J. Austral. Math. Soc. (*Series A*) 39 (1985), 159–168

A POSTERIORI ERROR ESTIMATES FOR ELLIPTIC BOUNDARY-VALUE PROBLEMS

W. L. CHAN

(Received 30 September 1979)

Communicated by G. R. Morris

Abstract

A posteriori error estimates for a class of elliptic unilateral boundary value problems are obtained for functions satisfying only part of the boundary conditions. Next, we give an alternative approach to the *a posteriori* error estimates for self-adjoint boundary value problems developed by Aubin and Burchard. Further, we are able to construct an alternative estimate with mild additional assumptions. An example of a linear differential operator of order $2k$ is given.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 35 J 30, 35 J 35, 35 J 40; secondary 65 N 15.

1. Introduction

In considering a mixed boundary value problem, Aubin (1972) has obtained a posteriori error estimates for functions satisfying only the natural boundary conditions and the forced boundary conditions respectively through the introduction of conjugate problems. In this note, we consider a similar problem for unilaterial boundary value problems. *A posteriori* error estimates are obtained for functions satisfying partially the given boundary conditions. Error bounds are also obtained in terms of another type of function, following a result obtained by Aubin (1972), page 287. Our method, however, does not need the introduction of any conjugate problems.

https://doi.org/10.1017/S1446788700022436 Published online by Cambridge University Press

2. Preliminaries and assumptions

Let Ω be a smooth bounded domain in $\mathbf{R}^n$ with boundary Γ. We introduce the Sobolev space of order k, denoted by $H^k(\Omega)$, consisting of real-valued functions in Ω such that

$$D^j u \in L^2(\Omega), \qquad 0 \leqslant |j| \leqslant k,$$

and $\gamma_j = \partial^j/\partial n^j$, $j \geqslant 0$, will be the trace operators mapping $H^k(\Omega)$ onto $H^{k-j-1/2}(\Gamma)$. The differential operator defined by

$$\Lambda u = \sum_{|p|,|q| \leqslant k} (-1)^{|q|} D^q\big(a_{pq}(x) D^p u\big)$$

is the formal operator associated with the bilinear form

$$a(u, v) = \sum_{|p|,|q| \leqslant k} \int_\Omega a_{pq}(x) D^p u D^q v \, dx.$$

There exist operators δ_{2k-j-1} mapping $H^k(\Omega, \Lambda)$ into $H^{-(k-j-1/2)}(\Gamma)$, $0 \leqslant j \leqslant k - 1$, where

$$H^k(\Omega, \Lambda) = \{u \in H^k(\Omega) \colon \Lambda u \in L^2(\Omega)\},$$

such that Green's formula

$$a(u, v) = (\Lambda u, v)_{L^2(\Omega)} + \sum_{0 \leqslant j \leqslant k-1} \langle \delta_{2k-j-1} u, \gamma_j v \rangle_{H^{k-j-1/2}(\Gamma)}$$

holds for all $u \in H^k(\Omega, \Lambda)$, $v \in H^k(\Omega)$. Here $(\cdot, \cdot)$ denotes inner product and $\langle \cdot , \cdot \rangle$ denotes duality pairing. We also make the following assumptions:

$$(1) \qquad \begin{cases} a(u, v) \leqslant M\|u\|_{H^k(\Omega)}\|v\|_{H^k(\Omega)} & \text{for all } u, v \in H^k(\Omega); \\ a(u, v) \geqslant c\|u\|^2_{H^k(\Omega)} & \text{for all } u \in H^k(\Omega). \end{cases}$$

Formulation of the problem.

Look for u satisfying

$$(2) \qquad \begin{cases} u \in H^k(\Omega), \\ \Lambda u = f, \\ \gamma_j u \geqslant 0, \quad \delta_{2k-j-1} u \geqslant 0, \langle \delta_{2k-j-1} u, \gamma_j u \rangle = 0, \quad 0 \leqslant j \leqslant k - 1, \end{cases}$$

where $f \in L^2(\Omega)$ is given. It is easy to verify that (2) is equivalent to

$$(3) \qquad \begin{cases} u \in H^k(\Omega), \\ \gamma_j u \geqslant 0, 0 \leqslant j \leqslant k - 1, \\ a(u, v - u) \geqslant (f, v - u) \text{ for all } v \text{ such that } \gamma_j v \geqslant 0, 0 \leqslant j \leqslant k - 1. \end{cases}$$

We see that (3) is a variational inequality and it is known that under the assumption (1), it has a unique solution (see Lions and Stampacchia (1967)).

3. A posteriori error estimates

THEOREM 1. *Let u be a solution of* (2), $v \in H^k(\Omega)$ *satisfy* $\gamma_j v \geqslant 0$, $0 \leqslant j \leqslant k-1$ *and* $w \in H^k(\Omega, \Lambda)$ *satisfy* $\delta_{2k-j-1} w \geqslant 0$, $0 \leqslant j \leqslant k-1$. *Then the following a posteriori error estimates hold*:

$$(4)\quad \begin{cases} \text{(i)} & \|v-u\|_{H^k(\Omega)} \leqslant \dfrac{1}{2c}\left\{ M\|v-w\|_{H^k(\Omega)} + \|\Lambda w - f\|_{L^2(\Omega)} + \Delta^{1/2} \right\}, \\ \text{(ii)} & \|u-w\|_{H^k(\Omega)} \leqslant \dfrac{1}{2c}\left\{ M\|v-w\|_{H^k(\Omega)} + \|\Lambda w - f\|_{L^2(\Omega)} \right. \\ & \qquad\qquad \left. + \left[\Delta + 4c(\Lambda w - f, v-w)\right]^{1/2} \right\}, \end{cases}$$

where $\Delta = [(M\|v-w\|_{H^k(\Omega)} + \|\Lambda w - f\|_{L^2(\Omega)})^2 + 4c\sum_{0 \leqslant j \leqslant k-1} \langle \delta_{2k-j-1} w, \gamma_j v\rangle]$.

PROOF.

$$\begin{aligned} c\|v-u\|^2_{H^k(\Omega)} &\leqslant a(v-u, v-u) \leqslant a(v, v-u) - (f, v-u) \\ &= a(v-w, v-u) + a(w, v-u) - (f, v-u) \\ &\leqslant M\|v-w\|_{H^k(\Omega)}\|v-u\|_{H^k(\Omega)} + \|\Lambda w - f\|_{L^2(\Omega)}\|v-u\|_{L^2(\Omega)} \\ &\quad + \sum_{0 \leqslant j \leqslant k-1} \langle \delta_{2k-j-1} w, \gamma_j v\rangle. \end{aligned}$$

It follows easily that (i) holds.

Also,

$$\begin{aligned} c\|u-w\|^2_{H^k(\Omega)} &\leqslant a(u-w, u-w) = a(u-w, v-w) + a(u-w, u-v) \\ &\leqslant a(u-w, v-w) + (\Lambda w - f, v-u) + \sum_{0 \leqslant j \leqslant k-1} \langle \delta_{2k-j-1} w, \gamma_j v\rangle \\ &\leqslant \left(M\|v-w\|_{H^k(\Omega)} + \|\Lambda w - f\|_{L^2(\Omega)}\right)\|u-w\|_{H^k(\Omega)} \\ &\quad + (\Lambda w - f, v-w) + \sum_{0 \leqslant j \leqslant k-1} \langle \delta_{2k-j-1} w, \gamma_j v\rangle. \end{aligned}$$

Hence (ii) follows.

4. Self-adjoint problems

Consider the following simple example. Let Ω be a smooth bounded subset of $\mathbf{R}^n$ and Γ its boundary. We are interested in the solution of the boundary value problem:

$$(5)\qquad \begin{cases} \text{(i)} & -\Delta u + \lambda u = f \quad \text{in } \Omega, \\ \text{(ii)} & u = g_1 \quad \text{on } \Gamma_1, \\ \text{(iii)} & \dfrac{\partial u}{\partial n} = g_2 \quad \text{on } \Gamma_2, \end{cases}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_1, Γ_2 are disjoint. Aubin and Burchard (1971) have obtained a posteriori error estimates for approximate solutions of (5) by constructing a boundary value problem conjugate to (5), associating with the splitting $-\Delta = -\mathrm{div}(\mathrm{grad})$. Alternatively, problem (5) can be viewed as the optimization problem: find u such that

$$\tfrac{1}{2}\int_\Omega \|\mathrm{grad}\, u\|^2 + \lambda\|u\|^2\, dx - \int_\Omega f\cdot u dx - \int_{\Gamma_2} g_2 \cdot u d\sigma(x)$$

is minimized, subject to $u = g_1$ on Γ_1.

Each function u, satisfying $\partial u/\partial n = g_2$ on Γ_2 will give a lower bound for this minimization problem. Making use of this bound, we can give an a posteriori estimate for (5) which turns out to be the same as that given by Aubin and Burchard. When two functions u_1, u_2, satisfying $\partial u/\partial n = g_2$ are given, we are able to derive an alternative estimate, making use of Schwarz's inequality.

We shall follow the notations of Aubin (1972), page 289. V, H and T are real Hilbert spaces and $\gamma \in L(V, T)$ satisfies

$$(6)\qquad \begin{cases} \text{(i)} & \gamma \text{ maps } V \text{ onto } T, \\ \text{(ii)} & V \subset H, \text{ the injection is continuous}, \\ \text{(iii)} & \mathrm{Ker}\,\gamma = V_0 \text{ is dense in } H. \end{cases}$$

Let E be another real Hilbert space and $P \in L(V, E)$, $Q \in L(E, E')$ and $G = QP$. The formal operator associated with the bilinear form (Pu, Gv) is $\Lambda = G^*P$ where $G^* = (G|_{V_0})' \in L(E, V_0')$. Then, there exists $\delta \in L(V, T')$ such that Green's formula

$$(7)\qquad (Pu, Gv) = (\Lambda u, v) + \langle \delta u, \gamma v\rangle$$

holds for all $u \in V(\Lambda) = \{u \in V\colon \Lambda u \in H\}$. We are also given a continuous projector σ_1 of T and define $\sigma_2 = 1 - \sigma_1$; $T_j = \sigma_j T$; $\gamma_j = \sigma_j\gamma$; $\delta_j = \sigma_j'\delta$; $j = 1, 2$.

Thus, (7) can be written as

$$(8)\qquad (Pu, Gv) = (\Lambda u, v) + \langle \delta_1 u, \gamma_1 v\rangle + \langle \delta_2 u, \gamma_2 v\rangle.$$

Consider the problem: find u satisfying

$$(9)\qquad \begin{cases} \text{(i)} & \Lambda u + \lambda u = f, \qquad \lambda > 0, \\ \text{(ii)} & \gamma_1 u = t_1, \\ \text{(iii)} & \delta_2 u = t_2, \end{cases}$$

where $f \in H$, $t_1 \in T_1$, $t_2 \in T_2'$ are given. Such a solution exists and is unique if, for instance, Q is E-elliptic and $\lambda > 0$. Our problem is given any $v, \hat{v} \in V$ satisfying $\gamma_1 v = t_1$, $\delta_2 \hat{v} = t_2$, find upper bounds for

$$(P(u-v), G(u-v)) + \lambda(u-v, u-v)$$

and

$$(P(u-\hat{v}), G(u-\hat{v})) + \lambda^{-1}\|G^*P(u-\hat{v})\|^2$$

without solving (9).

5. Alternative derivation of a posteriori error estimates

In this section we derive the *a posteriori* error estimates given by Aubin and Burchard under the additional assumption that Q is self-adjoint.

LEMMA 1. *Let Q be self-adjoint. Then*

$$(v_1, Qv_1) - (v_2, Qv_2) = (v_1 - v_2, Q(v_1 - v_2)) + 2(v_2, Q(v_1 - v_2)).$$

LEMMA 2. *Let Q be self-adjoint and positive definite. If u satisfies* (9), *then $v = u$ will minimize*

$$J(v) = \tfrac{1}{2}(Pv, QPv) + \tfrac{1}{2}\lambda(v, v) - (f, v) - \langle t_2, \gamma_2 v\rangle$$

subject to $\gamma_1 v = t_1$.

Furthermore

$$J(v) - J(u) = \tfrac{1}{2}\{(P(v-u), QP(v-u)) + \lambda(v-u, v-u)\}.$$

PROOF. Let u satisfy (9) and $\gamma_1 v = t_1$; then $u \in V(\Lambda)$. In view of Lemma 1 and Green's formula

$$\begin{aligned} J(v) - J(u) &= \tfrac{1}{2}\{(P(v-u), QP(v-u)) + \lambda(v-u, v-u)\} \\ &\quad + (Pu, QP(v-u)) + \lambda(u, v-u) - (f, v-u) - \langle t_2, \gamma_2(v-u)\rangle \\ &= \tfrac{1}{2}(P(v-u), QP(v-u)) + \tfrac{1}{2}\lambda(v-u, v-u) \geqslant 0. \end{aligned}$$

LEMMA 3. *If $\gamma_1 v = t_1$, $\delta_2 \hat{v} = t_2$, $\hat{v} \in V(\Lambda)$, then $J(v) \geqslant J_1(\hat{v})$, where*

$$J_1(\hat{v}) = \langle \delta_1 \hat{v}, t_1\rangle - \tfrac{1}{2}(P\hat{v}, G\hat{v}) - (2\lambda)^{-1}\|f - G^*P\hat{v}\|^2.$$

Furthermore

$$J(v) - J_1(\hat{v}) = \tfrac{1}{2}\left\{(P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2\right\}.$$

PROOF.

$$\begin{aligned}\tfrac{1}{2}&\left[(Pv, Gv) + (P\hat{v}, G\hat{v}) + \lambda(v,v) + \lambda^{-1}\|f - G^*P\hat{v}\|^2\right] \\ &= \tfrac{1}{2}\left\{(P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2\right\} \\ &\quad + (P\hat{v}, Gv) + (f - G^*P\hat{v}, v) \\ &= \tfrac{1}{2}\left\{(P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2\right\} \\ &\quad + (f, v) + \langle \delta_1\hat{v}, t_1\rangle + \langle t_2, \gamma_2 v\rangle.\end{aligned}$$

Hence $J(v) - J_1(\hat{v}) = \frac{1}{2}\{(P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2\} \geqslant 0$.

Setting $v = u$ in Lemma 3, we have

$$(10)\qquad J(u) - J_1(v) = \tfrac{1}{2}(P(u-\hat{v}), G(u-\hat{v})) + \tfrac{1}{2}\lambda^{-1}\|G^*P(u-\hat{v})\|^2.$$

THEOREM 2. *Suppose* $v, \hat{v} \in V$ *satisfy* $\gamma_1 v = t_1$, $\delta_2\hat{v} = t_2$, $\hat{v} \in V(\Lambda)$. *Then*

$$\begin{aligned}&(P(u-v), G(u-v)) + \lambda(u-v, u-v) \\ &\qquad \leqslant (P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2, \\ &(P(u-\hat{v}), G(u-\hat{v})) + \lambda^{-1}\|G^*P(u-\hat{v})\|^2 \\ &\qquad \leqslant (P(v-\hat{v}), G(v-\hat{v})) + \lambda^{-1}\|\lambda v + G^*P\hat{v} - f\|^2.\end{aligned}$$

PROOF. Since $J(v) - J(u) \leqslant J(v) - J_1(\hat{v})$ and $J(u) - J_1(\hat{v}) \leqslant J(v) - J_1(\hat{v})$, the results follow from Lemma 3 and Lemma 2.

6. An alternative estimate

In this section, we shall derive another error estimate for v under the assumption that we are given two functions satisfying $\delta_2\hat{v} = t_2$, by completing the square for $J_1(v)$ and then applying Schwarz's inequality. Note that if we fix $r \in V(\Lambda)$ satisfying $\delta_2 r = t_2$, it follows from Lemma 3 that

$$(11)\quad J(v) - \tfrac{1}{2}\left\{P(v-r), G(v-r) + \lambda^{-1}\|\lambda v + G^*Pr - f\|^2\right\} = \text{constant}$$

(independent of v).

Hence any solution to (11) will minimize $J_2(v) = \frac{1}{2}\{(P(v-r), G(v-r)) + \lambda^{-1}\|\lambda v + G^*Pr - f\|^2\}$. Now we give a lower bound for $J_2(v)$.

https://doi.org/10.1017/S1446788700022436 Published online by Cambridge University Press

LEMMA 4. *Suppose $\hat{v}$ satisfies $\delta_2\hat{v} = t_2$, $\hat{v} \in V(\Lambda)$. Then*

$$J_2(v) \geqslant (1/2K)\big(\langle\delta_1(\hat{v} - r), t_1 - \gamma_1 r\rangle - \lambda^{-1}(G^*P(\hat{v} - r), \lambda r + G^*Pr - f)\big)^2,$$

where $K = (P(\hat{v} - r), G(\hat{v} - r)) + \lambda^{-1}\|G^*P(\hat{v} - r)\|^2$.

PROOF. Since

$$\begin{aligned}(P(\hat{v} - r), G(v - r)) &+ \lambda^{-1}(-G^*P(\hat{v} - r), \lambda v + G^*Pr - f)\\ &= \langle\delta_1(\hat{v} - r), t_1 - \gamma_1 r\rangle - \lambda^{-1}(G^*P(\hat{v} - r), \lambda r + G^*Pr - f),\end{aligned}$$

the inequality then follows from Schwarz's inequality.

THEOREM 3. *Suppose $v, \hat{v}, r \in V$ satisfy $\gamma_1 v = t_1$, $\delta_1\hat{v} = \delta_1 r = t_2$, $\hat{v}, r \in V(\Lambda)$. Then*

$$\begin{aligned}(P(u - v), G(u - v)) &+ \lambda(u - v, u - v)\\ &\leqslant (P(v - r), G(v - r)) + \lambda^{-1}\|\lambda v + G^*Pr - f\|^2\\ &\quad -\tfrac{1}{K}\big(\langle\delta_1(\hat{v} - r), t_1 - \gamma_1 r\rangle - \lambda^{-1}(G^*P(\hat{v} - r), \lambda r + G^*Pr - f)\big)^2.\end{aligned}$$

PROOF. Denote the bound in Lemma 4 by $b(\hat{v})$. Then $J(v) - J(u) = J_2(v) - J_2(u) \leqslant J_2(v) - b(\hat{v})$.

The result then follows from Lemma 2.

One can easily show that the estimate given in Theorem 3 is related to that given in Theorem 2. Indeed, the difference of the two estimates is $2\{(J(v) - J_1(\hat{v})) - (J_2(v) - b(\hat{v}))\} = 2\{(J_1(r) - J_1(\hat{v})) + b(\hat{v})\}$. But

$$\begin{aligned}J_1(r) - J_1(\hat{v}) &= \tfrac{1}{2}\{(P(\hat{v} - r), G(\hat{v} - r)) + \lambda^{-1}\|G^*P(\hat{v} - r)\|^2\}\\ &\quad +\lambda^{-1}(G^*P(\hat{v} - r), G^*Pr + \lambda r - f) - \langle\delta_1(\hat{v} - r), t_1 - \gamma_1 r\rangle.\end{aligned}$$

Hence, the difference of the two estimates is

$$\begin{aligned}&\Big[(P(\hat{v} - r), G(\hat{v} - r)) + \lambda^{-1}\|G^*P(\hat{v} - r)\|^2\Big]^{-1}\\ &\quad\times\Big\{(P(\tilde{v} - r), G(\hat{v} - r)) + \lambda^{-1}\|G^*P(\hat{v} - r)\|^2\\ &\qquad +\lambda^{-1}(G^*P(\hat{v} - r), G^*Pr + \lambda r - f) - \langle\delta_1(v - r), t_1 - \gamma_1 r\rangle\Big\}^2\\ &= \Big[P(\hat{v} - r), G(\hat{v} - r) + \lambda^{-1}\|G^*P(\hat{v} - r)\|^2\Big]^{-1}\\ &\quad\times\Big\{(P(\hat{v} - r), G(\hat{v} - r)) + \lambda^{-1}(G^*P(\hat{v} - r), G^*P\hat{v} + \lambda r - f)\\ &\qquad - \langle\delta_1(\hat{v} - r), t_1 - \gamma_1 r\rangle\Big\}^2 \geqslant 0.\end{aligned}$$

https://doi.org/10.1017/S1446788700022436 Published online by Cambridge University Press

Now suppose we know two solutions of $\gamma_1 v = t_1$. We may proceed in an analogous way to obtain an estimate for $\hat{v}$ satisfying $\delta_2 \hat{v} = t_2$. Fix $s \in V(\Lambda)$ satisfying $\gamma_1 s = t_1$. If we set

$$J_3(\hat{v}) = \tfrac{1}{2}\Big(\big(P(s-\hat{v}), G(s-\hat{v})\big) + \lambda^{-1}\|\lambda s + G^*P\hat{v} - f\|^2\Big),$$

then

$$J_1(\hat{v}) + J_3(\hat{v}) = J(s) = \text{constant (independent of } \hat{v}).$$

We also have

$$\begin{aligned}(P(s-\hat{v}), G(s-v)) &+ \lambda^{-1}(\lambda s + G^*P\hat{v} - f, \lambda(s-v)) \\ &= (G^*Ps + \lambda s - f, s - v) + \langle \delta_2 s - t_2, \gamma_2(s-v)\rangle.\end{aligned}$$

Then by Schwarz's inequality,

$$J_3(\hat{v}) \geqslant \tfrac{1}{2Y}\big((G^*Ps + \lambda s - f, s - v) + \langle \delta_2 s - t_2, \gamma_2(s-v)\rangle\big)^2,$$

where $Y = (P(s-v), G(s-v)) + \lambda(s-v, s-v)$.

It follows that

$$\begin{aligned}(P(u-\hat{v}), G(u-\hat{v})) &+ \lambda^{-1}\|G^*P(u-\hat{v})\|^2 = 2(J(u) - J_1(\hat{v})) \\ &= 2(J_1(u) - J_1(\hat{v})) = 2(J_3(\hat{v}) - J_3(u)) \\ &\leqslant (P(s-\hat{v}), G(s-\hat{v})) + \lambda^{-1}\|\lambda s + G^*P\hat{v} - f\|^2 \\ &\quad - \tfrac{1}{Y}\big((G^*Ps + \lambda s - f, s - v) + \langle \delta_2 s - t_2, \gamma_2(s-v)\rangle\big)^2.\end{aligned}$$

Further, it can be easily shown that the difference between this estimate and that given in Theorem 2 is

$$\begin{aligned}&[P(s-v), G(s-v) + \lambda(s-v, s-v)]^{-1} \\ &\quad \times (P(s-v), G(s-v)) + \lambda(s-v, s-v) \\ &\quad - \big((G^*Ps + \lambda s - f, s - v) - \langle \delta_2 s - t_2, \gamma_2(s-v)\rangle\big)^2 \geqslant 0.\end{aligned}$$

7. Example

We now apply the results of the previous section to obtain a posteriori error estimates for approximate solutions of self-adjoint boundary value problems of a differential operator of order $2k$.

Again we follow the notations of Aubin (1972), as described in Section 2 with the additional assumption that $a_{pq}(x) = a_{qp}(x)$.

https://doi.org/10.1017/S1446788700022436 Published online by Cambridge University Press

Suppose we are given the following data:
(i) $f \in L^2(\Omega)$,
(ii) $g_j \in H^{k-j-1/2}(\Gamma)$, $0 \leqslant j \leqslant p-1$, $1 \leqslant p \leqslant k$,
(iii) $h_j \in H^{k-j-1/2}(\Gamma)$, $k \leqslant j \leqslant 2k-p-1$.
We consider the problem: find u that satisfies

$$(12) \qquad \begin{cases} \text{(i)} & \Lambda u + \lambda u = f, \quad \lambda > 0, \\ \text{(ii)} & \gamma_j u = g_j, \quad 0 \leqslant j \leqslant p-1, \\ \text{(iii)} & \delta_j u = h_j, \quad k \leqslant j \leqslant 2k-p-1. \end{cases}$$

Results of the previous sections can be applied to obtain

THEOREM 5. *Suppose u is a solution of (12), $v \in H^k(\Omega)$ satisfies $\gamma_j v = g_j$, $0 \leqslant j \leqslant p-1$ and $\hat{v}, r \in H^k(\Omega, \Lambda)$ satisfy $\delta_j \hat{v} = \delta_j r = h_j$, $k \leqslant j \leqslant 2k-p-1$. Then*

$$\begin{aligned} a(u-v, u-v) &+ \lambda(u-v, u-v)_{L^2(\Omega)} \\ &\leqslant a(v-r, v-r) + \lambda^{-1}\|\lambda v + \Lambda r - f\|^2_{L^2(\Omega)} \\ &\quad - \frac{1}{Z}\Big\{ \sum_{0 \leqslant j \leqslant p-1} \big\langle \delta_{2k-j-1}(\hat{v} - r), g_j - \gamma_j r \big\rangle_{H^{k-j-1/2}(\Gamma)} \\ &\qquad\qquad - \lambda^{-1}(\Lambda(\hat{v} - r), \lambda r + \Lambda r - f)_{L^2(\Omega)} \Big\}^2, \end{aligned}$$

where $Z = a(\hat{v} - r, \hat{v} - r) + \lambda^{-1}\|\Lambda(\hat{v} - r)\|^2_{L^2(\Omega)}$.

If $s \in H^k(\Omega, \Lambda)$, $v \in H^k(\Omega)$ satisfy $\gamma_j s = \gamma_j v = g_j$, $0 \leqslant j \leqslant p-1$, and if $\hat{v} \in H^k(\Omega, \Lambda)$ satisfy $\delta_j \hat{v} = h_j$, $k \leqslant j \leqslant 2k-p-1$, then

$$\begin{aligned} a(u-\hat{v}, u-\hat{v}) &+ \lambda^{-1}\|\Lambda(u-\hat{v})\|^2_{L^2(\Omega)} \\ &\leqslant a(s-\hat{v}, s-\hat{v}) + \lambda^{-1}\|\lambda s + \Lambda \hat{v} - f\|^2_{L^2(\Omega)} \\ &\quad - \frac{1}{W}\Big\{ (\Lambda s + \lambda s - f, s - v)_{L^2(\Omega)} \\ &\qquad\qquad + \sum_{k \leqslant j \leqslant 2k-p-1} \big\langle \delta_j s - h_j, \gamma_{2k-j-1}(s - v) \big\rangle_{H^{k-j-1/2}(\Gamma)} \Big\}^2, \end{aligned}$$

where $W = a(s-v, s-v) + \lambda(s-v, s-v)_{L^2(\Omega)}$.

REMARK. In Theorem 5, we have given a bound for $a(u-v, u-v)$ only. To obtain a bound for $\|u-v\|^2_{H^k(\Omega)}$ we must assume that $a(u, v)$ is elliptic and hence that there exists a constant M such that $M\|u-v\|^2 \leqslant a(u-v, u-v)$.

References

J. P. Aubin (1972), *Approximation of elliptic boundary-value problems*, (Wiley Interscience, New York).

J. P. Aubin and H. Burchard (1971), 'Some aspects of the method of the hypercircle applied to elliptic variational problems', *Proceedings of SYNSPADE*, edited B. Hubbard (Academic Press, New York).

J. L. Lions and G. Stampacchia (1967), 'Variational inequalities', *Comm. Pure Appl. Math.* **20**, 493–519.

Department of Mathematics
Science Centre
The Chinese University of Hong Kong
Shatin, N. T.
Hong Kong