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RESEARCH ARTICLE

Stable anisotropic minimal hypersurfaces in $\mathbf{R}^4$

Otis Chodosh [1] and Chao Li [2]

[1]Department of Mathematics, Stanford University, Building 380, Stanford, CA 94305, USA; E-mail: ochodosh@stanford.edu.
[2]Courant Institute, New York University, 251 Mercer St, New York, NY 10012, USA; E-mail: chaoli@nyu.edu.

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Abstract
We show that a complete, two-sided, stable immersed anisotropic minimal hypersurface in $\mathbf{R}^4$ has intrinsic cubic volume growth, provided the parametric elliptic integral is C^2-close to the area functional. We also obtain an interior volume upper bound for stable anisotropic minimal hypersurfaces in the unit ball. We can estimate the constants explicitly in all of our results. In particular, this paper gives an alternative proof of our recent stable Bernstein theorem for minimal hypersurfaces in $\mathbf{R}^4$. The new proof is more closely related to techniques from the study of strictly positive scalar curvature.

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1. Introduction

Consider $\Phi : \mathbf{R}^{n+1} \setminus \{0\} \to (0, \infty)$ a 1-homogeneous C^3_{loc} function (i.e. $\Phi(sv) = s\Phi(v)$ for $s > 0$). For $M^n \to \mathbf{R}^{n+1}$ a two-sided immersion (with chosen unit normal field $\nu(x)$), we can define the *anisotropic* *area functional*

$$\mathbf{\Phi}(M) = \int_M \Phi(\nu(x))\, d\mu.$$

Surfaces minimising the $\mathbf{\Phi}$-functional arise as the equilibrium shape of crystalline[1] materials, as well as scaling limits of Ising and percolation models (see [7, Chapter 5]). We say that M is $\mathbf{\Phi}$-*stationary* if $\frac{d}{dt}\big|_{t=0}\mathbf{\Phi}(M_t) = 0$ for all compactly supported variations of M (fixing ∂M) and that M is $\mathbf{\Phi}$-*stable* if in addition $\frac{d^2}{dt^2}\big|_{t=0}\mathbf{\Phi}(M_t) \geq 0$ for such variations. Note that if $\Phi(\nu) = |\nu|$, $\mathbf{\Phi}$ reduces to the n-dimensional area functional and a $\mathbf{\Phi}$-stable hypersurface is known as a *stable minimal hypersurface*. We say that $\mathbf{\Phi}$ is *elliptic* if the Φ-unit ball $\Phi^{-1}((0,1]) \cup \{0\}$ is uniformly convex.

This article is motivated by the following questions:

Question 1.1. For an anisotropic elliptic functional $\mathbf{\Phi}$, is the flat hyperplane $\mathbf{R}^n \subset \mathbf{R}^{n+1}$ the only complete, two-sided $\mathbf{\Phi}$-stationary and stable immersion in $\mathbf{R}^{n+1}$?

Question 1.2. If $M^n \to \mathbf{R}^{n+1}$ is a complete, two-sided $\mathbf{\Phi}$-stationary and stable immersion (for some anisotropic elliptic functional $\mathbf{\Phi}$), does M satisfy the intrinsic polynomial volume growth condition $\mathrm{Vol}(B_M(p,\rho)) \leq C\rho^n$?

By a well-known blowup argument, an affirmative answer to Question 1.1 yields a priori interior curvature estimates for $\mathbf{\Phi}$-stable immersions with boundary, and even for stable immersion with respect to a *parametric elliptic integrand* (where Φ is allowed to also depend on x). We also note that for minimal surfaces, one can derive *lower* polynomial growth bounds (both intrinsic and extrinsic), but for general $\mathbf{\Phi}$-stationary surfaces, no monotonicity type formula is known (cf. [3, 16, 17]) (on the other hand, stability can be used to derive a *lower* volume growth estimate; see Corollary 3.3 and [17]).

For the area functional, Question 1.1 (and thus, Question 1.2) has been completely resolved in the affirmative when $n = 2$ (independently) by Fischer-Colbrie and Schoen, do Carmo and Peng and Pogorelov [19, 23, 39] (see also [42]) and recently, when $n = 3$ by the authors [10]. In particular, we recall the result of Pogorelov (yielding a localised volume growth estimate).

Theorem 1.3 [39], cf. [51, Lemma 34], [36, Theorem 2]. *Suppose that $M^2 \to \mathbf{R}^3$ is a stable minimal immersion so that the intrinsic ball $B_M(p,R) \subset M$ has compact closure in M and is topologically a disk. Then*

$$|B_M(p,\rho)| \leq \frac{4}{3}\pi\rho^2.$$

On the other hand, Questions 1.1 and 1.2 remain open (even for the area functional) for $n = 4, 5, 6$. There exist nonflat stable minimal hypersurfaces (area minimisers) in $\mathbf{R}^8$ and beyond [4, 28] (thus answering Question 1.1 in the negative), but all known examples satisfy the conclusion of Question 1.2. Note that Schoen, Simon and Yau [41] (cf. [43, 47, 52]) have shown that when $n \leq 5$, a complete, two-sided stable minimal immersion satisfying the volume growth condition in Question 1.2 must be flat.

For arbitrary elliptic functionals, there are nonflat minimisers for $n \geq 3$ [34, 35], but as in the case of area, all known examples satisfy the intrinsic volume growth condition in Question 1.2. When $n = 2$, Question 1.1 is open for general elliptic functionals but is resolved in the affirmative assuming quadratic area growth (as shown by White [50]) or assuming the functional is sufficiently C^2-close[2] to area (as shown by Lin [30]; see also [29, 46]). Still, for $n = 2$, Colding and Minicozzi [14] have given a new proof of Theorem 1.3 that extends to show that Question 1.2 holds in the affimative for functionals sufficiently C^2-close to area. When $n \geq 3$, Question 1.1 is answered in the negative by considering the nonflat area minimising solutions constructed by Mooney and Yang [34] (see also [33, 35]). On the other hand, Winklmann [53] has resolved Question 1.1 in the affirmative for $n \leq 5$ under the assumptions that

[1]We note that in the crystalline setting, Φ is usually only Lipschitz continuous.

[2]Throughout this article, Φ *is $C^{k,\alpha}$-sufficiently close to area* will mean that $\|\Phi - 1\|_{C^{k,\alpha}(S^n)} \leq \varepsilon(n)$ for some fixed $\varepsilon(n) > 0$.

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the functional is sufficiently C^4-close to area and that the surface satisfies the growth condition from Question 1.2.

1.1. Main results

In this article, we consider the volume growth problem (Question 1.2) for $\mathbf{\Phi}$-stable hypersurfaces $\mathbf{R}^4$. In fact, the estimate we prove here is new even in the case of stable minimal hypersurfaces. As such, it yields an alternative approach to our recent result [10] (this is discussed further in Section 2).

We note that all constants in this paper can be given explicitly, see Remark 1.9.

Theorem 1.4. *Assume that Φ satisfies*

$$|v|^2 \leq D^2\Phi(\nu)(v,v) \leq \sqrt{2}|v|^2, \tag{1.1}$$

for all $v \in \nu^\perp$. Consider $M^3 \to \mathbf{R}^4$ a complete, two-sided, $\mathbf{\Phi}$-stationary and stable immersion. Suppose $0 \in M$ and M is simply[3] *connected. Then there exist explicit constants $V_0 = V_0(\|\Phi\|_{C^1(S^3)}), Q > 0$, such that*

(i) $|B_M(0,\rho)| \leq V_0\rho^3$, *for all $\rho > 0$.*
(ii) *For each connected component Σ_0 of $\partial B_M(0,\rho)$, we have*

$$\max_{x\in\Sigma_0} r(x) \leq Q \min_{x\in\Sigma_0} r(x),$$

where $r(x) := d_{\mathbf{R}^4}(0,x)$.

Note that (1.1) implies that $\nu \mapsto \Phi(\nu)$ is convex (since $D^2\Phi(\nu)(\nu,\nu) = 0$ by 1-homogeneity). As such, all $\mathbf{\Phi}$ considered in Theorem 1.4 satisfy the ellipticity condition mentioned previously.

We note that by combining Theorem 1.4 with [53], we obtain:

Corollary 1.5. *If Φ is C^4-sufficiently close to area, then any two-sided, complete $\mathbf{\Phi}$-stationary and stable immersion is flat.*

Remark 1.6. Although it is not explicitly done in [53], the 'sufficiently close' requirement can be quantified. Alternatively, we note that by combining Theorem 1.4 with a contradiction argument in the spirit of [46], Corollary 1.5 actually holds under the weaker assumption of $C^{2,\alpha}$-closeness (but with no numerical estimate of the required closeness).

We can also prove a localised version of Theorem 1.4 more in the spirit of Pogorelov's result (cf. Theorem 1.3). The estimate we prove here is slightly different, since it considers extrinsic balls, but is an interior[4] estimate. Even for stable minimal surfaces, we are not aware of such an estimate in $\mathbf{R}^3$ with explicit[5] constants, cf. Remark 1.9.

Theorem 1.7. *Suppose that Φ satisfies (1.1). Assume $M^3 \to B_1(0) \subset \mathbf{R}^4$ is a proper, two-sided $\mathbf{\Phi}$-stationary and stable immersion. Suppose $0 \in M$, M is simply connected and ∂M is connected. Then there exist explicit constants $\rho_0 \in (0,1), V_1 = V_1(\|\Phi\|_{C^1(S^3)})$, such that*

$$|M^*_{\rho_0}| \leq V_1,$$

where $M^*_{\rho_0}$ is the connected component of $M \cap B_{\mathbf{R}^4}(0,\rho_0)$ that contains 0.

[3]We note that a standard argument (cf. [23]) shows that if $M^3 \to \mathbf{R}^4$ is a complete, two-sided $\mathbf{\Phi}$-stable immersion, then so is the immersion from the universal cover.

[4]As observed in [27, Section 1], the bridge principle for stable minimal surfaces [31] implies that there cannot be an estimate for the area of a proper stable minimal immersion $M^2 \to B_1(0) \subset \mathbf{R}^3$, even if M is topologically constrained to be a disk.

[5]Given an area-free curvature estimate (available for minimal surfaces when $n = 2, 3$ [10, 42]), one can prove an extrinsic interior Pogorelov result [39] in the spirit of Theorem 1.7 by a straightforward contradiction argument (with no control on the constant). The method used here gives an alternative proof of this curvature estimate (and extends to certain elliptic integrands) and yields explicit (and not too large) constants.

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Remark 1.8. More generally, we can drop the requirement that M is simply connected and ∂M is connected. In this case, we have:

$$|M^*_{\rho_0}| \le V_1(b_1(M) + E),$$

where E is the number of boundary connected components of M and $b_1(M)$ is the first Betti number.

Remark 1.9. One may explicitly compute the constants V_0, V_1, Q, ρ_0 as follows. Let

$$\begin{aligned} c_0 &= \frac{1}{\sqrt{2} - \frac{1}{2}}, \\ \lambda &= \frac{3}{2}\left(\frac{1}{2} - \frac{3(c_0 - 1)}{8(\frac{1}{\sqrt{2}} - \frac{1}{2})}\right) = \frac{3(5 + 3\sqrt{2})}{56} \approx 0.495. \end{aligned}$$

Then we have

$$V_0 = \frac{8\pi e^{\frac{15\pi}{\lambda}} \|\Phi\|_{C^1(S^3)}}{3\lambda \min_{\nu \in S^3} \Phi(\nu)}, \quad Q = e^{\frac{7\pi}{\sqrt{\lambda}}}$$

and

$$\rho_0 = e^{-\frac{5\pi}{\sqrt{\lambda}}}, \quad V_1 = \frac{8\pi \|\Phi\|_{C^1(S^3)}}{3\lambda \min_{\nu \in S^3} \Phi(\nu)}.$$

1.2. Related work

We recall, here, some works (beyond those mentioned above) that are related to this paper. The regularity of hypersurfaces *minimising* parametric elliptic integrands has been studied in several places including [20, 21, 40, 44]. See also [1, 48] for estimates without the minimising hypothesis. Existence of critical points of parametric elliptic integrands has been considered in [15, 49, 50]. Finally, we note that stable solutions for the nonlocal area functional satisfy an a priori growth estimate (as in Question 1.2) in all dimensions [13] (see also [22]).

1.3. Notation

We will use the following notation:

- $B_{\mathbf{R}^{n+1}}(0, \rho) := \{x \in \mathbf{R}^{n+1} : |x| < \rho\}$.
- $r(x) = \mathrm{dist}_{\mathbf{R}^{n+1}}(0, x)$.
- $M^n \to \mathbf{R}^{n+1}$ is an immersion and g the induced Riemannian metric on M.
- D is the connection in $\mathbf{R}^{n+1}$, ∇ is the induced connection on M.
- μ is the volume form of g.
- $B_M(0, \rho) := \{x \in M : \mathrm{dist}_{M,g}(0, x) < \rho\}$.
- ν is a choice of unit normal vector field of M.
- The shape operator will be written $S = \nabla\nu$ and the second fundamental form written $A(X, Y) = S(X) \cdot Y$.
- The scalar curvature of g will be denoted by R.
- We will use the ℓ^2-norm to define C^k-norms, that is $\|f\|_{C^k} := (\sum_{j=0}^k \|D^{(j)} f\|^2_{C^0})^{\frac{1}{2}}$.

1.4. Organisation of the paper

In Section 2, we explain the techniques used in this paper in the special case of the area functional. The remaining part of the paper contains the details necessary for the generalisation to anisotropic

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integrands. We begin in Section 3 with some preliminary results. Section 4 contains a generalisation of the one-ended result for stable minimal hypersurfaces due to Cao, Shen and Zhu to the case of certain anisotropic integrands. We describe the conformally changed metric in Section 5 as introduced by Gulliver–Lawson and then combine these techniques with μ-bubbles to prove the main results in Section 5. Appendix A contains (well-known) computations of the first and second variation for elliptic integrands. Appendix B contains an auxiliary result comparing certain quadratic forms.

2. Volume growth for stable minimal hypersurfaces in $\mathbf{R}^4$

In this section, we illustrate how one may use stability to deduce area estimates for stable minimal immersions $M^3 \to \mathbf{R}^4$. We will defer certain ancillary results and computation to later sections (where they were carried out for general Φ-stationary and stable hypersurfaces) and instead focus on the geometric ideas and consequences.

The main result we will prove here is as follows:

Theorem 2.1. *Let $M^3 \to \mathbf{R}^4$ be a complete, two-sided, simply connected, stable minimal immersion, $0 \in M$. Then,*

$$|B_M(0,\rho)| \le \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{e^{\frac{30\pi}{\sqrt{3}}}}{6\sqrt{\pi}}\rho^3,$$

for all $\rho \ge 0$.

Combined with the work of Schoen et al. [41], this yields a new proof of our recent result [10]:

Corollary 2.2. *Any complete, two-sided, stable minimal immersion $M^3 \to \mathbf{R}^4$ is flat.*

In fact, we have the following localised volume estimate in the spirit of Theorem 1.3.

Theorem 2.3. *Let $M^3 \to \mathbf{R}^4$ be a two-sided, simply connected stable minimal immersion, with $0 \in M$, ∂M connected and $M \to B_{\mathbf{R}^4}(0,1)$ proper. Then,*

$$|M^*_{\rho_0}| \le \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{1}{6\sqrt{\pi}},$$

where $M^*_{\rho_0}$ is the connected component of $M \cap B_{\mathbf{R}^4}(0,r_0)$ that contains 0 and $\rho_0 = e^{-\frac{10\pi}{\sqrt{3}}}$.

Proof of Theorem 2.1. The first step is to consider a particular conformal deformation of (M,g). On $M \setminus \{0\}$, consider the conformally deformed metric $\tilde{g} = r^{-2}g$ (where we recall that r is the *Euclidean* distance to the origin and g is the induced metric on M). We use $\tilde{\nabla}$, $\tilde{\mu}$, $\tilde{\Delta}$ to denote the covariant derivative, the volume form and the Laplacian with respect to $\tilde{g}$, respectively. This conformal change was first used by Gulliver and Lawson [27] to study isolated singularities for minimal hypersurfaces in $\mathbf{R}^{n+1}$.

Remark 2.4. The relevance of the Gulliver and Lawson conformal deformation is a key insight in our work. Indeed, this allows us to apply tools from the study of *strictly* positive scalar curvature (cf. Remark 2.5). Our previous proof of Corollary 2.2 used tools from *nonnegative* scalar curvature (cf. [37, 38]).[6]

The computations in this part work for minimal immersions $M^n \to \mathbf{R}^{n+1}$ whenever $n \ge 3$. For $\lambda \in \mathbf{R}$, $\varphi \in C^1_0(M \setminus \{0\})$, consider the quadratic form

$$\mathcal{Q}(\varphi) := \int_M \left(|\tilde{\nabla}\varphi|^2_{\tilde{g}} + (\tfrac{1}{2}\tilde{R} - \lambda)\varphi^2\right) d\tilde{\mu},$$

[6]Added in proof: some time after this paper appeared, Catino et al. found a third proof of Corollary 2.2, based on a surprising connection between stability and *nonnegative Bakry–Émery Ricci curvature* [24].

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where $\tilde{R}$ is the scalar curvature of $\tilde{g}$. One computes (see Section 5 for details) that

$$\begin{aligned}\mathcal{Q}(r^{\frac{n-2}{2}}\varphi) &= \int_M \left(r^2|\nabla(r^{\frac{n-2}{2}}\varphi)|^2 + (\tfrac{1}{2}\tilde{R}-\lambda)r^{n-2}\varphi^2\right) r^{-n}\,d\mu \\ &= \int_M \left(|\nabla\varphi|^2 + \tfrac{1}{2}R\varphi^2 + \left(\frac{n}{2}\left(n - \frac{n+2}{2}|\nabla r|^2\right) - \lambda\right) r^{-2}\varphi^2\right) d\mu \\ &\geq \int_M \left(|\nabla\varphi|^2 + \tfrac{1}{2}R\varphi^2 + \left(\frac{n(n-2)}{4} - \lambda\right)\right) d\mu.\end{aligned}$$

By the (traced) Gauss equations, minimality of M implies that $|A_M|^2 = -R_g$. Thus, we can use stability of M to conclude

$$\int_M \left(|\nabla\varphi|^2 - |A|^2\varphi^2\right) d\mu \geq 0 \quad \Rightarrow \quad \int_M \left(|\nabla\varphi|^2 + \frac{1}{2}R\varphi^2\right) d\mu \geq 0,$$

for all $\varphi \in C_0^1(M)$. Note that we have used the fact that the scalar curvature of a minimal hypersurface in $\mathbf{R}^{n+1}$ has $R \leq 0$ and that $\frac{1}{2} < 1$. In particular, choosing $\lambda = \frac{n(n-2)}{4}$ above, we find that $\mathcal{Q}(\varphi) \geq 0$ for any $\varphi \in C_0^1(M \setminus \{0\})$. Using [23, Theorem 1], there exists $u \in C^\infty(M \setminus \{0\})$, $u > 0$ in the interior of $M \setminus \{0\}$, such that

$$\tilde{\Delta} u \leq -\frac{1}{2}\left(\frac{n(n-2)}{2} - \tilde{R}\right) u. \tag{2.1}$$

We note that (2.1) is an integral form of *strictly positive* scalar curvature.

In the second step, we restrict to the case of $n = 3$. We use warped μ-bubbles to derive geometric inequalities for 3-manifolds (N^3, g) admitting a positive function u with (2.1).

Remark 2.5. The μ-bubble technique was first used by Gromov [8, Section $5\frac{5}{6}$] (see also [25]). Warped μ-bubbles have previously been combined with minimal hypersurface techniques to study problems in scalar curvature and in minimal surfaces (see, e.g. [9, 11, 12, 26, 55, 56]). Precisely, suppose $n = 3$ and $\partial N \neq \emptyset$. Then there exists an open set Ω containing ∂N, $\Omega \subset B_{\frac{10\pi}{\sqrt{3}}}(\partial N)$, such that each connected component of $\partial\Omega \setminus \partial N$ is a 2-sphere with area at most $\frac{32\pi}{3}$ and intrinsic diameter at most $\frac{4\pi}{\sqrt{3}}$ (see Lemma 6.1).

Fix $\rho > 0$. By [6], $M \setminus B_M\left(0, e^{\frac{10\pi}{\sqrt{3}}}\rho\right)$ has only one unbounded component E. Denote by $M' = M \setminus E$. We apply Remark 2.5 to $N = M'$ and find $M_0 \subset M'$ with $\mathrm{dist}_{\tilde{g}}(\partial M_0, \partial M') \leq \frac{10\pi}{\sqrt{3}}$. The topological assumptions on M force ∂M_0 to be connected, so $|\partial M_0|_{\tilde{g}} \leq \frac{32\pi}{3}$ and ∂M_0 has intrinsic diameter $\leq \frac{4\pi}{\sqrt{3}}$. By comparing g-distance with $\tilde{g}$-distance (see (2) in Lemma 6.2), we find that

$$B_M(0,\rho) \subset M_0 \subset B_M(0, e^{\frac{10\pi}{\sqrt{3}}}\rho).$$

In particular, bounding intrinsic distance by extrinsic distance, we see that $\sup_{\partial M_0} r(x) \leq e^{\frac{10\pi}{\sqrt{3}}}\rho$. Thus, we have

$$|B_M(0,\rho)| \leq |M_0| \leq \frac{1}{6\sqrt{\pi}}|\partial M_0|_g^{\frac{3}{2}} \leq \frac{1}{6\sqrt{\pi}}(e^{\frac{10\pi}{\sqrt{3}}}\rho)^3|\partial M_0|_{\tilde{g}}^{\frac{3}{2}} \leq \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{e^{\frac{30\pi}{\sqrt{3}}}}{6\sqrt{\pi}}\rho^3,$$

where in the second step, we have used the isoperimetric inequality for minimal hypersurfaces in Euclidean spaces due to Brendle [5] (cf. [32]). This completes the proof. □

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We now consider the requisite changes needed to prove the local result:

Proof of Theorem 2.3. In the case where M is properly immersed in $B_1(0) \subset \mathbf{R}^4$, we proceed similarly as before and obtain a region M', such that $\mathrm{dist}_{\tilde{g}}(\partial M', \partial B_1(0)) \leq \frac{10\pi}{\sqrt{3}}$, $|\partial M'|_{\tilde{g}} \leq \frac{32\pi}{3}$ and $\partial M'$ is connected. Again, using Lemma 6.2, we conclude that

$$M^*_{\rho_0} \subset M',$$

where $\rho_0 = e^{-\frac{10\pi}{\sqrt{3}}}$ and $M^*_{\rho_0}$ is the connected component of $M \cap B_{\mathbf{R}^4}(0, \rho_0)$ that contains 0. Using [5] as above,

$$|M^*_{r_0}| \leq |M'| \leq \frac{1}{6\sqrt{\pi}} |\partial M'|^{\frac{3}{2}}_{g} \leq \frac{1}{6\sqrt{\pi}} |\partial M'|^{\frac{3}{2}}_{\tilde{g}} \leq \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{1}{6\sqrt{\pi}}.$$

This completes the proof. □

3. Preliminaries on anisotropic integrands

We now consider a general anisotropic elliptic integrand. For $M^n \to \mathbf{R}^{n+1}$ two-sided immersion, we can set

$$\mathbf{\Phi}(M) = \int_M \Phi(\nu(x))\, d\mu.$$

In this section, we discuss the first and second variation formulae, as well as some important consequences to be used later.

3.1. First variation

Recall that M is $\mathbf{\Phi}$-stationary means that $\frac{d}{dt}\big|_{t=0} \mathbf{\Phi}(M_t) = 0$ for all compactly supported variations M_t fixing ∂M. By (A.1), (A.2), (A.3) this is equivalent to

$$\mathrm{div}_M(D\Phi(\nu)) = \mathrm{tr}_M(\Psi(\nu) S_M) = 0,$$

which we can interpret as vanishing of the $\mathbf{\Phi}$-mean curvature. Here, $\Psi(\nu) : T\mathbf{R}^{n+1} \to T\mathbf{R}^{n+1}$ is defined by $\Psi(\nu) : X \mapsto D^2\Phi(\nu)[X, \cdot]$ and S_M is the shape operator of M.

By the calculation in Section A.3, we find that if M is $\mathbf{\Phi}$-stationary, then for any compactly supported (but not necessarily normal) vector field X along Σ, we have

$$\int_M \Phi(\nu)\, \mathrm{div}_M X + D_{D\Phi(\nu)^T} X \cdot \nu = \int_{\partial M} \Phi(\nu) X \cdot \eta + (X \cdot \nu) D\Phi(\nu) \cdot \eta. \tag{3.1}$$

By plugging the position vector field into (3.1), we obtain the following isoperimetric type inequality.

Corollary 3.1. *Suppose $M^n \to \mathbf{R}^{n+1}$ is $\mathbf{\Phi}$-stationary and the image of ∂M is contained in $B_{\mathbf{R}^{n+1}}(0, \rho)$ for some $\rho > 0$. Then*

$$|M| \leq \frac{\rho \|\Phi\|_{C^1(S^n)}}{n \cdot \min_{\nu \in S^n} \Phi(\nu)} |\partial M|.$$

Proof. Recall that $r(x) = \mathrm{dist}_{\mathbf{R}^{n+1}}(x, 0)$. Plug $X = \sum_{i=1}^{n+1} x_i e_i$, the position vector field in $\mathbf{R}^{n+1}$, into (3.1). Then $\mathrm{div}_M X = n$ and

$$D_{D\Phi(\nu)^T} X \cdot \nu = \sum_i (D_{D\Phi(\nu)^T} x_i)(e_i \cdot \nu) = \sum_i (D\Phi(\nu)^T \cdot e_i)(e_i \cdot \nu) = D\Phi(\nu)^T \cdot \nu = 0.$$

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On the other hand, $|X(x)| \leq r(x)$. Thus, we find (using ν, η orthonormal)

$$\int_M n\Phi(\nu) \leq \int_{\partial M} \|\Phi\|_{C^1(S^n)}|X| \leq \rho\|\Phi\|_{C^1(S^n)}|\partial M|.$$

This completes the proof. □

The next lemma generalises the traced Gauss equation $R = -|A|^2$ (valid for minimal hypersurfaces) to the case of $\mathbf{\Phi}$-stationary hypersurfaces in $\mathbf{R}^4$, under the assumption that $D^2\Phi(\nu)$ is sufficiently pinched.

Lemma 3.2. *Suppose Φ satisfies (1.1) and $M^3 \to \mathbf{R}^4$ is $\mathbf{\Phi}$-stationary. Then at each point on M, the induced scalar curvature satisfies $R \leq 0$ and*

$$-R \leq |A|^2 \leq -c_0 R, \tag{3.2}$$

where

$$c_0 = \frac{1}{\sqrt{2} - \frac{1}{2}} \approx 1.09.$$

Proof. Recall that $\mathbf{\Phi}$-stationarity can be written as $\mathrm{tr}_M(\Psi(\nu)S_M) = 0$. Diagonalising A_M at a given point, write k_i for the principal curvatures of M and e_i for corresponding principal directions. Thus, $\mathbf{\Phi}$-stationarity can be written as

$$0 = \sum_{i=1}^{3} a_i k_i,$$

where $a_i = D^2\Phi(\nu)[e_i, e_i]$. Without loss of generality, we can assume that $a_1 \leq a_2 \leq a_3$. Note that the pinching assumption (1.1) yields

$$1 \leq a_1 \leq a_2 \leq a_3 \leq \sqrt{2}.$$

We have $|A|^2 = \sum k_i^2$, $R = 2\sum_{i<j} k_i k_j$. Writing $k_3 = -\frac{a_1 k_1 + a_2 k_2}{a_3}$, we have

$$|A|^2 = Q_1(k_1, k_2) := \frac{a_1^2 + a_3^2}{a_3^2} k_1^2 + \frac{2a_1 a_2}{a_3^2} k_1 k_2 + \frac{a_2^2 + a_3^2}{a_3^2} k_2^2,$$

$$-R = Q_2(k_1, k_2) := \frac{2a_1}{a_3} k_1^2 + \frac{2(a_1 + a_2 - a_3)}{a_3} k_1 k_2 + \frac{2a_2}{a_3} k_2^2.$$

By the Gauss equation, we have $R + |A|^2 = H^2 \geq 0$, and hence, $|A|^2 \geq -R$. Moreover, whenever $(a_1 + a_2 - a_3)^2 < 4a_1 a_2$ (which is guaranteed by, for instance, $a_3 < 4a_1$), Q_2 is a positive definite quadratic form, and hence, $-R$ is nonnegative. Given that $\frac{a_3}{a_1}, \frac{a_3}{a_2} \in [1, \sqrt{2}]$, (3.2) follows from Appendix B. □

3.2. *Second variation*

Suppose now that $M^n \to \mathbf{R}^{n+1}$ is $\mathbf{\Phi}$-stationary and stable. In Section A.2, we derive the following second variation formula.

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathbf{\Phi}(M_t) = \int_M \langle \nabla u, \Psi(\nu)\nabla u\rangle - \mathrm{tr}_M\left(\Psi(\nu)S_M^2\right)u^2, \tag{3.3}$$

where $u\nu$ is the variation vector field. Note that stability and (3.3) imply that

$$\int_M |\nabla u|^2 - \Lambda |A|^2 u^2 \geq 0, \tag{3.4}$$

for all $u \in C^1_c(M \setminus \partial M)$. Here, Λ depends on the ellipticity of $\mathbf{\Phi}$. It is important to observe that if Φ satisfies (1.1), then $\Lambda \geq \frac{1}{\sqrt{2}}$, and in particular,

$$\int_M |\nabla u|^2 - \frac{1}{\sqrt{2}} |A|^2 u^2 \geq 0, \tag{3.5}$$

for all $u \in C^1_c(M \setminus \partial M)$.

3.3. Sobolev inequality and its consequences

In this section, we assume that $n \geq 3$, M^n is a two-sided $\mathbf{\Phi}$-stationary and stable hypersurface immersed in $\mathbf{R}^{n+1}$, where $\mathbf{\Phi}$ is a general anisotropic elliptic integral. The Michael–Simon Sobolev inequality [32] implies that for any $f \in C^1_c(M)$,

$$C_n \left(\int_M |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int_M |\nabla f| + |fH|$$

(see also [5]).

Replacing f by $f^{\frac{2(n-1)}{n-2}}$, we find:

$$C_n \left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-1}{n}} \leq \int_M \frac{2(n-1)}{n-2} |f|^{\frac{n}{n-2}} |\nabla f| + |f|^{\frac{2(n-1)}{n-2}} |H|. \tag{3.6}$$

By the Hölder inequality,

$$\int_M |f|^{\frac{2(n-1)}{n-2}} |H| \leq \left(\int_M f^2 H^2 \right)^{\frac{1}{2}} \left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{1}{2}}.$$

The Φ-stability inequality implies

$$\int_M f^2 H^2 \leq n \int_M f^2 |A|^2 \leq C(\Phi) \int_M |\nabla f|^2.$$

Now we use the Hölder inequality on the first term of the right hand of (3.6) and conclude the following Sobolev inequality:

$$\left(\int_M |f|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C(n, \Phi) \int_M |\nabla f|^2. \tag{3.7}$$

Corollary 3.3. *Suppose $M^n \to \mathbf{R}^{n+1}$ is $\mathbf{\Phi}$-stationary and stable. Assume that $B_M(p, \rho) \subset M$ has compact closure. Then,*

$$|B_M(p, \rho/2)| \geq C(n, \Phi) \rho^n.$$

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Proof. For any $u \in C^1(M)$, such that $u \geq 0$ and $\Delta u \geq 0$, the Sobolev inequality (3.7) and the standard Moser iteration implies that, for any $\theta \in (0, 1)$, $s > 0$,

$$\sup_{B_M(p,\theta\rho)} u \leq C(n, \theta, \Phi, s) \left(\rho^{-n} \int_{B_M(p,\rho)} u^s\right)^{1/s}.$$

The result follows by taking $u = 1$, $s = 1$ and $\theta = \frac{1}{2}$. □

Remark 3.4. The use of Sobolev inequality for volume lower bound was first used by Allard [2, Section 7.5].

Corollary 3.5. *Suppose $M^n \to \mathbf{R}^{n+1}$ is two-sided complete, $\mathbf{\Phi}$-stationary and stable, and K is a compact subset of M. Then each unbounded component of $M \setminus K$ has infinte volume.*

Proof. Let E be an unbounded component of $M \setminus K$. Suppose the contrary, that $|E| < V < \infty$. Choose ρ, such that $C(n, \Phi)\rho^n > V$. By completeness, there exists $p \in E$, such that $d_M(p, \partial E) > \rho$. Then we have

$$V > |E| > |B_M(p, \rho)| > C(n, \Phi)\rho^n > V,$$

a contradiction. This completes the proof. □

Combining (3.7) and Corollary 3.5, the same argument as used by Cao et al. [6] implies the following result:

Corollary 3.6. *If $M^n \to \mathbf{R}^{n+1}$ is a complete two-sided, $\mathbf{\Phi}$-stationary and stable immersion with at least two ends, then there is a bounded nonconstant harmonic function on M with finite Dirichlet energy.*

4. One-endedness

Through this section, we assume that $n = 3$, $M^3 \to \mathbf{R}^4$ is $\mathbf{\Phi}$-stationary and stable. By analysing harmonic functions on M, we will show that M has only one end, if $\mathbf{\Phi}$ satisfies (1.1) (following [6, 45]).

Lemma 4.1. *Suppose that M^3 is a complete two-sided, $\mathbf{\Phi}$-stationary and stable immersion in $\mathbf{R}^4$ and u is a harmonic function on M. Then*

$$(\Lambda - \tfrac{1}{\sqrt{2}}) \int_M \varphi^2 |A|^2 |\nabla u|^2 + \frac{1}{2} \int_M \varphi^2 |\nabla |\nabla u||^2 \leq \int_M |\nabla \varphi|^2 |\nabla u|^2, \tag{4.1}$$

for any $\varphi \in C_0^1(M)$. Here, $\Lambda = \Lambda(\Phi)$ is the constant in (3.4).

Proof. Fix $p \in M$. Let k_i be the principal curvatures, e_i be the corresponding orthonormal principal directions diagonalising A_M.

We first show that for any immersed hypersurface M^3 in $\mathbf{R}^4$, equipped with the induced metric, $p \in M$, and any unit vector $v \in T_pM$, we have

$$\mathrm{Ric}(v, v) \geq -\frac{1}{\sqrt{2}} |A|^2.$$

Write $v = \sum y_i e_i$. Then $\sum y_i^2 = 1$. By the Gauss equation, we have

$$\mathrm{Ric}(e_i, e_j) = \sum_k \mathrm{Rm}(e_i, e_k, e_k, e_j) = \sum_k \left(A(e_k, e_k)A(e_i, e_j) - A(e_i, e_k)A(e_j, e_k)\right),$$

and thus, $\mathrm{Ric}(e_i, e_j) = 0$ when $i \neq j$ and $\mathrm{Ric}(e_i, e_i) = \sum_{j \neq i} A(e_i, e_i)A(e_j, e_j)$. Therefore,

$$\mathrm{Ric}(v, v) = \sum_i \sum_{j \neq i} A(e_j, e_j)A(e_i, e_i) y_i^2 = k_1(k_2 + k_3)y_1^2 + k_2(k_3 + k_1)y_2^2 + k_3(k_1 + k_2)y_3^2.$$

By Cauchy-Schwarz and the inequality of arithmetic and geometric means,

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 \geq k_1^2 + \frac{1}{2}(k_2 + k_3)^2 \geq -\sqrt{2}k_1(k_2 + k_3) \\ \Rightarrow k_1(k_2 + k_3) \geq -\frac{1}{\sqrt{2}} \sum_i k_i^2 = -\frac{1}{\sqrt{2}}|A|^2. \end{aligned}$$

Similarly,

$$k_2(k_3 + k_1) \geq -\frac{1}{\sqrt{2}}|A|^2, \quad k_3(k_1 + k_2) \geq -\frac{1}{\sqrt{2}}|A|^2.$$

Therefore,

$$\mathrm{Ric}(v, v) \geq -\frac{1}{\sqrt{2}}|A|^2 \sum_i y_i^2 = -\frac{1}{\sqrt{2}}|A|^2. \tag{4.2}$$

Applying this to ∇u, we conclude that:

$$\mathrm{Ric}(\nabla u, \nabla u) \geq -\frac{1}{\sqrt{2}}|A_M|^2|\nabla u|^2.$$

Since M is Φ-stable, (3.4) yields

$$\int_M \Lambda|A|^2\varphi^2 \leq \int_M |\nabla\varphi|^2, \quad \forall\varphi \in C_0^1(M).$$

Replacing φ by $|\nabla u|\varphi$, we have:

$$\begin{aligned} \int_M \varphi^2|\nabla u|^2|A|^2 &\leq \int_M |\nabla\varphi|^2|\nabla u|^2 + 2\int_M \left(\varphi|\nabla u|\langle\nabla\varphi, \nabla|\nabla u|\rangle + \varphi^2|\nabla|\nabla u||^2\right) \\ &= \int_M |\nabla\varphi|^2|\nabla u|^2 - \int_M \varphi^2|\nabla u|\Delta|\nabla u|. \end{aligned} \tag{4.3}$$

By the improved Kato inequality,

$$|\nabla^2 u|^2 \geq \frac{3}{8}|\nabla u|^{-2}|\nabla|\nabla u|^2|^2.$$

Combined with the Bochner formula and (4.2), we have:

$$\begin{aligned} \Delta|\nabla u|^2 &= 2\,\mathrm{Ric}_M(\nabla u, \nabla u) + 2|\nabla^2 u|^2 \\ &\geq -\sqrt{2}|A|^2|\nabla u|^2 + \frac{3}{4}|\nabla u|^{-2}|\nabla|\nabla u|^2|^2. \end{aligned} \tag{4.4}$$

Thus,

$$\Delta|\nabla u| \geq -\frac{1}{\sqrt{2}}|A|^2|\nabla u| + \frac{1}{2}|\nabla u|^{-1}|\nabla|\nabla u||^2. \tag{4.5}$$

(4.1) follows from (4.3) and (4.5). □

Proposition 4.2. *Suppose Φ satisfies (1.1). Then any complete, two-sided Φ-stable immersion M^3 in $\mathbf{R}^4$ has only one end.*

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Proof. Suppose the contrary, that M has at least two ends. Then Corollary 3.6 implies that M admits a nontrivial harmonic function u with $\int_M |\nabla u|^2 \leq C < \infty$. For $\rho > 0$, take $\varphi \in C_c^1(M)$, such that $\varphi|_{B_M(0,\rho)} = 1$, $\varphi|_{B_M(0,2\rho)} = 0$ and $|\nabla\varphi| \leq \frac{2}{\rho}$. Then (4.1) implies that

$$\int_{B_M(0,\rho)} (\Lambda - \tfrac{1}{\sqrt{2}})|A|^2|\nabla u|^2 + \frac{1}{2}|\nabla|\nabla u||^2 \leq \frac{4}{\rho^2}\int_M |\nabla u|^2 \leq \frac{4C}{\rho^2}.$$

Here, $\Lambda \geq \frac{1}{\sqrt{2}}$ by (1.1). Sending $\rho \to \infty$, we conclude that

$$|\nabla|\nabla u||^2 \equiv 0.$$

In particular, this implies that $|\nabla u|$ is a constant. Since u is nonconstant, we have that $|\nabla u| > 0$. However, this implies that

$$\int_M 1 = \frac{1}{|\nabla u|^2}\int_M |\nabla u|^2 < \infty,$$

contradicting Corollary 3.5. □

5. A conformal deformation of metrics

Take $M^3 \to \mathbf{R}^4$ to be $\mathbf{\Phi}$-stable, where Φ satisfies (1.1). In this section, we carry out the conformal deformation technique used by Gulliver and Lawson [27] on M.

Consider the function $r(x) = \mathrm{dist}_{\mathbf{R}^{n+1}}(0, x)$ on M and the position vector field $\vec{X}$. Then $\Delta\vec{X} = \vec{H}$. Thus, $\Delta(r^2) = \Delta(\sum x_i^2) = 2\vec{X}\cdot\Delta X + 2|\nabla\vec{X}|^2 = 2\vec{X}\cdot\vec{H} + 2n$. We find:

$$\Delta r = \frac{n}{r} + H(\hat{x}\cdot\nu) - \frac{|\nabla r|^2}{r},$$

here, $\hat{x} = \frac{\vec{X}}{|\vec{X}|}$ is the normalised position vector.

Suppose that $w > 0$ is a smooth function on $M^n \setminus \{0\}$. On $M \setminus \{0\}$, define $\tilde{g} = w^2 g$. For $\lambda \in \mathbf{R}$, $\varphi \in C_c^1(M \setminus \{0\})$, consider the quadratic form

$$\mathcal{Q}_w(\varphi) = \int_M \left(|\tilde{\nabla}\varphi|_{\tilde{g}}^2 + (\tfrac{1}{2}\tilde{R} - \lambda)\varphi^2\right) d\tilde{\mu},$$

where $\tilde{\nabla}$, $\tilde{R}$, $\tilde{\mu}$ are the gradient, the scalar curvature and the volume form with respect to $\tilde{g}$, respectively. One relates the geometric quantities in g and $\tilde{g}$ as follows:

$$|\nabla\varphi|_g^2 = w^2|\tilde{\nabla}\varphi|_{\tilde{g}}^2, \quad d\mu = w^{-n}d\tilde{\mu}.$$

Moreover, we have

$$w^2\tilde{R} = R - 2(n-1)\Delta\log w - (n-1)(n-2)|\nabla\log w|^2.$$

Denote by $\tilde{\mathcal{Q}}_w(\varphi) := \mathcal{Q}_w(w^{\frac{2-n}{2}}\varphi)$. We compute:

$$\begin{aligned}
&\tilde{\mathcal{Q}}_w(\varphi)\\
&= \int_M \left(w^{-2}|\nabla(w^{\frac{2-n}{2}}\varphi)|_g^2 + (\tfrac{1}{2}\tilde{R} - \lambda)w^{2-n}\varphi^2\right) w^n d\mu\\
&= \int_M \left(w^{n-2}|w^{\frac{2-n}{2}}\nabla\varphi - \tfrac{n-2}{2}\varphi w^{-\frac{n}{2}}\nabla w|_g^2 + (\tfrac{1}{2}w^2\tilde{R} - w^2\lambda)\varphi^2\right) d\mu
\end{aligned}$$

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$$
\begin{aligned}
&= \int_M \left(|\nabla\varphi - \tfrac{n-2}{2}\varphi\nabla\log w|_g^2 + (\tfrac{1}{2}w^2\tilde{R} - w^2\lambda)\varphi^2 \right) d\mu \\
&= \int_M \left(|\nabla\varphi|_g^2 - \frac{n-2}{2}\left\langle \nabla(\varphi^2), \nabla\log w \right\rangle_g + \frac{(n-2)^2}{4}|\nabla\log w|_g^2\varphi^2 + (\tfrac{1}{2}w^2\tilde{R} - w^2\lambda)\varphi^2 \right) d\mu \\
&= \int_M \left(|\nabla\varphi|_g^2 + \left(\frac{n-2}{2}\Delta\log w + \frac{(n-2)^2}{4}|\nabla\log w|_g^2 + \tfrac{1}{2}w^2\tilde{R} - w^2\lambda \right)\varphi^2 \right) d\mu \\
&= \int_M \left(|\nabla\varphi|_g^2 + \tfrac{1}{2}R\varphi^2 - \left(\frac{n}{2}\left(\Delta\log w + \tfrac{(n-2)}{2}|\nabla\log w|_g^2 \right) + w^2\lambda \right)\varphi^2 \right) d\mu.
\end{aligned}
$$

We now choose $w = r^{-1}$ on $M \setminus \{0\}$. Note that (dropping the g subscript on the norm of the gradient)

$$
\begin{aligned}
\Delta\log w + \frac{n-2}{2}|\nabla\log w|^2 &= -\frac{\Delta r}{r} + \frac{n}{2}\frac{|\nabla r|^2}{r^2} \\
&= -\frac{n}{r^2} - \frac{H(\hat{x}\cdot\nu)}{r} + \frac{n+2}{2}\frac{|\nabla r|^2}{r^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{\mathcal{Q}}_w(\varphi) &= \int_M \left(|\nabla\varphi|^2 + \tfrac{1}{2}R\varphi^2 + \left(\frac{n}{2}\left(n + rH(\hat{x}\cdot\nu) - \tfrac{n+2}{2}|\nabla r|^2 \right) - \lambda \right) r^{-2}\varphi^2 \right) d\mu \\
&\geq \int_M \left(|\nabla\varphi|^2 + \tfrac{1}{2}R\varphi^2 + \left(\frac{n}{2}\left(n - \tfrac{1}{2}\beta r^2H^2 - \tfrac{1}{2\beta} - \tfrac{n+2}{2}|\nabla r|^2 \right) - \lambda \right) r^{-2}\varphi^2 \right) d\mu \qquad (5.1) \\
&= \int_M \left(|\nabla\varphi|^2 + (\tfrac{1}{2}R - \tfrac{n}{4}\beta H^2)\varphi^2 + \left(\frac{n}{2}\left(n - \tfrac{1}{2\beta} - \tfrac{n+2}{2}|\nabla r|^2 \right) - \lambda \right) r^{-2}\varphi^2 \right) d\mu,
\end{aligned}
$$

for $\beta > 0$ to be chosen later.

By the Gauss equation and Lemma 3.2,

$$
H^2 = |A|^2 + R \leq (1 - c_0)R.
$$

Combining with $|\nabla r| \leq 1$, we have

$$
\tilde{\mathcal{Q}}_w(\varphi) \geq \int_M \left(|\nabla\varphi|^2 + (\tfrac{1}{2} + \tfrac{n}{4}\beta(c_0 - 1))R\varphi^2 + \left(\frac{n}{2}(\tfrac{n-2}{2} - \tfrac{1}{2\beta}) - \lambda \right) r^{-2}\varphi^2 \right) d\mu. \qquad (5.2)
$$

On the other hand, (3.5) and (3.2) imply that for every $\varphi \in C_c^1(M)$,

$$
\int_M \left(|\nabla\varphi|^2 + \frac{1}{\sqrt{2}}R\varphi^2 \right) d\mu \geq 0.
$$

Note that $R \leq 0$. Thus, by choosing

$$
\beta = \frac{4(\frac{1}{\sqrt{2}} - \frac{1}{2})}{n(c_0 - 1)}, \quad \lambda = \frac{n}{2}\left(\frac{n-2}{2} - \frac{1}{2\beta} \right) = \frac{n}{2}\left(\frac{n-2}{2} - \frac{n(c_0 - 1)}{8(\frac{1}{\sqrt{2}} - \frac{1}{2})} \right),
$$

we have that $\tilde{\mathcal{Q}}_w(\varphi) \geq 0$ for all $\varphi \in C_c^1(M \setminus \{0\})$. We summarise these in the following proposition.

Proposition 5.1. *Suppose $n \geq 3$, (M^n, g) is an immersed hypersurface in $\mathbf{R}^{n+1}$, $\Lambda, c_0 \in \mathbf{R}$, such that:*

$$
\int_M (|\nabla\varphi|^2 + \Lambda R\varphi^2)dV_M \geq 0, \quad \forall\varphi \in C_c^1(M),
$$

$$
\Lambda > \tfrac{1}{2}, \quad c_0 \geq 1, \quad |A|^2 \leq -c_0 R_M.
$$

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Then the conformally deformed manifold $(M \setminus \{0\}, \tilde{g} = r^{-1}g)$ *satisfies*

$$\lambda_1(-\tilde{\Delta} + \tfrac{1}{2}\tilde{R}) \geq \lambda,$$

where $\lambda = \frac{n}{2}\left(\frac{n-2}{2} - \frac{n(c_0-1)}{8(\Lambda-\frac{1}{2})}\right)$.

6. Volume estimates

We first recall a diameter bound for warped μ-bubbles in 3-manifolds satisfying $\lambda_1(-\Delta + \frac{1}{2}R) \geq \lambda > 0$.

Lemma 6.1 (Warped μ-bubble area and diameter bound). *Let* (N^3, g) *be a* 3*-manifold with compact connected boundary satisfying*

$$\lambda_1(-\Delta + \tfrac{1}{2}R) \geq \lambda > 0. \tag{6.1}$$

Suppose there exists $p \in N$*, such that* $d_N(p, \partial N) \geq \frac{5\pi}{\sqrt{\lambda}}$*. Then there exists a connected open set* Ω *containing* ∂N*,* $\Omega \subset B_{\frac{5\pi}{\sqrt{\lambda}}}(\partial N)$*, such that each connected component of* $\partial\Omega \setminus \partial N$ *is a* 2*-sphere with area at most* $\frac{8\pi}{\lambda}$ *and intrinsic diameter at most* $\frac{2\pi}{\sqrt{\lambda}}$*.*

Proof. This is an application of estimates for the warped μ-bubbles (see, e.g. [9, Section 3]). Since N satisfies (6.1), there exists $u \in C^\infty(N)$, $u > 0$ in $\mathring{N}$, such that

$$\Delta_N u \leq -\tfrac{1}{2}(2\lambda - R_N)u. \tag{6.2}$$

Take $\varphi_0 \in C^\infty(M)$ to be a smoothing of $d_N(\cdot, \partial N)$, such that $|\operatorname{Lip}(\varphi_0)| \leq 2$ and $\varphi_0 = 0$ on ∂N. Choose $\varepsilon \in (0, \frac{1}{2})$, such that ε, $\frac{4}{\sqrt{\lambda}}\pi + 2\varepsilon$ are regular values of φ_0. Define

$$\varphi = \frac{\varphi_0 - \varepsilon}{\frac{4}{\sqrt{\lambda}} + \frac{\varepsilon}{\pi}} - \frac{\pi}{2},$$

$\Omega_1 = \{x \in N : -\frac{\pi}{2} < \varphi < \frac{\pi}{2}\}$ and $\Omega_0 = \{x \in N : -\frac{\pi}{2} < \varphi \leq 0\}$. We have that $|\operatorname{Lip}(\varphi)| < \frac{\sqrt{\lambda}}{2}$. In Ω_1, define $h(x) = -\frac{1}{2}\tan(\varphi(x))$. By a direct computation, we have

$$\lambda + h^2 - 2|\nabla h| \geq 0. \tag{6.3}$$

Minimise

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} u \, d\mathcal{H}^2 - \int_{\Omega_1} (\chi_\Omega - \chi_{\Omega_0}) h u \, d\mathcal{H}^3,$$

among Caccioppoli sets, Ω in Ω_1 with $\Omega \Delta \Omega_0$ is compactly contained in Ω_1. By [9, Proposition 12], a minimiser $\tilde{\Omega}$ exists and has regular boundary. We take Ω to be the connected component of $\{x \in N : 0 \leq \varphi_0(x) \leq \varepsilon\} \cup \tilde{\Omega}$ that contains ∂N (in other words, we disregard any component of $\tilde{\Omega}$ that is disjoint from ∂N). We verify that Ω satisfies the conclusions of Lemma 6.1. Indeed, for any connected component Σ of $\partial\Omega \cap \Omega_1$, the stability of $\mathcal{A}$ implies [9, Lemma 14]:

$$\begin{aligned}\int_\Sigma &|\nabla\psi|^2 u - \tfrac{1}{2}(R_N - \lambda - 2K_\Sigma)\psi^2 u + (\Delta_N u - \Delta_\Sigma u)\psi^2 \\ &- \tfrac{1}{2}u^{-1}\langle \nabla_N u, \nu\rangle^2 \psi^2 - \tfrac{1}{2}(\lambda + h^2 + 2\langle \nabla_N h, \nu\rangle)\psi^2 u \geq 0, \quad \forall \psi \in C^1(\Sigma).\end{aligned} \tag{6.4}$$

Taking $\psi = u^{-\frac{1}{2}}$ and using (6.2), (6.3), we conclude that

$$\lambda|\Sigma| \leq 2\int_\Sigma K_\Sigma dA \leq 8\pi \quad \Rightarrow \quad |\Sigma| \leq \frac{8\pi}{\lambda}.$$

Note that we have used Gauss–Bonnet, which also implies that Σ is a 2-sphere. The diameter upper bound follows from [9, Lemmas 16 and 18]. □

For the next lemma, recall that $r(x) = \mathrm{dist}_{\mathbf{R}^m}(0, x)$.

Lemma 6.2. *Below, $k \geq 2$ and N^k is a compact connected manifold, possibly with boundary.*

1. *Consider an immersion $N^k \to \mathbf{R}^m \setminus \{0\}$. Consider $p, q \in N$ with $d_{\tilde g}(p,q) \leq D$, where $\tilde g = r^{-2}g$ and g is the induced metric on N. Then $r(p) \leq e^D r(q)$.*
2. *Consider an immersion $\varphi : N^k \to \mathbf{R}^m$ with $0 \in \varphi(N)$. Consider $p, q \in N \setminus \varphi^{-1}(0)$ with $d_{\tilde g}(p,q) \leq D$. Write g for the induced metric on N, and let $\bar r(x) = d_g(\varphi^{-1}(0), x)$ denote the intrinsic distance on N. Then $\bar r(p) \leq e^D \bar r(q)$.*

Proof. We first establish (1). Choose a curve $\gamma : [0, L] \to N$, parametrised by $\tilde g$-unit speed, connecting p and q, such that $L \leq D + \varepsilon$. Using $|\nabla r|_g \leq 1$, we compute

$$\begin{aligned}
\log r(q) - \log r(p) &= \int_0^L \frac{d}{dt}\log r(\gamma(t))dt \\
&= \int_0^L r(\gamma(t))^{-1} g(\nabla r, \gamma'(t))dt \\
&\leq \int_0^L r(\gamma(t))^{-1}|\nabla r|_g|\gamma'(t)|_g dt \\
&\leq \int_0^L r(\gamma(t))^{-1}|\gamma'(t)|_g dt \\
&= \int_0^L |\gamma'(t)|_{\tilde g} dt = L \leq D + \varepsilon.
\end{aligned}$$

Thus, $r(q) \leq e^{D+\varepsilon} r(p)$. The result follows by sending $\varepsilon \to 0$.

For (2), we begin by noting that $|\nabla \bar r|_g = 1$ and $r(x) \leq \bar r(x)$ for any $x \in N$. Thus, arguing as above

$$\log \bar r(q) - \log \bar r(p) \leq \int_0^L \bar r(\gamma(t))^{-1}|\gamma'(t)|_g dt \leq \int_0^L r(\gamma(t))^{-1}|\gamma'(t)|_g dt = L.$$

The proof is completed as above. □

Proof of Theorem 1.4. Let $r = \mathrm{dist}_{\mathbf{R}^4}(\cdot, 0)$ and $\bar r = \mathrm{dist}_{M,g}(\cdot, 0)$, and consider $\tilde g = r^{-2}g$. Fix $\rho > 0$, and consider the geodesic ball $B_M(0, e^{\frac{5\pi}{\sqrt\lambda}}\rho)$. By Proposition 4.2, $M \setminus B_M(0, e^{\frac{5\pi}{\sqrt\lambda}}\rho)$ has only one unbounded connected component E. Denote by $M' = M \setminus E$. We claim that $\partial M' = \partial E$ is connected. Indeed, since M' and E are both connected, if $\partial M'$ has more than one connected component, then one can find a loop in M intersecting one component of $\partial M'$ exactly once, contradicting that M is simply connected. Applying Lemma 6.1 to $(M' \setminus \{0\}, \tilde g)$, we find a connected open set Ω in the $\frac{5\pi}{\sqrt\lambda}$ neighborhood of $\partial M'$, such that each connected component of $\partial\Omega \setminus \partial M'$ has area bounded by $\frac{8\pi}{\lambda}$ and diameter bounded by $\frac{2\pi}{\sqrt\lambda}$ (we emphasise here that the distance, area and diameter are with respect to $\tilde g$). Let M_0 be the connected component of $M' \setminus \Omega$ that contains 0.

We make a few observations about M_0. First, we claim that $M \setminus M_0$ is connected. To see this, let M_1 be the union of connected components of $M' \setminus \Omega$ other than M_0. Then $M \setminus M_0 = M_1 \cup \Omega \cup E$. Note that each connected component of M_1 shares a common boundary with Ω. Since Ω is connected, so is $M_1 \cup \Omega$. Next, we claim that M_0 has only one boundary component: otherwise, since both M_0 and $M \setminus M_0$ are connected, as before, we can find a loop in M intersecting a connected component of ∂M_0 exactly once, contradicting that M is simply connected.

Denote by $\Sigma = \partial M_0$. By (2) in Lemma 6.2, $\min_{x\in\Sigma} \bar{r}(x) \geq \rho$. Since $B_M(0,\rho)$ is connected, this implies that $B_M(0,\rho) \subset M_0$. On the other hand, by comparing intrinsic to extrinsic distance, we see that $\max_{x\in\Sigma} r(x) \leq e^{\frac{5\pi}{\sqrt{\lambda}}}\rho$, so

$$|\Sigma|_g = \int_\Sigma d\mu = \int_\Sigma r^2 d\tilde{\mu} \leq e^{\frac{10\pi}{\sqrt{\lambda}}}\rho^2|\Sigma|_{\tilde{g}} \leq \frac{8\pi}{\lambda}e^{\frac{10\pi}{\sqrt{\lambda}}}\rho^2.$$

Thus, Corollary 3.1 implies that

$$|B_M(0,\rho)|_g \leq |M_0|_g \leq \frac{\|\Phi\|_{C^1}}{3\min_{\nu\in S^3}\Phi(\nu)}e^{\frac{5\pi}{\sqrt{\lambda}}}\rho|\partial M_0|_g \leq \frac{8\pi e^{\frac{15\pi}{\lambda}}\|\Phi\|_{C^1}}{3\lambda\min_{\nu\in S^3}\Phi(\nu)}\rho^3.$$

This proves the first part of the assertion.

Now consider a connected component Σ_0 of $\partial B_M(0,\rho)$, and let E be the connected component of $M \setminus B_M(0,\rho)$, such that ∂E contains Σ_0. Since M is simply connected, we must have that $\partial E = \Sigma_0$. Apply Lemma 6.1 to $M \setminus E$, and obtain a connected surface Σ, such that $\mathrm{dist}_{\tilde{g}}(\Sigma_0, \Sigma) \leq \frac{5\pi}{\sqrt{\lambda}}$ and $\mathrm{diam}_{\tilde{g}}(\Sigma) \leq \frac{2\pi}{\sqrt{\lambda}}$ (the proof that Σ is connected follows a similar argument as used above). By the triangle inequality, we have that $\mathrm{diam}_{\tilde{g}}(\Sigma_0) \leq \frac{7\pi}{\sqrt{\lambda}}$. Thus, Lemma 6.2 implies that

$$\max_{x\in\Sigma_0} r(x) \leq e^{\frac{7\pi}{\sqrt{\lambda}}} \min_{x\in\Sigma_0} r(x).$$

This proves the assertion. □

Proof of Theorem 1.7. The proof is very similar to that of Theorem 1.4. We apply Lemma 6.1 to $(M \setminus \{0\}, \tilde{g} = r^{-2}g)$ and find a region Ω in the $\frac{5\pi}{\sqrt{\lambda}}$ neighborhood of ∂M, such that each connected component of $\Omega \setminus \partial M$ has area bounded by $\frac{8\pi}{\lambda}$ (again, the distance and area are with respect to $\tilde{g}$). Let M' be the connected component of $M \setminus \Omega$ that contains $\{0\}$. Then $\partial M'$ is connected.

Denote by $\Sigma = \partial M'$ and $\rho_0 = e^{-\frac{5\pi}{\sqrt{\lambda}}}$. By (1) in Lemma 6.2, $\min_{x\in\Omega} r(x) \geq \rho_0$. In particular, this implies that $M^*_{r_0} \subset M'$. We have

$$|\Sigma|_g = \int_\Sigma d\mu = \int_\Sigma r^2 d\tilde{\mu} \leq |\Sigma|_{\tilde{g}} \leq \frac{8\pi}{\lambda}.$$

Therefore, Corollary 3.1 implies that

$$|M^*_{\rho_0}|_g \leq |M'| \leq \frac{\|\Phi\|_{C^1}}{3\min_{\nu\in S^3}\Phi(\nu)}|\Sigma|_g \leq \frac{8\pi\|\Phi\|_{C^1}}{3\lambda\min_{\nu\in S^3}\Phi(\nu)}.$$

This completes the proof. □

Remark 6.3. In the more general case where we do not assume that M is simply connected or has one end (or boundary component), similar proofs work out. The only modification here is that ∂M_0 in the proof of Theorem 1.4 (or $\partial M'$ in the proof of Theorem 1.7) has connected components bounded by $b_1(M) + E$, where E is the number of ends if M is complete, and is the number of boundary components if $M \subset B_1(0)$. Thus, we have

$$|B_{M,R}(0)| \leq V_0(b_1(M) + E),$$

if M is complete and

$$|M^*_{\rho_0}| \le V_1(b_1(M) + E),$$

if $M \subset B_1(0)$.

A. First and second variation

We derive first and second variations of $\mathbf{\Phi}$ with emphasis on our geometric applications (see also [18, Appendix A] and [54, Section 2]). For $M^n \to \mathbf{R}^{n+1}$ a two-sided immersion, set

$$\mathbf{\Phi}(M) := \int_M \Phi(\nu)$$

for $\Phi : \mathbf{R}^{n+1} \to (0, \infty)$ an elliptic integrand.

A.1. First variation

Consider a 1-parameter family of surfaces M_t with normal speed at $t = 0$ given by $u\nu$ (with $u \in C^1_c(M \setminus \partial M)$). Recall that $\dot\nu = -\nabla u$. We find

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathbf{\Phi}(M_t) &= \int_M (Hu\Phi(\nu) - D_{\nabla u}\Phi(\nu)) \\
&= \int_M \left(H\Phi(\nu) + \operatorname{div}_M(D\Phi(\nu)^T)\right) u \\
&= \int_M (H\Phi(\nu) + \operatorname{div}_M(D\Phi(\nu) - (D_\nu\Phi(\nu))\nu))\, u \\
&= \int_M (H\Phi(\nu) + \operatorname{div}_M(D\Phi(\nu)) - (D_\nu\Phi(\nu))H)\, u.
\end{aligned}$$

Now, we note that we have that $D\Phi(\nu) \cdot \nu = \Phi(\nu)$ by the Euler theorem for homogeneous functions. Thus, we find that

$$\left.\frac{d}{dt}\right|_{t=0} \mathbf{\Phi}(M_t) = \int_M \operatorname{div}_M(D\Phi(\nu))u. \tag{A.1}$$

Thus,

$$H_\Phi = \operatorname{div}_M(D\Phi(\nu)), \tag{A.2}$$

vanishes if and only if M is a critical point of $\mathbf{\Phi}$. Let us rewrite this as follows (with $\{e_i\}_{i=1}^n$ a local orthonormal frame for M):

$$\begin{aligned}
\operatorname{div}_M(D\Phi(\nu)) &= \sum_{i=1}^n (D_{e_i} D\Phi(\nu)) \cdot e_i \\
&= \sum_{i=1}^n D^2\Phi(\nu)[D_{e_i}\nu, e_i] \\
&= \sum_{i=1}^n D^2\Phi(\nu)[S_\Sigma(e_i), e_i],
\end{aligned}$$

for S_M the shape operator of M. Let us define $\Psi(\nu) : T\mathbf{R}^{n+1} \to T\mathbf{R}^{n+1}$ by $\Psi(\nu) : X \mapsto D^2\Phi(\nu)[X, \cdot]$ (this is just the $(1,1)$-tensor associated to $D^2\Phi(\nu)$ via the Euclidean metric).

Then, we find

$$H_\Phi = \operatorname{tr}_M(\Psi(\nu)S_M). \tag{A.3}$$

Note that for $\Phi(\nu) = |\nu|$, we have

$$D\Phi(\nu) = |\nu|^{-1}\nu, \Psi(\nu) = |\nu|^{-1}\operatorname{Id} - |\nu|^{-3}\nu \otimes \nu^\flat,$$

so in particular, when $|\nu| = 1$, we find $\Psi(\nu)|_{T_p\Sigma} = \operatorname{Id}_{T_p\Sigma}$. Thus, this recovers the usual mean curvature.

A.2. Second variation

Recall the tube formula:

$$\dot{S} = -\nabla^2 u - S^2 u$$

(where we are regarding $\nabla^2 u$ as a $(1,1)$-tensor via g_M). Note also that the trace of a $(1,1)$-tensor is independent of the metric. Thus, we find

$$\dot{H}_\Phi = \operatorname{tr}_M(-\Psi(\nu)\nabla^2 u - \Psi(\nu)S_M^2 u + \Psi(\nu)' S_M).$$

Note that

$$\Psi(\nu)' = -(D_{\nabla u}\Psi)(\nu).$$

Hence,

$$\dot{H}_\Phi = \operatorname{tr}_M(-\Psi(\nu)\nabla^2 u - \Psi(\nu)S_M^2 u - (D_{\nabla u}\Psi)(\nu)S_M).$$

Integration on M gives

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathbf{\Phi}(M_t) = \int_M \langle \nabla u, \Psi(\nu)\nabla u\rangle - \operatorname{tr}_M\left(\Psi(\nu)S_M^2\right)u^2. \tag{A.4}$$

Thus, stability implies that

$$\int_M |\nabla u|^2 - \Lambda |A|^2 u^2 \geq 0, \quad \forall u \in C_c^1(M \setminus \partial M). \tag{A.5}$$

Here, Λ depends on the ellipticity of Φ. In particular, if Φ satisfies (1.1), then (3.3) implies that for Φ-stable surfaces M, we have

$$\int_M |\nabla u|^2 - \frac{1}{\sqrt{2}}|A_M|^2 u^2 \geq 0, \quad \forall u \in C_c^1(M \setminus \partial M). \tag{A.6}$$

Note that when $\Phi(X) = |X|$, we have seen that $\Psi(Y) = |Y|^{-1}\operatorname{Id} - |Y|^{-3}Y \otimes Y^\flat$. Hence,

$$D_X\Psi(\nu) = \left.\frac{d}{dt}\right|_{t=0}\Psi(\nu + X) = 2(X \cdot \nu)\operatorname{Id} - X \otimes \nu^\flat - \nu \otimes X^\flat.$$

In particular, $D_{\nabla u}\Psi(\nu)|_{T_pM} = 0$. Thus, we recover the standard second variation formula in this case.

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A.3. First variation through vector fields

We also deduce the first variation formula of $\mathbf{\Phi}$ through variations that are not necessarily normal to M. We compute as follows:

$$
\begin{aligned}
&\int_M \Phi(\nu)\operatorname{div}_M X\\
&= \int_M \Phi(\nu)\operatorname{div}_M X^T + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M \operatorname{div}_M(\Phi(\nu)X^T) - \nabla(\Phi(\nu))\cdot X^T + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M \operatorname{div}_M(\Phi(\nu)X^T) - D_{D\Phi(\nu)^T}\nu\cdot X^T + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M \operatorname{div}_M(\Phi(\nu)X^T) - D_{D\Phi(\nu)^T}X^T\cdot\nu + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M \operatorname{div}_M(\Phi(\nu)X^T) + D_{D\Phi(\nu)^T}(X\cdot\nu) - D_{D\Phi(\nu)^T}X\cdot\nu + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M \operatorname{div}_M(\Phi(\nu)X^T) + \operatorname{div}((X\cdot\nu)D\Phi(\nu)^T) - (X\cdot\nu)\operatorname{div}_\Sigma D\Phi(\nu)^T\\
&\qquad - D_{D\Phi(\nu)^T}X\cdot\nu + \Phi(\nu)(X\cdot\nu)H\\
&= \int_M -(X\cdot\nu)\operatorname{div}_M D\Phi(\nu)^T - D_{D\Phi(\nu)^T}X\cdot\nu + \Phi(\nu)(X\cdot\nu)H\\
&\qquad + \int_{\partial M}\Phi(\nu)X\cdot\eta + (X\cdot\nu)D\Phi(\nu)\cdot\eta\\
&= \int_M -(X\cdot\nu)\operatorname{div}_M D\Phi(\nu) - D_{D\Phi(\nu)^T}X\cdot\nu + \int_{\partial M}\Phi(\nu)X\cdot\eta + (X\cdot\nu)D\Phi(\nu)\cdot\eta.
\end{aligned}
$$

Thus, we find that if $H_\Phi = 0$, then

$$
\int_M \Phi(\nu)\operatorname{div}_M X + D_{D\Phi(\nu)^T}X\cdot\nu = \int_{\partial M}\Phi(\nu)X\cdot\eta + (X\cdot\nu)D\Phi(\nu)\cdot\eta. \tag{A.7}
$$

B. Some computations for quadratic forms

In this section, we explicitly compute the constant c_0 that appeared in Lemma 3.2. The approach is elementary.

Lemma B.1. *Let $a_1 \le a_2 \le a_3$ be positive constants, such that $\frac{a_3}{a_1} \le \sqrt{2}$. Consider quadratic forms*

$$
Q_1(k_1,k_2) = \frac{a_1^2 + a_3^2}{a_3^2}k_1^2 + \frac{2a_1a_2}{a_3^2}k_1k_2 + \frac{a_2^2 + a_3^2}{a_3^2}k_2^2,
$$

$$
Q_2(k_1,k_2) = \frac{2a_1}{a_3}k_1^2 + \frac{2(a_1 + a_2 - a_3)}{a_3}k_1k_2 + \frac{2a_2}{a_3}k_2^2.
$$

Then we have $Q_1 \le c_0 Q_2$, where

$$
c_0 = \frac{1}{\sqrt{2} - \frac{1}{2}} \approx 1.09.
$$

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Proof. Write $\alpha = \frac{a_1}{a_3}$, $\beta = \frac{a_2}{a_3}$, with $2^{-\frac{1}{2}} \leq \alpha \leq \beta \leq 1$. Then

$$\begin{aligned} Q_1(k_1,k_2) &= (1+\alpha^2)k_1^2 + 2\alpha\beta k_1 k_2 + (1+\beta^2)k_2^2 \\ &= (1+\alpha^2)\left(k_1 + \frac{\alpha\beta}{1+\alpha^2}k_2\right)^2 + \frac{1+\alpha^2+\beta^2}{1+\alpha^2}k_2^2. \end{aligned} \tag{B.1}$$

Under the substitution $x = k_1 + \frac{\alpha\beta}{1+\alpha^2}k_2$, $y = k_2$, we have $k_1+k_2+(-\alpha k_1 - \beta k_2) = (1-\alpha)x + \frac{1-\beta-\alpha\beta+\alpha^2}{1+\alpha^2}y$. Thus, by Cauchy-Schwartz,

$$\begin{aligned} (Q_1-Q_2)(k_1,k_2) &= (k_1+k_2-\alpha k_1 - \beta k_2)^2 \\ &= \left((1-\alpha)x + \frac{1-\beta-\alpha\beta+\alpha^2}{1+\alpha^2}y\right)^2 \\ &\leq c_1\left((1+\alpha^2)x^2 + \frac{1+\alpha^2+\beta^2}{1+\alpha^2}y^2\right) = c_1 Q_1(k_1,k_2), \end{aligned}$$

where $c_1 = \frac{(1-\alpha)^2}{1+\alpha^2} + \left(\frac{1-\beta-\alpha\beta+\alpha^2}{1+\alpha^2}\right)^2 \cdot \frac{1+\alpha^2}{1+\alpha^2+\beta^2}$. This gives $Q_1 \leq \frac{1}{1-c_1}Q_2$. Using $2^{-\frac{1}{2}} \leq \alpha \leq \beta \leq 1$, we have:

$$c_1 \leq \frac{(1-2^{-\frac{1}{2}})^2}{1+\frac{1}{2}} + \left(\frac{1-2^{-\frac{1}{2}}}{1+\frac{1}{2}}\right)^2 \cdot \frac{1+\frac{1}{2}}{2} = \frac{3}{2} - \sqrt{2}.$$

The result follows. □

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