlag, New York, 2002.
**[36]** D. Kerr and H. Li. *Ergodic Theory: Independence and Dichotomies (Springer Monographs in Mathematics)*. Springer, Cham, 2016.
**[37]** G. W. Mackey. Ergodic transformation groups with a pure point spectrum. *Illinois J. Math.* **8** (1964), 593–600.
**[38]** S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory (Universitext)*. Springer, New York, 2012.
**[39]** J. T. Moore and S. Solecki. A Boolean action of $C(M, U(1))$ without a spatial model and a re-examination of the Cameron–Martin theorem. *J. Funct. Anal.* **263**(10) (2012), 3224–3234.
**[40]** J. Peterson. Ergodic theory. *Lecture Notes*, 2011, available at https://math.vanderbilt.edu/peters10/index2.html.
**[41]** S. Popa. Cocycle and orbit equivalence superrigidity for malleable actions of $w$-rigid groups. *Invent. Math.* **170**(2) (2007), 243–295.
**[42]** I. E. Segal. Equivalences of measure spaces. *Amer. J. Math.* **73** (1951), 275–313.
**[43]** E. Szemerédi. On sets of integers containing no $k$ elements in arithmetic progression. *Acta Arith.* **27** (1975), 199–245.
**[44]** T. Tao. *Poincaré's Legacies: Pages from Year Two of a Mathematical Blog*. Vol. I. American Mathematical Society, Providence, RI, 2009.
**[45]** T. Tao. Real analysis relative to a finite measure space. *Blog article*, July 15, 2014, available at https://terrytao.wordpress.com.
**[46]** T. Ziegler. Universal characteristic factors and Furstenberg averages. *J. Amer. Math. Soc.* **20** (2007), 53–97.
**[47]** R. J. Zimmer. Ergodic actions with a generalized spectrum. *Illinois J. Math.* **20**(4) (1976), 555–588.
**[48]** R. J. Zimmer. Extension of ergodic group actions. *Illinois J. Math.* **20**(3) (1976), 373–409.