A graph related to the Euler ϕ function

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1. *Introduction*

In mathematics, graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph G is a pair $G = (V, E)$, where V and E are the vertex set and the edge set of G, respectively. The order and size of G is the number of vertices and edges of G, respectively. The degree or valency of a vertex u in a graph G (loopless), denoted by $\deg(u)$, is the number of edges meeting at u. If, for every vertex v in G, $\deg(v) = k$, we say that G is a k-regular graph. The cycle of order n is denoted by C_n and is a connected 2-regular graph. The path graph of order n is denoted by P_n and obtain by deleting an edge of C_n. A tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected undirected graph without cycle. A leaf (or pendant vertex) of a tree is a vertex of the tree of degree 1. An edge of a graph is said to be pendant if one of its vertices is a pendant vertex. A complete bipartite graph is a graph G with $V = X \cup Y$ and $X \cap Y = \varnothing$ such that every vertex of the set (part) X is connected to every vertex of the set (part) Y. If $\left|X\right| = m$ and $\left|Y\right| = n$, then this graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called the star graph which has $n + 1$ vertices. The distance between two vertices u and v of G, denoted by $d(u, v)$, is defined as the minimum number of edges of the walks between them. The complement of graph G is denoted by $\overline{G}$ and is a graph with the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if, and only if, they are not adjacent in G. For more information on graphs, refer to [1].

One of the topics in number theory is the Euler ϕ function, which is denoted by $\phi(n)$ and is the number of positive integers less than or equal to n and relatively prime to n. Leonhard Euler introduced the function in 1763. However, he did not at that time choose any specific symbol to denote it. Two facts about the Euler ϕ function are useful for evaluating $\phi(n)$. First, if p is prime and $k \in \mathbb{N}$, then $\phi\left(p^k\right) = p^k - p^{k-1}$ and second, ϕ is multiplicative; that is, if m and n are relatively prime, then $\phi(mn) = \phi(m)\phi(n)$ (see [2]). So $\phi(n)$ is even for all $n \geqslant 3$. Suppose that $n = p_1^{t_1}p_2^{t_2}\dots p_k^{t_k}$ and $\phi(n) = 2$, so

$$2 = p_1^{t_1-1}\left(p_1 - 1\right)p_2^{t_2-1}\left(p_2 - 1\right)\dots p_k^{t_k-1}\left(p_k - 1\right).$$

The only primes p_i in the factorisation of n must be such that $p_i - 1$ divides 2. So the only primes possible for n are 2 and 3 and so we have $n = 2^a3^b$ where $a, b \geqslant 0$ and $2 = 2^{a-1}(1)\,3^{b-1}(2)$. If $b > 1$ then $3 \mid 2$ which is a contradiction. So we have the following cases:

(i) $b = 1$: then $n = 2^a3$ and $\phi(n) = 2^{a-1}(2 - 1)\,3^{1-1}(2)$. If $a > 1$ then $\phi(n) = 2^a \neq 2$. If $b = 1$ then $\phi(n) = (2 - 1)(3 - 1) = 2$ and we have a solution which is $n = 6$. If $a = 0$ then $\phi(n) = (3 - 1) = 2$ and we have another solution which is $n = 3$.

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(ii) $b = 0$: then we have $n = 2^a$ and $\phi(n) = 2^{a-1} = 2$ which implies $a = 2$. So we have another solution which is $n = 4$.

Therefore there is no $n \neq 3, 4, 6$ such that $\phi(n) = 2$ and we have just three solutions for $\phi(n) = 2$. Let us state these facts as a lemma:

Lemma

(i) $\phi(n)$ is even for all $n \geqslant 3$.

(ii) The equation $\phi(n) = 2$ is true only for $n = 3, 4, 6$.

Gauss's theorem states that the sum of $\phi(d)$ over the divisors d of n is n. In other words,

$$\sum_{d \mid n} \phi(d) = n.$$

The number 15 has interesting properties that

$$15 = \phi(15) + \phi(\phi(15)) + \phi(\phi(\phi(15))) + \phi(\phi(\phi(\phi(15)))).$$

Loomis, Plytage and Polhill in [3] asked for which numbers does this happen? Let $\phi^0(n) = n$, $\phi^1(n) = \phi(n)$ and $\phi^i(n) = \phi(\phi^{i-1}(n))$. Then we can iterate ϕ to create the sequence $\{n, \phi(n), \phi^2(n), \dots\}$. Parts of this sequence can be shown as a graph (see [3]). Following Pillai [2], let $R(n)$ denote the smallest integer k such that $\phi^k(n)$, in other words, $R(n)$ is the number of steps it takes the sequence beginning with n to reach 1. Loomis, Plytage and Polhill made the following definitions:

> Define $\Phi(n)$ by $\Phi(n) = \sum_{i=1}^{R(n)} \phi^i(n)$. Define a perfect totient number as a number n for which $\Phi(n) = n$. They proved that a prime power p^k is a PTN if, and only if, $p = 3$. Also they showed that if n is a PTN and $4n + 1$ is prime, then $3(4n + 1)$ is also a PTN.

In the next section, we introduce a new graph related to Euler ϕ function and investigate the properties of this graph. Also, we consider some specific graphs and some chemical trees such as G_ϕ-graph in Section 3.

2. *A graph related to Euler ϕ function*

In this section we state the definition of a new graph which is related to the Euler ϕ function and investigate its properties.

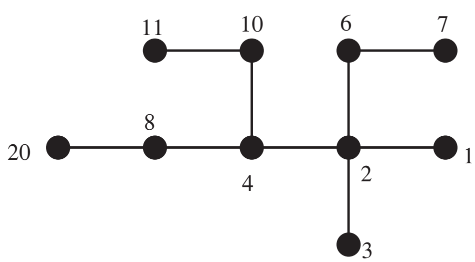

FIGURE 1: Graph related to the Euler ϕ function of Example 1

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Definition

For any set of natural numbers A, let $A_\phi = \{\phi^k(n) \mid n \in A, k \in \mathbb{N} \cup \{0\}\}$. The graph related to the Euler ϕ function is denoted by $G_\phi(A)$ and is a graph with vertex set $V = A_\phi$ and edge set $E = \{\{r, s\} \mid r, s \in V, \phi(r) = s\}$. We say a graph H is a G_ϕ-graph if there exists a set of natural numbers A, such that $H = G_\phi(A)$.

Remark

The graph $G_\phi(A)$ is a directed graph, because if $\phi(r) = s$, then $(r, s) \in E$, and since $\phi(s) \neq r$ so $(s, r) \notin E$. But in this paper, we consider the graph $G_\phi(A)$ without direction. Also note that since $\phi(n) < n$, the sequence $\{n, \phi(n), \phi^2(n), \dots\}$ is strictly decreasing and $\phi^k(n) < n$. This sequence reaches 1 after a finite number of steps.

Example 1

Let $A = \{3, 7, 11, 20\} \subseteq \mathbb{N}$. We have

$$\phi(20) = 8, \phi(8) = \phi^2(20) = 4, \phi(4) = \phi^3(20) = 2, \phi(2) = \phi^4(20) = 1,$$

$$\phi(11) = 10, \phi(10) = \phi^2(11) = 4, \phi(4) = \phi^3(11) = 2, \phi(2) = \phi^4(11) = 1,$$

$$\phi(7) = 6, \phi(6) = \phi^2(7) = 2, \phi(2) = \phi^3(7) = 1 \text{ and}$$

$$\phi(3) = 2, \phi(2) = \phi^2(3) = 1.$$

So we have $A_\phi = \{1, 2, 3, 4, 6, 7, 8, 10, 11, 20\}$ and therefore the graph $G_\phi(A)$ is the graph in Figure 1.

The following theorem gives two pieces of information about the structure of the graph $G_\phi(A)$:

Theorem 1

(i) For any set of natural numbers A, the graph $G_\phi(A)$ is a tree.

(ii) The distance between two vertices i $(i \in G_\phi(A))$ and 1 is $d(i,1) = R(i)$.

Proof:

(i) Suppose that $G_\phi(A)$ is not a tree. Therefore there is a cycle of length at least 3. Without loss of generality, suppose that the length of cycle is 3. So there are vertices a, b and c and the cycle is $abca$. By the definition of $G_\phi(A)$, we suppose that $\phi(a) = b$, $\phi(b) = c$ and $\phi(c) = a$. Therefore

$$\phi(a) = b, \phi^2(a) = \phi(b) = c \text{ and } \phi^3(a) = \phi(c) = a,$$

which is a contradiction with the fact in the Remark that notes that for every $k \geqslant 1$, and $n \neq 1$, $\phi^k(n) < n$.

(ii) It follows from the definitions.

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Here we ask a main question: Which trees can be a G_ϕ-graph? In the following theorem, we show that the path graph P_n can be a G_ϕ-graph.

Theorem 2

The path graph P_n ($n > 1$) is a G_ϕ-graph.

Proof

Since $\phi^{n-1}(2^{n-1}) = \phi^{n-2}(2^{n-2}) = \phi^{n-3}(2^{n-3}) = \ldots = 1$, it suffices to consider $A\left\{2^{n=1}\right\}$. So $A_\phi = \left\{1, 2, 4, \ldots, 2^{n-2}, 2^{n-1}\right\}$ and $G_\phi = P_n$.

The following result is an immediate result of Theorem 2:

Corollary: For every natural number m, there exists a set of natural numbers A, such that $G_\phi(A)$ has size m.

Also we have the following theorem:

Theorem 3

For every natural number n, there exist a set of natural numbers A, such that $|A| = n$ and $G_\phi(A)$ has size n.

Proof

It suffices to consider $A = \left\{1, 2, 4, \ldots, 2^{n-1}\right\}$. The rest is similar to the proof of Theorem 2.

Theorem 4

Let G be the G_ϕ-graph related to a set A of size n. If t is the number of its leaves, then $1 \leqslant t \leqslant n + 1$. Also there is a set B such that the number of leaves of $G_\phi(B)$ is t and $|B| = n$.

Proof

Suppose that there are t leaves. Since there is a leaf at vertex 1, there are $t - 1$ leaves with $t - 1$ distinct pendant vertices all of which are labelled by distinct elements of A, so $n = A \geqslant t - 1$. For second part, consider the odd prime numbers $p_1, p_2, \ldots, p_{t-1}$ with 1 in B and add $n - t$ members of the set

$$\{\phi(p_1), \phi^2(p_1), \phi^3(p_1), \ldots, \phi(p_2), \phi^2(p_2), \phi^3(p_2), \ldots$$
$$\ldots, \phi(p_{t-1}), \phi^2(p_{t-1}), \phi^3(p_{t-1}), \ldots, 2\}$$

to B, and for being confident about the size of this set it is suffices to consider enough large prime numbers.

Here we give the following example (regarding to the proof of the Theorem 4).

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Example 2

Let $n = 5$ and $t = 3$. Consider a set A containing three odd numbers $A = \{1, 7, 11\}$ and add two numbers 2 and 4 from

$$\left\{\phi(11) = 10, \phi^2(11) = 4, \phi(7) = 6, \phi^2(7) = 2\right\}$$

to it. So $B = \left\{1, 2, 4, 7, 11\right\}$. As we see in Figure 2, $G_\phi(B)$ has 3 leaves with $|B| = 5$ as desired.

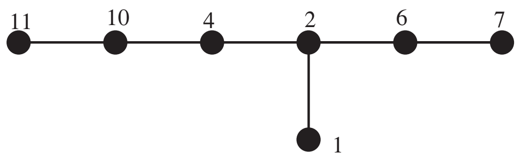

FIGURE 2: Graph related to Example 2

Theorem 5

The star graph $K_{1,n}$ cannot be a G_ϕ-graph, for $n > 4$.

Proof

One of the leaves of a G_ϕ graph should be 1. It is easy to see that there is only one number which satisfies the equation $\phi(n) = 1$ and that is 2 and so the neighbour of 1 is 2. Since by the Lemma there are only three solutions for the equation $\phi(n) = 2$, we can not have more than four adjacent vertices with 2 and therefore there is no star graph for $n > 4$.

A banana tree, $B(n, m)$ is a graph obtained by connecting one leaf of each n copies of an m-star graph to a new vertex. By Theorem 5 we have the following result:

Corollary

The banana tree $B(n, m)$ cannot be a G_ϕ-graph for $m \geqslant 6$.

Remark

The star graph $K_{1,n}$ is a G_ϕ-graph for $n < 4$. It suffices to consider $\{1, 2\}$ for $n = 1$, $\{1, 2, 3\}$ for $n = 2$, $\{1, 2, 3, 4\}$ for $n = 3$ and $\{1, 2, 3, 4, 6\}$ for $n = 4$.

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added [4].

Theorem 6

The centipede graph $P_n \circ K_1$ is a G_ϕ-graph, for $n > 1$.

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Proof

It suffices to consider $A = \{1, 2, 4, \dots, 2^n, 12, 24, 48, \dots, 3(2^n)\}$. So $G_\phi(A)$ is $P_n \circ K_1$ as shown in Figure 3. Therefore $P_n \circ K_1$ is a G_ϕ-graph.

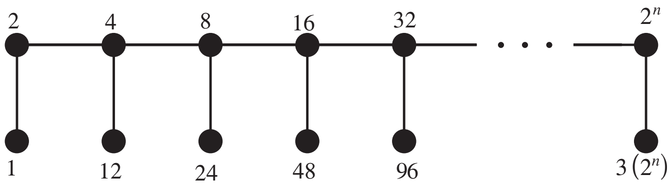

FIGURE 3: Graph related to Theorem 6

Theorem 7

If H is a G_ϕ-graph, then $H \circ \overline{K_n}$ is not a G_ϕ-graph, for $n > 3$.

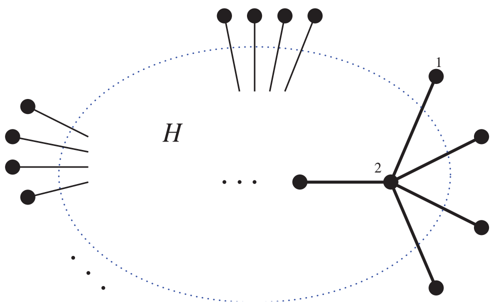

FIGURE 4: Graph $H \circ \overline{K_n}$ related to the proof of Theorem 7

Proof

Suppose that H is a G_ϕ-graph and without loss of generality let $n = 4$. We consider graph $H \circ \overline{K_n}$. If $H \circ \overline{K_n}$ is a G_ϕ-graph, then one of its leaves should be 1 and its neighbour should be 2 (see Figure 4). Then the vertex 2 has at least five neighbours one of them being 1. On the other hand the equation $\phi(x) = 2$ has only three solutions which are 3, 4 and 6. So, by assigning these three numbers to the three leaves adjacent to vertex 2, at least one vertex is left without assigning any number. So there is no set of natural numbers such that $H \circ \overline{K_4}$ be a G_ϕ-graph. By the same argument we conclude that $H \circ \overline{K_n}$ is not a G_ϕ-graph for $n > 4$.

3. *Some specific chemical trees as G_ϕ-graphs*

In chemical graphs, the vertices of the graph correspond to the atoms of the molecule, and the edges represent the chemical bonds. In this section, we consider some well-known molecules and investigate them as G_ϕ-graphs. We start with the following easy theorem:

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Theorem 8

(i) Methane, ethane and propane are G_ϕ-graphs.

(ii) Butane and isobutane are G_ϕ-graphs.

(iii) Pentane and isopentane are G_ϕ-graphs.

Proof

(i) Methane is a G_ϕ-graph with $A_\phi = \{3, 4, 6\}$, ethane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 8, 12\}$ and propane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 12, 15, 16, 20\}$ (see Figure 5).

(ii) Butane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 12, 15, 16, 20\}$ and Isobutane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 13, 15, 20, 21, 24, 28\}$, as we see in Figure 6.

(iii) Pentane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 12, 15, 17, 20, 48, 64, 80, 96\}$ and Isopentane is a G_ϕ-graph with $A_\phi = \{3, 5, 6, 12, 13, 15, 20, 28, 32, 40, 48\}$, as we see in Figure 7.

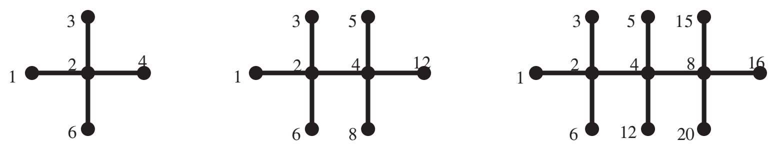

FIGURE 5: Methane, ethane and propane, respectively

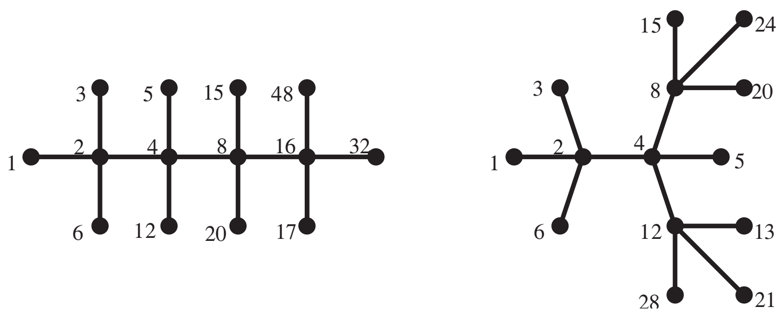

FIGURE 6: Butane and isobutane, respectively

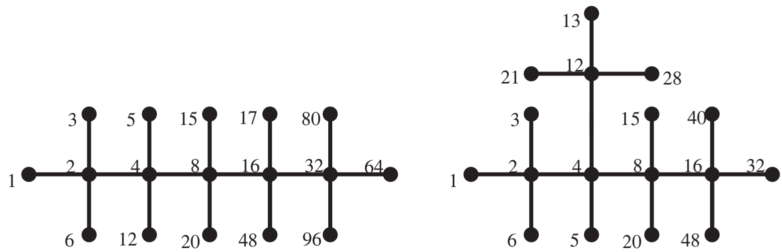

FIGURE 7: Pentane and isopentane, respectively

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Theorem 9

Neopentane is not a G_ϕ-graph.

Proof

Consider neopentane as seen in Figure 8. The vertex labelled 1 should be one of the leaves and the vertex labelled 2 should be its adjacent vertex. Now we consider the vertex u. As see in the Lemma, the vertex u should be 3, 4 or 6. By a similar argument to the proof of Theorem 5, we conclude that the equation $\phi(n) = 3$ has no solution. The numbers 7, 9, 14, 18 are the solutions of $\phi(n) = 6$ and 5, 8, 10, 12 are the solutions of $\phi(n) = 4$. So u is not 3. Let $u = 6$. So we should choose three numbers from 7, 9, 14, 18 to give to v, w and x. Each of them has three neighbours. But $\phi(n) = 14$ and $\phi(n) = 7$ have no solutions. Thus 6 is not suitable for u. By the same argument we also conclude that 4 is not suitable for u, since $\phi(n) = 10$ has only two solutions and $\phi(n) = 5$ has no solutions. Therefore we can not give the vertex u any number. Hence neopentane is not a G_ϕ-graph.

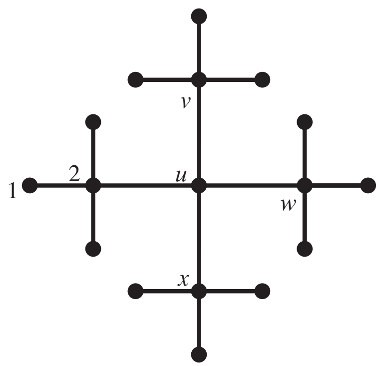

FIGURE 8: Neopentane related to the proof of Theorem 8

Theorem 3.3

The Alkanes C_nH_{2n+2} are G_ϕ-graphs.

Proof

For $n = 1, 2, 3$ we have methane, ethane and propane respectively (Figure 5). For $n = 4$ we have butane (Figure 6) and for $n = 5$ we have Pentane (figure 7). So for every $n > 5$ it suffices to consider

$$A = \{3, 6, 5, 12, 20, 24, 40, 80, \dots, 5(2^{2n-1}), 48, 96, 3(2^n), 2^{n+1}\}.$$

As we see in Figure 9, $G_\phi(A)$ is C_nH_{2n+2}.

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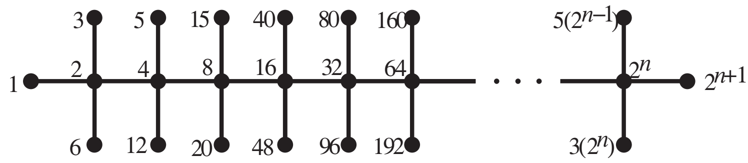

FIGURE 9: Alkanes C_nH_{2n+2} for every $n > 5$ related to the proof of Theorem 10

The graph D_2 in Figure 10 is the tree structure of a nanostar molecule has grown 2 stages (see [5, 6, 7]). We state and prove the following result.

Theorem 11

The graph D_2 is a G_ϕ-graph.

Proof

It suffices to consider $A = \{3, 256, 376, 384, 564\}$. As we see in Figure 10, D_2 is a G_ϕ-graph.

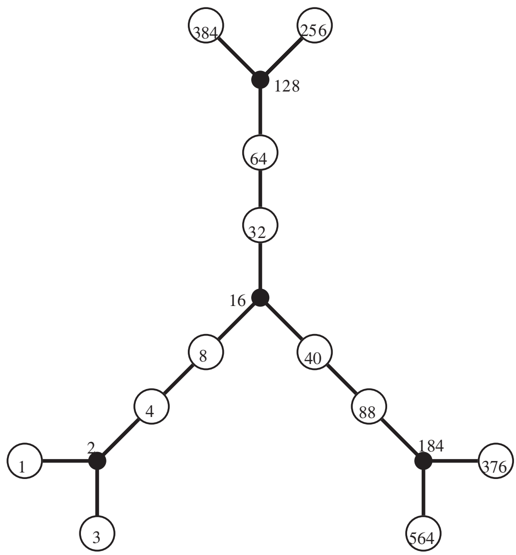

FIGURE 10: Nanostar D_2 related to Theorem 11

We think that there is no nanostar molecule which has grown at least four stages, as a G_ϕ-graph, but until now all attempts to prove this have failed. So we end this paper by proposing the following conjecture.

Conjecture

There is no nanostar molecule which has grown at least four stages, as a G_ϕ-graph.

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4. *Acknowledgements*

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