A GENERAL PERRON INTEGRAL, II

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1. Introduction. This paper continues work begun in a previous paper of the same title **(7)**, which will be called I; results from I will be referred to as Theorem 1.4, Axiom 1.1 etc. The notation used in the present paper will, except where noted, be that of I, to which reference should be made for further details.

In § 2, certain ideas presented in I are modified to give a neater and more general theory and then some new results of this theory are added. The remaining two sections develop some of the examples mentioned in I, § 5.

2. The axioms. In I, X was eventually a finite-dimensional vector space with a Bauer harmonic structure **(2)**, satisfying Axioms 1.1–1.6; further, for all $x \in X$, $\mathfrak{N}(x)$ was a fundamental system of convex, regular neighbourhoods of x. Replacing Axiom 1.1 by Axiom 1 below and using this new axiom to modify the definition of the generalized derivative (I, (2)) most of these restrictions can be removed.

2.1. X is then a locally compact space with a Bauer harmonic structure and for all $x \in X$, $\mathfrak{N}(x)$ is a fundamental system of regular neighbourhoods.

Axiom 1. *There exists a real function p, locally strictly hypoharmonic in X and $\mu(V; x)$-summable for all V and all $x \in V$.*

Throughout this paper, p (p' etc.) will denote such a function. Since p is $\mu(V; x)$-summable, $\Delta p(x; V)$ is defined and p being real implies $\Delta p(x; V) < \infty$ (I, (1)). Further, the statement that p is locally strictly hypoharmonic just says that $\Delta p(x; V) > 0$, for all V small enough; see I and **(3)**.

Axiom 1 is often a consequence of properties of the harmonic structure. Thus if Axiom K_D is satisfied, every real hypoharmonic function is $\mu(V; x)$-summable for all V and all $x \in V$ **(3**, Theorem 1**)**. If, in addition, X has a countable base and Axiom T_+ holds, there exists a real continuous function strictly hypoharmonic on X **(3**, Theorem 9**)**. It should be remarked that Axiom T_+ always hold if the theory is restricted to a relatively compact U **(2**, remark after Theorem 40**)**.

2.2. If f is a numerical function on X, the *generalized upper derivative of f with respect to p, relative to $\mathfrak{N}$, at x*, is

(1) $$\uparrow Df(x; \mathfrak{N}; p) = \limsup_{\mathfrak{N}(x)} \uparrow \Delta f(x; V)/\Delta p(x; V).$$

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The discussion in I can be applied here with only obvious modifications. When no ambiguity can result the symbol $\mathfrak{N}$ or the symbol p will be omitted from the left-hand side of (1). Clearly, when defined,

$$Df(x; p)Dp(x; p') = Df(x; p').$$

If the derivatives are defined in the t-harmonic structure, then the notation $t\text{-}\uparrow Df(x; \mathfrak{N}; p)$ etc., will be used; this is a slight change from I. If $p' = p/t$, then p' satisfies Axiom 1 in the t-harmonic structure and using results in I we easily see that

$$t\text{-}\uparrow Df(x; p') = \uparrow D(tf)(x; p), \qquad \text{etc.} \tag{2}$$

Clearly Theorem 1.4 remains valid and since $Dp(x; \mathfrak{N}; p) = 1$ for all $x \in X$ and all choices of $\mathfrak{N}$, Theorem 1.5 also holds, the role of v in its proof being taken by $-p$.

LEMMA 1. *If Axiom 1 is assumed and f attains a non-positive minimum at x, then $t\text{-}\downarrow Df(x; \mathfrak{N}; p') \geqslant 0$.*

Proof. This is just the last part of the proof of Theorem 1.5. Thus this derivative satisfies what Dynkin **(11)** calls *the minimum principle.*

2.3. We now discuss Theorem 1.7. If $\mathfrak{E} = \mathfrak{E}(U)$ is a collection of subsets of U, then we say that a numerical function f on U is $\downarrow\mathfrak{E}$-even if

$$\text{for all } E \in \mathfrak{E},\ \sup_{x\in X} \downarrow Df(x) = \sup_{x\in X\sim E} \downarrow Df(x).$$

If $-f$ is $\downarrow\mathfrak{E}$-even, then f is said to be $\uparrow\mathfrak{E}$-even and Theorem 1.5 can be extended immediately to the following theorem.

THEOREM 2. *If Axiom 1 is assumed and f is a $\downarrow\mathfrak{E}$-even numerical l.s.c. function on x and if for some $\mathfrak{N}$, $E \in \mathfrak{E}(X)$, $\downarrow Df(x; \mathfrak{N}) \leqslant 0$ for all $x \in X \sim E$, then $f \in \mathfrak{H}^*(X)$.*

This theorem corresponds to Theorem 1.7; the latter result can be considered as characterizing a class of $\downarrow\mathfrak{E}$-even functions when $\mathfrak{E}$ consists of enumerable subsets of X.

To see this, let us suppose that X is a metric space with metric r. Then, changing slightly the notation in I, we say that a numerical function f is $\downarrow\mathfrak{N}$-*smooth at* x if

$$\liminf_{\mathfrak{N}(x)} [\downarrow\Delta f(x; V)/\rho(x; V)] \leqslant 0, \tag{3}$$

where

$$\rho(x; V) = \{\textstyle\int_{V^*} [\mu(V, x)/r(x, z)]dz\}^{-1}. \tag{4}$$

If in (3) the limit exists and is equal to zero, then we say that f is $\mathfrak{N}$-*smooth at* x; if f has such a property at all points of a set A, we shall say that it

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possesses the property on A. The function ρ in (4) is to replace s in Axiom 1.4. Axiom 1.4 will always be considered as modified in this way.

LEMMA 3. *If p is $\mathfrak{N}$-smooth at x and* $\downarrow Df(x; \mathfrak{N}; p) \leqslant 0$, *then f is $\downarrow\mathfrak{N}$-smooth at x.*

Proof. If in fact $\downarrow Df(x; \mathfrak{N}; p) < 0$, then the proof is immediate. If not, then write $f = (f - p) + p$ and note that the sum of a function $\downarrow\mathfrak{N}$-smooth at x and one $\mathfrak{N}$-smooth at x is $\downarrow\mathfrak{N}$-smooth at x.

THEOREM 4. *If Axioms* 1, 1.2, 1.3, *and* 1.4, *modified as above, are assumed in X, a finite-dimensional vector space with $\mathfrak{N}$ convex, and if $\mathfrak{E}$ is the class of enumerable subsets of X and f a numerical l.s.c. $\downarrow\mathfrak{N}$-smooth function on X, then f is $\downarrow\mathfrak{E}$-even.*

Proof. If $E \in \mathfrak{E}$, let $\lambda = \sup_{x \in X \sim E} \downarrow Df(x; \mathfrak{N})$. If $\lambda = +\infty$ there is nothing to prove. Suppose then that λ is finite; without loss of generality we may assume that $\lambda = 0$, by considering $f - \lambda p$, for instance. Then the proof is just that of Theorem 1.7 with obvious modifications, using, at the point where Axiom 1.4 is introduced, Hölder's inequality with $p = q = 2$ instead of $p = 1$, $q = \infty$ as in I. In fact this proves that if $\sup_{x \in X \sim E} \downarrow Df(x; \mathfrak{N}) \leqslant \alpha$, then $\sup_{x \in X} \downarrow Df(x; \mathfrak{N}) \leqslant \alpha$. Hence, finally, if $\lambda = -\infty$,

$$\sup_{x \in X \sim E} \downarrow Df(x; \mathfrak{N}) \leqslant \alpha \qquad \text{for all } \alpha,$$

and so $\sup_{x \in X} \downarrow Df(x; \mathfrak{N}) = -\infty$.

2.4. The modifications now needed in the major and minor functions and in the $\mathfrak{H}$-integral are mostly fairly obvious and only a few points will be mentioned. If X is as in § 2.1 and satisfies Axiom 1, let U be a fixed relatively compact set with eventually nearly all of the points of U^* regular (see I). The classes $\mathfrak{N}(x)$, $x \in U$, $\mathfrak{E} = \mathfrak{E}(U)$, $\mathfrak{Z} = \mathfrak{Z}(U)$ (see I) of subsets of U are given. Let $\downarrow\mathfrak{S} = \downarrow\mathfrak{S}(U)$ be a collection of $\downarrow\mathfrak{E}$-even functions closed to finite sums and write $f \in \uparrow\mathfrak{S}$ if $-f \in \downarrow\mathfrak{S}$. In I no use is made of $\downarrow\mathfrak{S}$, which makes it more general than a direct particularization of the present theory. In fact it is not difficult to show that this slight generalization is always possible in cases in which the evenness concept derives from (3). The present situation is akin to that in **(15)**.

AXIOM 2. (i) *$\mathfrak{E}$ is closed to finite unions,* (ii) *$\mathfrak{Z}$ is closed to enumerable unions,* (iii) $\mathfrak{E} \subset \mathfrak{Z}$.

This replaces Axiom 1.5. The definition of the class of minor functions of f on U (I, § 3) now reads $m \in \downarrow\mathfrak{M}(f)$ if and only if

(i) $m \in \mathfrak{C}(\bar{U})$,
(ii) $\uparrow Dm(x) < \infty$ if $x \in U \sim E$, $E \in \mathfrak{E}$,
(iii) $\uparrow Dm(x) \leqslant f(x)$ if $x \in U \sim E \sim Z$, $E \in \mathfrak{E}$, $Z \in \mathfrak{Z}$,
(iv) $m \,|\, U \in \uparrow\mathfrak{S}$,
(v) $m(z) \geqslant 0$ for nearly all $z \in U^*$.

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2.5. With X as in § 2.4 certain results not given in I but belonging to the general theory will now be proved.

THEOREM 5. *If f is a $\downarrow\mathfrak{E}$-even numerical l.s.c. function on X and if $\downarrow Df(x;\mathfrak{N};p)$ is locally bounded below on $X \sim E$, $E \in \mathfrak{E}$, then f is δ-hyperharmonic on X.*

Proof. This generalizes a result of Arsove **(1**, Theorem 19**)**, whose proof applies here.

THEOREM 6. *If $p \in \mathfrak{C}(X)$, then for all $f \in \mathfrak{C}(X)$, V, and $x \in V$*

$$\min_{y\in V} \uparrow Df(y;\mathfrak{N};p) \leqslant \Delta f(x;V)/\Delta p(x;V) \leqslant \max_{y\in V} \downarrow Df(y;\mathfrak{N};p). \tag{6}$$

Proof. This generalizes a result of Denjoy **(9**, § 6**)**. As usual, there is no loss in generality in assuming $1 \in \mathfrak{H}(X)$; see Theorem 1.5. Let

$$g(y) = \lambda\Delta p(y;V) - \Delta f(y;V)$$

for all $y \in \bar{V}$. Then for all $z \in V^*$, $g(z) = 0$; choose λ so that $g(x) = 0$, that is, $\lambda = \Delta f(x;V)/\Delta p(x;V)$. Hence g assumes a non-positive minimum in V, at y say. So by Lemma 1, $\downarrow Dg(y;\mathfrak{N};p) \geqslant 0$; since $1 \in \mathfrak{H}(X)$, this use of Lemma 1 is justified. Simple calculations show that $\downarrow Dg(y) \leqslant \lambda - \downarrow Df(y)$, and hence $\downarrow Df(y) \geqslant \lambda$, which implies the right-hand side of (5). A similar argument completes the proof.

COROLLARY 7. *With the same hypotheses as Theorem* 6,

$$\max_{y\in V} \uparrow Df(y;\mathfrak{N};p) = \max_{y\in V} \downarrow Df(y;\mathfrak{N};p)$$

and

$$\min_{y\in V} \uparrow Df(y;\mathfrak{N};p) = \min_{y\in V} \downarrow Df(y;\mathfrak{N};p).$$

COROLLARY 8. *With the same hypotheses as Theorem* 6, *if for some $\mathfrak{N}$*

$$\downarrow Df(x;\mathfrak{N};p) \leqslant 0 \leqslant \uparrow Df(x;\mathfrak{N};p) \qquad \textit{for all } x \in X,$$

then $f \in \mathfrak{H}(X)$.

Proof. The proofs of these corollaries follow those of Denjoy **(9**, 1.6 and 1.8**)**. Corollary 8 is just Corollary 1.6 but the proof here is more direct.

COROLLARY 9. *With the same hypotheses as Theorem* 6, *and in a space X in which a Riemann integral is defined, $\downarrow Df(x;\mathfrak{N};p)$ and $\uparrow Df(x;\mathfrak{N};p)$ are Riemann integrable together, with the same values for their integrals.*

Proof. This is immediate from Corollary 7; see also **(9)**.

It may be remarked that with the concept of $\downarrow\mathfrak{E}$-evenness, Theorem 6 and Corollaries 7 and 8 may be improved slightly. Whether for $u \in \mathfrak{H}^*(x)$ $Du(x;\mathfrak{N};p)$ exists, except on some small set, is an interesting open question; it is known to be true in the classical cases; see **(22)**.

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3. A one-dimensional example. Let X be an open subset of the real line and $\mathfrak{H}(X)$ be the collection of real functions h for which

$$Rh(x) = h''(x) + r(x)h'(x) + s(x)h(x) = 0, \qquad \text{for all } x \in X. \tag{6}$$

X, r, s are so restricted that the axioms of the Bauer harmonic structure hold; for this it is sufficient to suppose that X is a sufficiently small bounded open interval and r, s locally Lipschitz; see **(2, 13, 21)**. Then in fact Axioms K_D, T_+ are satisfied **(2, 3, 13)**, and so Axiom 1 holds. The remaining axioms will be discussed later. A base of regular sets consists of all open intervals and the only negligible set is the empty set. Further **(13)**, if $u \in \mathfrak{C}^2(x)$, then $u \in \mathfrak{H}^*(x)$ if and only if $Ru \leqslant 0$; and if $Ru < 0$, then u is locally strictly hyperharmonic.

3.1. Let α, β be independent functions in $\mathfrak{H}(X)$, $V = (c, d)$ and $x \in V$; then simple calculations show that

$$H_f(x) = H_f^V(x) = \frac{[f(c), \beta(d)]}{[\alpha(c), \beta(d)]}\alpha(x) + \frac{[\alpha(c), f(d)]}{[\alpha(c), \beta(d)]}\beta(x), \tag{7}$$

$$\mu(V; x) = \frac{[\alpha(x), \beta(d)]}{[\alpha(c), \beta(d)]}\epsilon_c + \frac{[\alpha(c), \beta(x)]}{[\alpha(c), \beta(d)]}\epsilon_d, \tag{8}$$

$$\Delta f(x; V) = \frac{[f(c), \alpha(x), \beta(d)]}{[\alpha(c), \beta(d)]}, \tag{9}$$

where

$$[f(x), g(y)] = \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix},$$

$$[f(x), g(y), h(z)] = \begin{vmatrix} f(x) & f(y) & f(z) \\ g(x) & g(y) & g(z) \\ h(x) & h(y) & h(z) \end{vmatrix},$$

$$\epsilon_y = \text{unit mass at } y.$$

If α, β are chosen so that $\alpha(c) = \beta(d) = 0$, as is always possible, call them γ, δ; then the above formulae simplify to

$$H_f^V(x) = [f(d)/\gamma(d)]\gamma(x) + [f(c)/\delta(c)]\delta(x), \qquad \text{etc.}$$

Also write

$$G^V(x, t) = \begin{cases} \dfrac{[\alpha(t), \beta(d)][\alpha(c), \beta(x)]}{[\alpha(c), \beta(d)][\alpha'(x), \beta(x)]}, & c \leqslant x \leqslant t, \\[2ex] \dfrac{[\alpha(c), \beta(t)][\alpha(x), \beta(d)]}{[\alpha(c), \beta(d)][\alpha'(x), \beta(x)]}, & t \leqslant x \leqslant d, \end{cases} \tag{10}$$

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or in particular

$$G^V(x, t) = \begin{cases} \dfrac{\gamma(x), \delta(t)}{[\gamma'(x), \delta(x)]}, & c \leqslant x \leqslant t, \\ \dfrac{\gamma(t), \delta(x)}{[\gamma'(x), \delta(x)]}, & t \leqslant x \leqslant d. \end{cases}$$

Then $G^V(x, t)$ is the Green's function and is positive. If $f \in \mathfrak{C}(V)$, then the equation $RF = f$, with boundary conditions $F(c) = F(d) = 0$, is satisfied by

$$F(t) = -\int_V G^V(x, t) f(x)\, dx. \tag{11}$$

Hence if $f \in \mathfrak{C}(X)$ and $f > 0$, (11) defines a non-positive locally strictly hypoharmonic function in $\mathfrak{C}^2(X)$. In particular, we take the function p of Axiom 1 to be

$$p(t) = -\int_X G^X(x, t)\, dx. \tag{12}$$

If then $\mathfrak{N}$ is given, (1) defines a generalized derivative with respect to p. Two choices of $\mathfrak{N}$ are of interest:

$$\mathfrak{N}_\nu(x) = \{(x - h, x + k);\ h, k \text{ positive and small enough}\},$$
$$\mathfrak{N}_\sigma(x) = \{(x - h, x + h);\ h \text{ positive and small enough}\}.$$

If the choice is unimportant, we shall write $\mathfrak{N}$. By (9) with

$$V = (x - h, x + k),$$

$$\frac{\Delta f(x; V)}{\Delta p(x; V)} = \frac{[f(x - h), \alpha(x), \beta(x + k)]}{[p(x - h), \alpha(x), \beta(x + k)]}. \tag{13}$$

Before considering Axioms 2 and 1.6 it will be convenient to obtain some further elementary identities that will be useful later. It is always possible to choose α, β so that $\alpha'(x) = \beta(x) = 0, \alpha(x) = \beta'(x) = 1$ **(21)**; if this is done, call the functions ξ, η respectively. Then it is easily seen that

$$[f(x - h), \alpha(x), \beta(x + k)] = [\alpha(x), \beta'(x)][f(x - h), \xi(x), \eta(x + k)], \tag{14}$$

$$\begin{aligned}[f(x - h), \xi(x), \eta(x + k)] = {} & \{kf(x - h) + hf(x + k) - (h + k)f(x)\} \\ & + \tfrac{1}{2}r(x)\{h^2(f(x + k) - f(x)) + k^2(f(x) - f(x - h))\} \\ & + \tfrac{1}{2}s(x)hk(h + k)f(x) + O(h)o(k^2) + o(h^2)O(k),\end{aligned} \tag{15}$$

$$\begin{aligned}\mu(V; x) = \frac{1}{A}\Big\{ & \frac{k}{h + k}\,\epsilon_{x-h}(1 - \tfrac{1}{2}kr(x) + o(k)) \\ & + \frac{h}{h + k}\,\epsilon_{x+k}(1 + \tfrac{1}{2}hr(x) + o(h))\Big\},\end{aligned} \tag{16}$$

where

$$V = (x - h, x + k),$$
$$A = 1 + \frac{1}{2}(h - k)r(x) - \frac{1}{2}hks(x) + \frac{1}{h + k}(o(h^2) + o(k^2)).$$

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Using (15) and (16), the ratio on the left-hand side of (3) becomes

$$(17)\qquad \frac{\Delta f(x;V)}{\rho(x;V)} = \left\{\frac{f(x+k)-f(x)}{k}(2+hr(x)) + \frac{f(x)-f(x-h)}{h}(-2+k\sigma(x)) + (h+k)s(x)f(x) + (h+k)o(1)\right\}O(1).$$

If $h = k$, these results will be numbered (14-σ), etc.

3.2. We are now in a position to consider Axioms 2 and 1.6, taking for $\mathfrak{E}$ the family of enumerable sets, for $\mathfrak{Z}$ the sets of (Lebesgue) measure zero, and for $\downarrow\mathfrak{S}$ the collection of $\mathfrak{N}$-smooth functions. Instead of using Theorem 4, we discuss $\mathfrak{E}$ and $\downarrow\mathfrak{S}$ directly, avoiding discussion of Axioms 1.2–1.4.

From (17) it follows that $\mathfrak{N}_\nu$-smoothness implies the de la Vallée–Poussin condition (K) **(9**, p. 17**)**. Then from Denjoy **(9**, p. 37**)**, it is seen that we can take for $\downarrow\mathfrak{S}$ the $\mathfrak{N}_\nu$-smooth functions. Similarly (17-σ) shows that $\mathfrak{N}_\sigma$-smoothness implies smoothness, **(23)**, and again it is known that we can take for $\downarrow\mathfrak{S}$ the $\mathfrak{N}_\sigma$-smooth functions; see **(15**, **23**, p. 328**)**.

Now let Z be any subset in X of measure zero, written $|Z| = 0$; and let z be a non-positive u.s.c. summable function such that for all $x \in Z$, $z(x) = -\infty$. Such a function can be constructed as follows. Let $G_n \supset \bar{G}_{n+1}$, G_n open, $|G_n| < 1/n$, $n = 1, 2, \ldots$, and

$$\bigcap_{n=1}^{\infty} G_n \supset Z;$$

then write

$$z(x) = -\sum_{n=1}^{\infty} 1_{\bar{G}_n}(x),$$

where 1_A denotes the indicator of A. Then if

$$v_Z(t) = -\int_X G^X(x,t)z(x)\,dt,$$

Rudin's result **(21**, (4.5)**)** shows that $Dv_Z(t;\mathfrak{N};p) = -\infty$ for all $t \in Z$. Rudin's work is applicable to the symmetric case but easily extends to the general case. Hence Axiom 2 holds in both cases with this choice of $\mathfrak{E}$, $\downarrow\mathfrak{S}$, and $\mathfrak{Z}$.

LEMMA 10. *If $u \in \mathfrak{H}^*(V)$ and is bounded below, then*

$$\Delta u(t;V) \geqslant -\int_V G^V(x;t)\downarrow Du(x;\mathfrak{N};p)\,dx.$$

Proof. This is just the main part of Rudin's Lemma 5.5 **(21)** in the present notation.

COROLLARY 11. *With u as in Lemma 10, if for any V and $k > 0$ we write $E(V;k) = \{x; x \in V$ and $\downarrow Du(x;\mathfrak{N};p) \leqslant -k\}$ and if $\bar{V}_1 \subset V$, then $k|E(V_1;k)| \leqslant B < \infty$, where B is a constant depending on V, V_1, u only.*

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Proof. Let $C = \sup_{x\in V} H_u{}^V(x) - \inf_{x\in V} u(x)$. Then from Lemma 10

$$\infty > C \geqslant -\int_V G^V(x,t) \downarrow Du(x;\mathfrak{N};p)\,dx$$

$$\geqslant k\int_{E(V_1;k)} G^V(x,t)\,dx \geqslant k\,|E(V_1;k)| \min_{\substack{x\in V_1\\ t\in V}} G^V(x,t),$$

which proves the result.

This corollary generalizes a result of James and Gage (**16**, (7.2)) and implies Axiom 1.6, since the set Z of that axiom has $|Z| \leqslant B_n/a$, where $B_n \to 0$ as $n \to \infty$. Thus the theory of the derivatives and integrals derived in § 1 applies in both of the present cases. The integral is a generalization of the James P^2-integral (**15**, **16**).

3.3. Further properties of the generalized derivative can be obtained in this special case. We say that a numerical function f has a *second-order Peano derivative at* x, $f_2(x)$, if we can write

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f_2(x) + o(h^2).$$

This definition is easily modified to define one-sided, upper and lower derivatives (**23**). If $f_2(x)$ exists, write

$$\tilde{R}f(x) = f_2(x) + r(x)f'(x) + s(x)f(x).$$

We say that f has a *first-order de la Vallée–Poussin derivative at* x, $f'_\sigma(x)$, if we can write

$$\tfrac{1}{2}\{f(x+h) - f(x-h)\} = hf'_\sigma(x) + o(h),$$

and to have one of second order, $f''_\sigma(x)$, if

$$\tfrac{1}{2}\{f(x+h) + f(x-h)\} = f(x) + \tfrac{1}{2}h^2 f''_\sigma(x) + o(h^2).$$

If both $f'_\sigma(x)$ and $f''_\sigma(x)$ exist, write

$$R_\sigma f(x) = f''_\sigma(x) + r(x)f'_\sigma(x) + s(x)f(x).$$

It should be remarked that if $\tilde{R}f(x)$ or $R_\sigma f(x)$ exists, then it is finite. If $Rf(x)$ exists and is finite, then both of $\tilde{R}f(x)$ and $R_\sigma f(x)$ exist and all three numbers are equal; the converse is false in general, although valid under certain mild restrictions (**8**).

If now $\tilde{R}f(x)$ exists, then (15) gives

$$(18)\qquad [f(x-h), \xi(x), \eta(x+k)] = \tfrac{1}{2}hk(h+k)\tilde{R}f(x) + O(h)o(k^2) + o(h^2)O(k),$$

and if $R_\sigma(x)$ exists (15-σ) gives

$$(18\text{-}\sigma)\qquad [f(x-h), \xi(x), \eta(x+h)] = h^3 R_\sigma f(x) + o(h^3).$$

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In particular from (12), (18), and (18-σ),

$$(19)\quad [p(x-h), \xi(x), \eta(x+k)] = \tfrac{1}{2}hk(h+k) + O(h)o(k^2) + o(h^2)O(k),$$

$$(19\text{-}\sigma)\qquad [p(x-h), \xi(x), \eta(x+h)] = h^3 + o(h^3).$$

THEOREM 12. (a) *If* $\tilde{R}f(x)$ *exists, then* $Df(x; \mathfrak{N}_\nu; p)$ *exists and has the same value.* (b) *If* $R_\sigma f(x)$ *exists, then* $Df(x; \mathfrak{N}_\sigma; p)$ *exists and has the same value.*

Proof. This is immediate, using (1), (13), (18) or (18-σ), and (19) or (19-σ).

It is easily seen from (15-σ) that if $f'_\sigma(x)$ exists and $Df(x; \mathfrak{N}_\sigma; p)$ exists and is finite, then $R_\sigma f(x)$ exists; a converse of part (a) is proved later (Theorem 15 below). These results show that the generalized derivative (1) is, in these cases, a generalization of (6). The main theorem of **(21)** gives conditions under which the generalized derivative reduces to R; it can, therefore, be taken as a generalization of certain results of Burkill **(8)**.

Now let $f_+(x), f^+(x), f_-(x), f^-(x)$ denote the four Dini derivatives of f at x.

THEOREM 13. *If* $f \in \mathfrak{C}(X)$ *and*

(a) *if* $f^+(x) > f_-(x)$, *then* $\uparrow Df(x; \mathfrak{N}_\nu; p) = \infty$,

(b) *if* $f_+(x) > f^-(x)$, *then* $\uparrow Df(x; \mathfrak{N}_\nu; p) = \infty$,

(c) *if* $f^-(x) > f_+(x)$, *then* $\downarrow Df(x; \mathfrak{N}_\nu; p) = -\infty$,

(d) *if* $f_-(x) > f^+(x)$, *then* $\downarrow Df(x; \mathfrak{N}_\nu; p) = -\infty$.

(*It is known that hypotheses* (b) *and* (d) *can only occur on an enumerable set,* **(9)**).

COROLLARY 14. (a) *If* $\uparrow Df(x; \mathfrak{N}_\nu; p)$ *is finite, then* $-\infty < f^+(x) = f_-(x) < \infty$.

(b) *If* $\downarrow Df(x; \mathfrak{N}_\nu; p)$ *is finite, then* $-\infty < f^-(x) = f_+(x) < \infty$.

(c) *If both* $\uparrow Df(x; \mathfrak{N}_\nu; p)$ *and* $\downarrow Df(x; \mathfrak{N}_\nu; p)$ *are finite, then* $-\infty < f'(x) < \infty$.

(d) *If either* $\uparrow Df(x; \mathfrak{N}_\nu; p)$ *or* $\downarrow Df(x; \mathfrak{N}_\nu; p)$ *is finite, then exactly one first-order two-sided derivative exists and it is finite.*

(e) *If the hypothesis* (d) *holds at all points of* X, *then* f *is* ACG* *on* X.

Proofs. These results generalize theorems of Denjoy **(9**, § 3**)** and the proofs follow his. In particular the proof of Theorem 13 follows from a consideration of

$$(20)\qquad \frac{\Delta f(x; V)}{\Delta p(x; V)} = \frac{1}{h+k}\left\{\frac{f(x+k)-f(x)}{k}[2+hr(x)]\right.$$
$$\left. + \frac{f(x)+f(x-h)}{h}[-2+kr(x)] + (h+k)s(x)f(x) + (h+k)o(1)\right\}[1+o(1)],$$

a formula resulting from (15) and (19).

THEOREM 15. *If* $f \in \mathfrak{C}(X)$ *and* $Df(x; \mathfrak{N}_\nu; p)$ *exists and is finite, then* $\tilde{R}f(x)$ *exists.*

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Proof. It follows from Corollary 14(c) that $f'(x)$ exists and is finite; hence if $h \to 0$ in (20), we have

$$\frac{2}{k}\left\{\frac{f(x+k)-f(x)}{k} - f'(x)\right\} + r(x)f'(x) + s(x)f(x) = Df(x; \mathfrak{N}_\nu; s) + o(1).$$

Thus the right-hand second-order Peano derivative exists at x with value $Df(x; \mathfrak{N}_\nu; p)$. By letting $k \to 0$ in (20) we can show similarly that the left-hand second-order Peano derivative exists with the same value. This completes the proof.

Modifying the above method would enable us to prove

THEOREM 16. *If $f \in \mathfrak{C}(X)$ and*

$$-\infty < \downarrow Df(x; \mathfrak{N}_\nu; p) \leqslant \uparrow Df(x; \mathfrak{N}_\nu; p) < \infty,$$

then $\uparrow \tilde{R}f(x) = \uparrow Df(x; \mathfrak{N}_\nu; p)$ *and* $\downarrow \tilde{R}f(x) = \downarrow Df(x; \mathfrak{N}_\nu; p)$.

If $f'(x)$ exists, write $\uparrow f_2(x) = \sup\{(f')^+(x), (f')^-(x)\}$ and

$$\uparrow Rf(x) = \uparrow f_2(x) + r(x)f'(x) + s(x)f(x).$$

Then we have

THEOREM 17. *If f is differentiable on X, then $\uparrow Rf(x) \geqslant \uparrow Df(x; \mathfrak{N}_\nu; p)$ for all $x \in X$.*

Proof. This is a generalization of results due to de la Vallée–Poussin and Denjoy (**9**, § 46). Following Denjoy, if $h > 0$, $k > 0$ are chosen small enough, it can be shown that

$$[f(x+k) - f(x)]/k \leqslant f'(x) + \tfrac{1}{2}k \uparrow f_2(x) + o(k), \qquad 2 + kr(x) \geqslant 0.$$

$$[f(x) - f(x-h)]/h \geqslant f'(x) - \tfrac{1}{2}h \uparrow f_2(x) + o(h), \qquad -2 + hr(x) \leqslant 0.$$

So, by (20),

$$\Delta f(x; V)/\Delta p(x; V) \leqslant \uparrow Rf(x) + o(1),$$

which gives the result.

3.4. Finally we show that under reasonable conditions the general $\mathfrak{H}$-integral associated with the present theory (see I), reduces to a Riemann, Lebesgue, or Perron integral. If (11) exists as a finite Riemann integral, we say that f is *G^V-Riemann integrable*, with similar definitions for other types of integrals.

If a derivative relative to $\mathfrak{N}_\nu$ is G^V-Riemann integrable, then the $\mathfrak{H}$-integral is a Riemann integral and can be calculated by elementary means using a method based on that of Denjoy (**9**, § 7).

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LEMMA 18. *If $1 \leqslant m \leqslant n$ and f_i, α_i, β_i, $i = 1, \ldots, n$, are real numbers, then*

$$[f_1, \alpha_m, \beta_n] = \sum_{i=1}^{n} A_i [f_{i-1}, \alpha_i, \beta_{i+1}], \tag{21}$$

where

$$A_i = \begin{cases} \dfrac{[\alpha_m, \beta_n][\alpha_1, \beta_i]}{[\alpha_{i-1}, \beta_i][\alpha_i, \beta_{i+1}]}, & i = 1, \ldots, m, \\ \dfrac{[\alpha_1, \beta_m][\alpha_i, \beta_n]}{[\alpha_{i-1}, \beta_i][\alpha_i, \beta_{i+1}]}, & i = m, \ldots, n, \end{cases}$$

provided all the A_i, $i = 1, \ldots, n$, are defined.

Proof. Elementary calculations give (21). Note that $A_1 = A_n = 0$ and that the two forms of A_m are identical.

LEMMA 19. *Let $x_1 < x_2 < \ldots < x_n$ be points of X, $\alpha_i = \alpha(x_i)$, $\beta_i = \beta(x_i)$, $i = 1, \ldots, n$, where α, β are independent solutions of* (6), *$V_i = (x_{i-1}, x_{i+1})$, $i = 2, \ldots, n-1$, $V = (x_1, x_n)$; then if $2 \leqslant m \leqslant n-1$,*

$$\Delta f(x_m; V) = \sum_{i=2}^{n-1} B_i \Delta f(x_i; V_i), \tag{22}$$

where

$$B_i = \begin{cases} \dfrac{[\alpha_m, \beta_n][\alpha_1, \beta_i][\alpha_{i-1}, \beta_{i+1}]}{[\alpha_1, \beta_n][\alpha_{i-1}, \beta_i][\alpha_i, \beta_{i+1}]}, & i = 2, \ldots, m, \\ \dfrac{[\alpha_1, \beta_m][\alpha_i, \beta_n][\alpha_{i-1}, \beta_{i+1}]}{[\alpha_1, \beta_n][\alpha_{i-1}, \beta_i][\alpha_i, \beta_{i+1}]}, & i = m, \ldots, n. \end{cases} \tag{23}$$

Proof. This is immediate from (21) and (9). Equation (22) generalizes the formulae (**9**, (2.7)).

THEOREM 20. *If f is a derivative relative to $\mathfrak{N}_\nu$, of F and if f is G^V-Riemann integrable, then for $t \in V$*

$$F(t) = H_F{}^V(t) - \int_V G^V(x, t) f(x)\, dx = \int_{V, F, t} f.$$

Proof. The equality on the right is just Theorem 1.16. From Corollary 9 it is unnecessary to specify which derivative relative to $\mathfrak{N}_\nu$ is taken (upper, lower, or other). The full details of the proof will not be given as it follows that of Denjoy.

Let $V = (c, d)$ be subdivided using a subdivision in which t is a point, i.e. $c = x_1 < \ldots < x_m = t < \ldots < x_n = d$. Then by (22) and the definition of $\Delta F(x; V)$ we have

$$F(t) = H_F{}^V(t) - \sum_{i=2}^{n-1} C_i \frac{\Delta F(x_i; V_i)}{\Delta p(x_i; V_i)} = H_F{}^V(t) - \sum_{i=2}^{n-1} C_i f_i,$$

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where for $i = 2, \ldots, n-1$, V_i is as in Lemma 19, $C_i = B_i \Delta p(x_i; V_i)$, B_i given by (23), and f_i is some number between $\sup_{x \in V_i} f(x)$ and $\inf_{x \in V_i} f(x)$; the last equality is obtained using Theorem 6. By Lemma 19 and by (14),

$$C_i = \frac{[\alpha_m, \beta_n][\alpha_1, \beta_i][\alpha_i, \beta'_i][p(x_{i-1}), \xi(x_i), \eta(x_{i+1})]}{[\alpha_1, \beta_m][\alpha_{i-1}, \beta_i][\alpha_i, \beta_{i+1}]}, \qquad i = 2, \ldots, m.$$

By (19) and noting that

$$[\alpha_{i-1}, \beta_i] = [\alpha_i, \beta'_i](x_i - x_{i-1}) + O(|x_i - x_{i-1}|^2)$$

we get, for $i = 2, \ldots, m$,

$$C_i = \frac{1}{2}\frac{[\alpha_m, \beta_n][\alpha_1, \beta_i]}{[\alpha_1, \beta_n][\alpha'_i, \beta_i]}(x_{i+1} - x_i) + \text{terms of smaller order.}$$

Similarly, if $i = m, \ldots, n-1$,

$$C_i = \frac{1}{2}\frac{[\alpha_1, \beta_m][\alpha_i, \beta_n]}{[\alpha_1, \beta_n][\alpha'_i, \beta_i]}(x_{i+1} - x_i) + \text{terms of smaller order.}$$

Hence, using (10),

$$F(t) = H_F{}^V(t) - \tfrac{1}{2}\sum G^V(x_i, t) f_i (x_{i+1} - x_i) + \text{terms of smaller order.}$$

Noting that, except for intervals eventually of small order, the summation covers V twice, the result follows by taking the limit in the usual way.

THEOREM 21. *If $f \in \mathfrak{F}(V)$* (see I) *and there exists a G^V-Lebesgue integrable g such that $f \geqslant g$ almost everywhere, then f is G^V-Lebesgue integrable and*

$$\text{(24)} \qquad \int_{V,\Phi,t} f = H_\Phi{}^V(t) - \int_V G^V(x, t) f(x)\, dx.$$

(This theorem and an obvious generalization of Theorem 1.16 imply the main result of **(21)**.)

Proof. It is sufficient to consider the case $g = 0$. Since 0 is then a minor function of f on V, if M is any major function of f on V, it is a real continuous hypoharmonic function on V by Lemma 1.8(i). Hence, by **(21**, Lemma 5.5**)**, $\downarrow DM$ is G^V-Lebesgue integrable. Further, by the definition of M, $\downarrow DM \geqslant f$ almost everywhere and so f is G^V-Lebesgue integrable. The rest of the theorem follows from the following more general theorem.

THEOREM 22. *If f is G^V-Perron integrable, then $f \in \mathfrak{F}(V)$ and* (24) *holds.*

Proof. The proof is a generalization of a method due to Denjoy **(9**, § 11, 12**)**; essentially, the generalization consists of constructing a second-order majorant without first constructing a first-order one, to use Denjoy's terminology.

It follows from Denjoy's work **(9**, § 11 and appendix 2.3**)** that we need only consider the case of f G^V-Lebesgue integrable. Further, as in I, it suffices to take $\Phi = 0$ in (24).

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The proof consists of constructing a sequence of major functions of f on V converging uniformly to the right-hand side of (24); since a similar construction will give a sequence of minor functions with the same limit, the definition of $\mathfrak{J}(V)$ implies (24).

Let $\gamma = \max_{t\in V}\int_V G^V(x, t)\, dx$, $\quad e_n = \{x; x \in V \text{ and } n \leqslant f(x) < n + 1\}$, E_n = an open neighbourhood of e_n in V such that

$$|E_n \sim e_n| < [1/(n+1)^3], \quad \phi_n = (n+1)1_{\bar{E}n}, \quad f_n = f1_{e_n}, n = 0, 1, 2, \ldots .$$

Define

$$m_n(t) = -\int_V G^V(x, t)\phi_n(x)\, dx.$$

Then since $\phi_n \geqslant 0$ and u.s.c. it follows that m_n is a non-positive hypoharmonic function on V; cf. Theorem 1.4 and **(21**, (4.5)**)**. If $t \in e_n$ and V' is a neighbourhood of t in E_n, then by **(21**, (4.4)**)**

$$\begin{aligned}\Delta m_n(t; V') &= -\int_{V'} \Delta G^V(x, \cdot)(t; V')\phi_n(x)\, dx\\ &= -(n+1)\int_{V'} \Delta G^V(x, \cdot)(t; V')\, dx = (n+1)\Delta p(t; V').\end{aligned}$$

So $\Delta m_n(t; V')/\Delta p(t; V') = n + 1 > f(t)$. Since in any case this latter ratio is non-negative, it exceeds f at any point of e_n or at any point where f is non-positive, provided that at least V' is small enough.

Let $M_0(t) = \sum_{n\geqslant 0} m_n(t)$, a convergent series for all $t \in V$; then it is easily seen that for all $t \in V$, $\downarrow DM_0(t) \geqslant f(t)$. The rest of the conditions being easily seen to hold, M_0 is a major function of f on V. Further, if

$$f(x; 0) = \sup(f(x), 0),$$

$$M_0(t) = -\int_V G^V(x, t)f(x, 0)\, dx + \sum_{n\geqslant 0}\left[-\int_V G^V(x, t)\{\phi_n(x) - f(x)\}\, dx\right].$$

Hence simple calculations show that

$$0 \geqslant M_0 + \int_V G^V(x, t)f(x; 0)\, dx \geqslant -3\gamma.$$

Repeat the above construction for $f^* = (f + n)/\epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$, $n = 1, 2, \ldots$. Then define $M_n(t) = \epsilon_n M_0^*(t) - np(x)$; again since

$$\downarrow DM_n(t) = \epsilon_n \downarrow DM_0^*(t) - n \geqslant \epsilon_n f^*(t) - n \geqslant f(t),$$

we have that M_n is a major function of f on V. Further if

$$f(x; n) = \sup(f(x), -n),$$

then $[f(t; n) + n]/\epsilon_n = f^*(x; 0)$ so

$$0 \geqslant M_0^*(t) + \int_V G^V(x, t)f^*(x; 0)\, dx \geqslant -3\gamma$$

implies that

$$0 \geqslant M_n(t) + \int_V G^V(x, t)f(x; n)\, dx \geqslant -3\epsilon_n \gamma \qquad \text{for } n = 0, 1, 2, \ldots .$$

Letting $n \to \infty$ we have that $M_n(t)$ converges uniformly to $-\int_V G^V(x, t)f(x)$ dx, as was to be proved.

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4. The solutions of an elliptic partial differential equation.

4.1. Since many of the methods used in this section are very similar to those used in the relevant parts of § 3, full details will not be given. Let X be a domain in R^n, $n \geqslant 1$, and $\mathfrak{H}(X)$ be the collection of functions h satisfying

$$(25) \qquad Lh(x) = \sum_{i,j=1}^{n} a_{ij}(x)h''_{ij}(x) + \sum_{i=1}^{n} b_i(x)h'_i(x) + c(x)h(x) = 0$$

for all $x \in X$, where $x = (x_1, \ldots, x_n)$; $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, n)$; the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)y_i y_j$$

is positive definite for all $x \in X$;

$$h''_{ij} = \partial^2 h/\partial x_i\, \partial x_j,\ h'_i = \partial h/\partial x_i \quad (i, j = 1, \ldots, n).$$

If all the functions a_{ij}, b_i, c are locally Lipschitz and X is small enough, then it is known that the axioms of the Bauer harmonic structure are satisfied **(2, 3, 13)**. Further they are satisfied with Axioms T_+, and K_D, so Axiom 1 holds. The regular sets certainly include all open balls in X and in fact all domains with smooth enough frontier. As in the previous section, if $u \in \mathfrak{C}^2(X)$, then $u \in \mathfrak{H}^*(X)$ if and only if $Lu \leqslant 0$; if $Lu < 0$, u is strictly locally hyperharmonic. The negligible sets are the classical polar sets **(13)** and subsets of a V^* are negligible if and only if they are of $\mu(V; x)$-measure zero for all $x \in V$. Further, for all $V \subset X$, there exists a non-negative Green's function $G^V(x; t)$ such that if $f \in \mathfrak{C}(X)$ the equation $LF = f$, with boundary condition $F(z) = 0$ for all $z \in V^*$, is satisfied by (11). So p can again be chosen as in (12) and if $\mathfrak{N}$ is given, (1) defines a generalized derivative.

If $F \in \mathfrak{C}^2(X)$ and $LF = f$, then by (11)

$$F(t) = H_F{}^V(t) - \int_V G^V(x, t)f(x)\,dx, \qquad t \in V;$$

so

$$\frac{\Delta F(t;\, V)}{\Delta p(\,t; V)} = \frac{\displaystyle\int_V G^V(x, t)f(x)\,dx}{\displaystyle\int_V G^V(x, t)\,dx} = f(t) + \epsilon,$$

where $\lim_{\mathfrak{N}(t)} \epsilon = 0$. Hence the generalized derivative $DF(t; \mathfrak{N}; p)$ is an extension of the operator LF and has the same class of harmonic functions.

We now restrict attention to the symmetric case when for all $x \in X$, $\mathfrak{N}(x)$ consists of the open balls in X with x as centre. To consider Axioms 2 and 1.6 we take $\mathfrak{E}$ to be the family of enumerable subsets, $\mathfrak{Z}$ to be the sets of measure zero, and $\downarrow\mathfrak{S}$ the collection of functions $\mathfrak{N}$-smooth on x.

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First suppose that L is the Laplacian. Then arguments similar to those in § 3 justify the choice of $\mathfrak{Z}$ and show that Axiom 1.6 is justified; in particular analogues of Lemma 10 and Corollary 11 can be proved using (**19**, 3.7 and 4.2.11) rather than (**21**).

To extend this to general L it is necessary to generalize the results of Rudin (**19**); this will be taken up elsewhere. As to the choice of $\mathfrak{E}$ and $\downarrow\mathfrak{S}$, this can be justified in certain cases by use of Theorem 3. First note that since X has an enumerable base, it is sufficient to show that the result holds locally. In particular it can then be assumed that $c \leqslant 0$ (**13**). To see that Theorem 3 applies we must check Axioms 1.2, 1.3, and 1.4. Axiom 1.2 is immediate since $h \in \mathfrak{H}(X)$ implies that $h \in \mathfrak{C}^2(X)$, and Axiom 1.3 is certainly valid locally. If now L is uniformly elliptic with $a_{i,j} \in \mathfrak{C}^{2,\lambda}(X)$, $b_i \in \mathfrak{C}^{1,\lambda}(X)$, then a generalization of the Poisson integral representation of harmonic functions exists (**18**, Theorem 10.1; **12**, Theorem 6; **13**, Theorem 35.1; and **5**, (6.5)). Thus in the case $n = 2$ if V is a ball of radius ρ and centre x, it follows from (**12**) and (**5**) that

$$H_f^V(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\rho, \theta) f(x + \rho e^{i\theta})\, d\theta,$$

where $k(\rho, \theta) = 1 + \rho k_1(\theta) + o(\rho) k_2(\rho, \theta)$. This implies Axiom 1.4, provided that $k_1 > 0$, or $k_1 = 0$ and $k_2 > 0$. The cases of $n > 2$ follow in a similar manner.

Thus, except for a determination of $\mathfrak{Z}$, the present theory applies to solutions of certain equations of type (25), at least if the coefficients are smooth enough. The unsymmetric derivatives seem harder to discuss since a Poisson integral representation is lacking. In particular it would be of interest to connect certain of them with the generalized derivatives discussed in (**18**).

4.2. In a certain sense the above case is typical. Let us consider a Bauer harmonic structure satisfying Axioms K_D and T_+ on a locally compact space with a denumerable base in which situation Axiom 1 holds. Bauer (**4**) has pointed out that a result of Meyer (**17**) implies that for every regular V there can be constructed a sub-Markovian Feller semi-group on the bounded Borel functions on V, with the excessive functions of the semi-group being just the non-negative superharmonic functions on V. With this semi-group can be associated a right-continuous left-quasi-continuous strong Markov process with values in the Alexandroff compactification of V (**11**, Chapter III). The characteristic operator of this process is an extension of the weak infinitesimal generator of the semi-group (**11**, Chapter V). If the process is continuous, the characteristic operator is a local differential operator (**11**, Chapter V) obeying the minimum principle (**11**, 5.12), and coinciding with a generalized derivative of the present theory introduced into the original harmonic structure (**11**, (5.27) and Theorem 13.7). If, further, the process is a diffusion process, its characteristic operator is an extension of an elliptic operator L with $c \leqslant 0$.

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This will occur if V is a differentiable manifold such that at all of its points there exists a coordinate system for which polynomials of degrees 0, 1, 2 in these coordinates are in the domain of the characteristic operator **(11)**.

Thus a very large class of the generalized derivatives in the theory are extensions of elliptic operators. The present approach is more direct than through the associated Markov processes and slightly more general, not being tied to the measurability conditions usual in that subject. The exact relationship between the generalized derivatives and the corresponding characteristic operators (or the infinitesimal generators) when the Markov process is not continuous has still to be determined.

5. Other examples.

5.1. It follows from **(2**, 6.1 and 6.3**)** that if X is a small enough domain of R^{n+1}, $n \geqslant 1$, and $\mathfrak{H}(X)$ is the collection of functions h satisfying

$$Ph(x) = Lh(x) - h'_{n+1}(x) = 0, \qquad \text{for all } x \in X, \tag{26}$$

where L is defined by (25) and subject to the restrictions introduced there, then the axioms of a Bauer harmonic structure are satisfied in a form that implies Axiom 1. The base of regular sets can be taken to be the $(n + 1)$-dimensional open equal-sided simplexes with one side in the hyperplane $x_{n+1} = t_0$ and the opposite vertex having the $(n + 1)$th coordinate $t_1 > t_0$. If V is then a regular domain, it is known that there exists a Green's function in V with the usual properties **(10)**. Thus p can again be chosen as in (12) and the resulting generalized derivative is an extension of P. The further refinements of the theory (sets $\mathfrak{E}$, $\mathfrak{Z}$, $\downarrow\mathfrak{S}$, and Axiom 1.6) require results concerning parabolic equations corresponding to **(19)**.

5.2. The existence of a Green's function is, under fairly general hypotheses, equivalent to the existence of a strictly positive non-constant superharmonic function **(14)**. This situation has been studied quite generally under the name of Green spaces **(6)**. A Green space is an example of a Bauer harmonic structure **(2**, 6.4**)** satisfying Axiom 1. Hence there is a generalized derivative on a Green space that can be used to characterize its harmonic and hyperharmonic functions.

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