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Equations of motion in Poincaré-Četaev variables with constraint multipliers

Q.K. Ghori

Suslov's constraint multipliers are used to derive the equations of motion of dynamical systems (holonomic or nonholonomic) in the form of Poincaré-Četaev equations and in the canonical form. For holonomic systems defined by redundant variables, the constraint multipliers occuring in the canonical equations are determined and a modification of the Hamilton-Jacobi Theorem for integrating the canonical equations is presented.

1. Introduction

The method of constraint multipliers going back to Suslov [7] allows the reduction of Lagrange's equations of motion of a holonomic dynamical system to the ordinary canonical equations which can be integrated by the Hamilton-Jacobi Theorem. Employing such multipliers, Šul'gin [5], Šahaĭdarova [4], and others have published equations of motion of holonomic systems in redundant generalised coordinates. In his recent paper [6], Šul'gin has extended these equations to the case of linear nonholonomic systems.

We shall be concerned with the generalisations of these results in the Poincaré-Četaev variables. We begin with a conservative dynamical system whose position at any time t is specified by the variables $x_1, x_2, \ldots, x_n$. As in [2], let the set of operators $X_0, X_1, \ldots, X_n$

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with commutators

(1) $(X_0, X_p) = C_{0pq}X_q$, $(X_p, X_q) = C_{pqr}X_r$ $(p, q, r = 1, 2, \ldots, n)$

define the infinitesimal displacements of the system; and let the parameters $\eta_1, \eta_2, \ldots, \eta_n$ and $\omega_1, \omega_2, \ldots, \omega_n$ characterize the real and possible displacements, so that the variation of an arbitrary function $f(x_1, \ldots, x_n; t)$ in a real and possible displacement of the system is determined by the relation

(2) $df = [X_0(f)+\eta_p X_p(f)]dt$, $\delta f = \omega_p X_p(f)$ $(p = 1, 2, \ldots, n)$

and the differential constraints (holonomic or linear nonholonomic) are expressed by m $(< n)$ equations

(3) $F_\alpha = A_{\alpha p}\eta_p + A_{\alpha 0} = 0$ $(\alpha = 1, 2, \ldots, m; p = 1, 2, \ldots, n)$,

the ω's satisfying the relations

$$\frac{\partial F_\alpha}{\partial \eta_p}\omega_p = 0 \quad (\alpha = 1, 2, \ldots, m; p = 1, 2, \ldots, n) .$$

Here C_{0pq}, C_{pqr}, $A_{\alpha p}$, and $A_{\alpha 0}$ are functions of $x_1, x_2, \ldots, x_n, t$, and the convention of summing over a repeated suffix is adopted.

2. Equations of motion with constraint multipliers

It has been shown in [3] that the motion of the dynamical system under consideration, for which the kinetic potential is

$$L(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n; t) ,$$

is determined by the differential equations

(4) $$\frac{d}{dt}\frac{\partial L}{\partial \eta_p} - C_{0pq}\frac{\partial L}{\partial \eta_q} - C_{qpr}\eta_q\frac{\partial L}{\partial \eta_r} - X_p(L) - \lambda_\alpha\frac{\partial F_\alpha}{\partial \eta_p} = 0$$

$$(\alpha = 1, 2, \ldots, m; p, q, r = 1, 2, \ldots, n) ,$$

where $\lambda_1, \ldots, \lambda_m$ are the Lagrange undetermined multipliers.

According to Suslov [7], we introduce the constraint multipliers M_α by the relations

$$dM_\alpha = -\lambda_\alpha dt \quad (\alpha = 1, 2, \ldots, m) .$$

We also note from (2) and (3) that

$$X_p(F_\alpha) = \eta_q X_p(A_{\alpha q}) + X_p(A_{\alpha 0}) ,$$

and

$$\frac{d}{dt}\frac{\partial F_\alpha}{\partial \eta_p} = X_0(A_{\alpha p}) + \eta_q X_q(A_{\alpha p}) .$$

In view of the last relations, equations (4) assume the form

(5) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \eta_p} + M_\alpha \frac{\partial F_\alpha}{\partial \eta_p}\right) - C_{0pq}\frac{\partial L}{\partial \eta_q} - C_{pqr}\eta_q \frac{\partial L}{\partial \eta_r} - X_p(L) - M_\alpha X_p(F_\alpha)$$
$$= M_\alpha\left(\Omega^\alpha_{0p} + \eta_q \Omega^\alpha_{qp}\right) \quad (\alpha = 1, 2, \ldots, m;\ p, q, r = 1, 2, \ldots, n) ,$$

where

(6) $$\Omega^\alpha_{0p} = X_0(A_{\alpha p}) - X_p(A_{\alpha 0}) , \quad \Omega^\alpha_{qp} = X_q(A_{\alpha p}) - X_p(A_{\alpha q}) .$$

The equations (5) are the Poincaré-Četaev equations of motion of the nonholonomic system with constraint multipliers. The $(n+m)$ equations (5) and (3) are sufficient to determine the $(n+m)$ unknown quantities $x_1, x_2, \ldots, x_n$, $M_1, M_2, \ldots, M_m$ as functions of t.

Let us assume the vanishing of the nonholonomy terms Ω^α_{0p} and Ω^α_{qp}, occuring in equations (5). It follows that the constraint equations (3) are integrable and the system is holonomic. In such a case the x's and t are connected by relations in the finite form

(7) $$f_\alpha(x_1, x_2, \ldots, x_n; t) = 0 \quad (\alpha = 1, 2, \ldots, m) ,$$

and equations (3) may be taken to be equivalent to

$$F_\alpha = \frac{df_\alpha}{dt} = X_0(f_\alpha) + \eta_p X_p(f_\alpha) = 0 .$$

Consequently we have

(8) $$A_{\alpha p} = X_p(f_\alpha) , \quad A_{\alpha 0} = X_0(f_\alpha) ,$$

and, in view of (1), the following relations hold:

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$$(9)\qquad \Omega^{\alpha}_{0p} = \left(X_0X_p-X_pX_0\right)f_\alpha = c_{0pq}X_q\left(f_\alpha\right) = 0\ ,$$

$$\Omega^{\alpha}_{qp} = \left(X_qX_p-X_pX_q\right)f_\alpha = c_{qpr}X_r\left(f_\alpha\right) = 0\ .$$

The preceding analysis shows that, for a holonomic system defined by redundant variables, the equations of motion with constraint multipliers are

$$(10)\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_p} + M_\alpha \frac{\partial F_\alpha}{\partial \eta_p}\right) - c_{0pq}\frac{\partial L}{\partial \eta_q} - c_{qpr}\eta_q\frac{\partial L}{\partial \eta_r} - X_p(L) - M_\alpha X_p\left(F_\alpha\right) = 0$$

$$(\alpha = 1, 2, \ldots, m;\ p, q, r = 1, 2, \ldots, n)\ .$$

3. Canonical equations

In order to pass from equations (5) for the motion of a nonholonomic system to the canonical equations, we introduce new variables y_p by the relations

$$(11)\qquad y_p = \frac{\partial L}{\partial \eta_p} + M_\alpha \frac{\partial F_\alpha}{\partial \eta_p}\quad (p = 1, 2, \ldots, n)\ .$$

Let us assume that in the $(n+m)$ equations (11) and (3) the Jacobian of the $(n+m)$ functions

$$\frac{\partial L}{\partial \eta_p} + M_\alpha \frac{\partial F_\alpha}{\partial \eta_p}\ ,\ F_\alpha$$

with respect to the η's and M's is different from zero. We can then solve these equations to obtain

$$(12)\qquad \begin{aligned} \eta_p &= \eta_p\left(x_1, \ldots, x_n; y_1, \ldots, y_n; t\right)\ ,\\ M_\alpha &= M_\alpha\left(x_1, \ldots, x_n; y_1, \ldots, y_n; t\right)\ . \end{aligned}$$

Varying the function L in accordance with (2) and using (3) and (11), we get

$$\delta L = \omega_p X_p(L) + \frac{\partial L}{\partial \eta_p}\,\delta\eta_p$$

$$= \omega_p\left(\frac{dy_p}{dt} - M_\alpha X_p(F_\alpha) - C_{0pq}(y_q - M_\alpha A_{\alpha q}) - \eta_q C_{qpr}(y_r - M_\alpha A_{\alpha r}) - M_\alpha\left(\Omega^\alpha_{0p} + \eta_q\Omega^\alpha_{qp}\right)\right) + $$
$$+ \left(y_p - M_\alpha \frac{\partial F_\alpha}{\partial \eta_p}\right)\delta\eta_p$$

which reduces to

$$(13)\quad \delta L + M_\alpha \delta F_\alpha = \omega_p\Big[\frac{dy_p}{dt} - C_{0pq}y_q - C_{qpr}\eta_q y_r +$$
$$+ M_\alpha\left(C_{0pq}A_{\alpha q} + \eta_q C_{qpr}A_{\alpha r} - \Omega^\alpha_{0p} - \eta_q\Omega^\alpha_{qp}\right)\Big] + y_p\delta\eta_p \ .$$

Let us introduce the function

$$H(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = y_p\eta_p - L \ .$$

In the functions F_α , we replace the η's by their values obtained from (12) and denote the resulting function by $H_\alpha(x_1, \ldots, x_n; y_1, \ldots, y_n; t)$, so that $\delta F_\alpha = \delta H_\alpha$ and the constraint equations (3) become

$$(14)\qquad H_\alpha(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = 0 \quad (\alpha = 1, 2, \ldots, m) \ .$$

Varying the function H and using (13), we find that

$$\delta H - M_\alpha \delta H_\alpha = \eta_p\delta y_p - \omega_p\Big[\frac{dy_p}{dt} - C_{0pq}y_q - C_{qpr}\eta_q y_r +$$
$$+ M_\alpha\left(C_{0pq}A_{\alpha q} + \eta_q C_{qpr}A_{\alpha r} - \Omega^\alpha_{0p} - \eta_q\Omega^\alpha_{qp}\right)\Big] \ .$$

On the other hand, we have

$$\delta H - M_\alpha\delta H_\alpha = \omega_p\left(X_p(H) - M_\alpha X_p(H_\alpha)\right) + \left(\frac{\partial H}{\partial y_p} - M_\alpha \frac{\partial H_\alpha}{\partial y_p}\right)\delta y_p \ .$$

It follows that

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$$\eta_p = \frac{\partial H}{\partial y_p} - M_\alpha \frac{\partial H_\alpha}{\partial y_p} ,$$

$$(15) \quad \frac{dy_p}{dt} = -X_p(H) + M_\alpha X_p(H_\alpha) + C_{0pq} y_q + C_{qpr} \eta_q y_r - M_\alpha \left(C_{0pq} A_{\alpha q} + \eta_q C_{qpr} A_{\alpha r} - \Omega^\alpha_{0p} - \eta_q \Omega^\alpha_{qp} \right)$$

$$(\alpha = 1, 2, \ldots, m; \; p, q, r = 1, 2, \ldots, n) .$$

In case the dynamical system is holonomic satisfying conditions (8) and (9), the equations (15) reduce to the form

$$\eta_p = \frac{\partial H}{\partial y_p} - M_\alpha \frac{\partial H_\alpha}{\partial y_p} ,$$

$$(16) \quad \frac{dy_p}{dt} = -X_p(H) + M_\alpha X_p(H_\alpha) + C_{0pq} y_q + C_{qpr} \eta_q y_r$$

$$(\alpha = 1, 2, \ldots, m; \; p, q, r = 1, 2, \ldots, n) .$$

Finally we define a function K by the relation

$$K = H - M_\alpha H_\alpha .$$

To transform equations (15) we note that along a trajectory the constraint equations (14) hold, so that we may write

$$(17) \qquad M_\alpha \frac{\partial H_\alpha}{\partial y_p} = \frac{\partial}{\partial y_p} (M_\alpha H_\alpha) , \quad M_\alpha X_p(H_\alpha) = X_p(M_\alpha H_\alpha) .$$

Consequently the equations (15) for a nonholonomic system assume the form

$$\eta_p = \frac{\partial K}{\partial y_p} ,$$

$$(18) \quad \frac{dy_p}{dt} = -X_p(K) + C_{0pq} y_q + C_{qpr} \eta_q y_r - M_\alpha \left(C_{0pq} A_{\alpha q} + \eta_q C_{qpr} A_{\alpha r} - \Omega^\alpha_{0p} - \eta_q \Omega^\alpha_{qp} \right)$$

$$(\alpha = 1, 2, \ldots, m; \; p, q, r = 1, 2, \ldots, n) .$$

In the case of a holonomic system, the canonical equations (16) take the form

$$\eta_p = \frac{\partial K}{\partial y_p} ,$$

(19) $$\frac{dy_p}{dt} = -X_p(K) + C_{0pq}y_q + C_{qpr}\eta_q y_r \quad (p, q, r = 1, 2, \ldots, n) .$$

If the x's are assumed to be generalised coordinates and $\eta_p = \dot{x}_p$, then all the C_{0pq}, C_{qpr} vanish. In this special case equations (19) reduce to the equations obtained by Šahaĭdarova [4] and equations (18) are identical with those published by Šul'gin [6].

In the rest of this work we limit ourselves to a holonomic system whose motion in the presence of integrable constraints of the form (3) or (14) is governed by the equations (16) or (18).

4. Determination of the constraint multipliers

Consider the motion of a holonomic system which is subjected to constraints of the form (14), the equations governing the motion being given by (16). We shall determine the constraint multipliers M_α as the solution of a system of m linear equations.

For the sake of simplicity, let us assume the constraints to be stationary. Then equations (14) have the form

(20) $$H_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \quad (\alpha = 1, 2, \ldots, m) ,$$

and the canonical equations (16) reduce to the form

$$\eta_p = \frac{\partial H}{\partial y_p} - M_\alpha \frac{\partial H_\alpha}{\partial y_p} ,$$

(21) $$\frac{dy_p}{dt} = -X_p(H) + M_\alpha X_p(H_\alpha) + C_{qpr}\eta_q y_r$$
$$(\alpha = 1, 2, \ldots, m;\ p, q, r = 1, 2, \ldots, n) .$$

Differentiating (20) with respect to the time, we obtain

$$\eta_p X_p(H_\alpha) + \frac{\partial H_\alpha}{\partial y_p}\frac{dy_p}{dt} = 0 .$$

Substituting for η_p and $\frac{dy_p}{dt}$ from (21), we have

$$(22) \quad \left[\frac{\partial H}{\partial y_p} - M_\beta \frac{\partial H_\beta}{\partial y_p}\right] X_p\left(H_\alpha\right) + \frac{\partial H_\alpha}{\partial y_p}\left[-X_p(H)+M_\beta X_p\left(H_\beta\right)+C_{qpr}y_r\left(\frac{\partial H}{\partial y_q} - M_\beta \frac{\partial H_\beta}{\partial y_q}\right)\right] = 0$$

$$(\alpha,\ \beta = 1,\ 2,\ \ldots,\ m;\ p,\ q,\ r = 1,\ 2,\ \ldots,\ n)\ .$$

Let us define the Poisson bracket $(f,\ g)$ by the relation

$$(23) \quad (f,\ g) = \left[\frac{\partial f}{\partial y_p} X_p g - \frac{\partial g}{\partial y_p} X_p f\right] + C_{qpr}y_r \frac{\partial f}{\partial y_q} \frac{\partial g}{\partial y_p}$$

$$(p,\ q,\ r = 1,\ 2,\ \ldots,\ n)\ .$$

In view of (23), the equations (22) are equivalent to

$$\left(H,\ H_\alpha\right) - M_\beta\left(H_\beta,\ H_\alpha\right) = 0 \quad (\alpha,\ \beta = 1,\ 2,\ \ldots,\ m)\ .$$

These equations are a set of m linear equations to find $M_1,\ M_2,\ \ldots,\ M_m$. Substituting their values in (21), we have $2n$ equations to find the η's and y's.

5. Hamilton-Jacobi Theorem

We again consider holonomic systems whose motion in the presence of constraint equations (3) or (14) is described with redundant variables by canonical equations of the form (19) or with constraint multipliers by equations of the form (16). For such systems, the integration of the equations of motion can be effected by a method analogous to the well-known Hamilton-Jacobi method.

In order to formulate the Hamilton-Jacobi Theorem for the canonical equations (16), we consider, as in [1, 3], the partial differential equation

$$(24) \qquad X_0(S) + H\left(x_1,\ \ldots,\ x_n;\ X_1(S),\ \ldots,\ X_n(S);\ t\right) + \phi = 0\ .$$

The function ϕ is to be determined in such a way that if $S\left(x_1,\ \ldots,\ x_n;\ a_1,\ \ldots,\ a_n;\ t\right)$, containing n arbitrary constants $a_1,\ \ldots,\ a_n$, is a complete integral of (24), then the integrals of equations (16) are given by

$$(25) \qquad y_p = X_p(S)\ ,$$

$$(26) \qquad b_p = A_p(S) \quad (p = 1,\ 2,\ \ldots,\ n)\ ,$$

where the A_p define the set of infinitesimal operators for the a's , and b_p are new arbitrary constants.

Let us suppose that the complete integral $S(x_1, \ldots, x_n; a_1, \ldots, a_n; t)$ is substituted in (24). Then, applying the operator A_p to (24) and using (25), we get

$$A_pX_0(S) + \frac{\partial H}{\partial y_p} A_pX_q(S) + \frac{\partial \phi}{\partial y_p} A_pX_q(S) = 0 \quad (q = 1, 2, \ldots, n) ,$$

which, in view of the first set of equations (16), becomes

$$(27) \qquad A_pX_0(S) + \eta_q A_pX_q(S) + M_\alpha \frac{\partial H_\alpha}{\partial y_q} A_pX_q(S) + \frac{\partial \phi}{\partial y_q} A_pX_q(S) = 0 .$$

Again, differentiating (26) with respect to the time, we have

$$(28) \qquad X_0A_p(S) + \eta_q X_qA_p(S) = 0 .$$

Since S is a complete integral, we have

$$X_0A_p(S) = A_pX_0(S) , \quad A_pX_q(S) = X_qA_p(S) ,$$

and the determinant $|X_qA_p(S)| \neq 0$. It follows from (27) and (28) that

$$\left(M_\alpha \frac{\partial H_\alpha}{\partial y_q} + \frac{\partial \phi}{\partial y_q}\right) X_qA_p(S) = 0 ,$$

which, in view of (17), is equivalent to

$$\frac{\partial}{\partial y_q} \left(M_\alpha H_\alpha + \phi\right) X_qA_p(S) = 0 .$$

As the determinant of the coefficients is non-vanishing, the only solution of the last equations is the trivial solution. This implies that

$$(29) \qquad \phi = -M_\alpha H_\alpha + \psi(x_1, \ldots, x_n; t) .$$

Next we again apply the operator X_p to (24) with ϕ given by (29) and use (25). Then we obtain

$$X_pX_0(S) + X_p(H) + \frac{\partial H}{\partial y_q} X_pX_q(S) - X_p\left(M_\alpha H_\alpha\right) - \frac{\partial}{\partial y_q} \left(M_\alpha H_\alpha\right) X_pX_q(S) + X_p(\psi) = 0 ,$$

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which, by virtue of (17) and the first set of equations (16), becomes

$$X_p X_0(S) + X_p(H) + \eta_q X_p X_q(S) - M_\alpha X_p(H_\alpha) + X_p(\psi) = 0 .$$

Finally, differentiating (25) with respect to the time, we get

$$\frac{dy_p}{dt} = X_0 X_p(S) + \eta_q X_q X_p(S) .$$

From the last two equations it follows that

$$\frac{dy_p}{dt} = (X_0, X_p)S + \eta_q (X_q, X_p)S - X_p(H) + M_\alpha X_p(H_\alpha) - X_p(\psi) = 0 ,$$

or, in view of (1) and (25),

$$(30) \qquad \frac{dy_p}{dt} = -X_p(H) + M_\alpha X_p(H_\alpha) + C_{0pq} y_q + C_{qpr} \eta_q y_r - X_p(\psi) .$$

A comparison of (16) and (30) shows that $X_p(\psi) = 0$ for $p = 1, 2, \ldots, n$. It follows that ψ is a function of t only and can be taken as zero by modifying S. Consequently

$$\phi = -M_\alpha H_\alpha \quad (\alpha = 1, 2, \ldots, m) .$$

This leads to the theorem analogous to the Hamilton-Jacobi Theorem, which may be thus stated. *If* $S = S(x_1, \ldots, x_n; a_1, \ldots, a_n; t)$ *is a complete integral of the partial differential equation*

$$X_0(S) + H(x_1, \ldots, x_n; X_1(S), \ldots, X_n(S); t) - M_\alpha H_\alpha = 0 \qquad (\alpha = 1, 2, \ldots, m) ,$$

then the integrals of the canonical equations (16) *are given by the equations* (25) *and* (26).

It may be remarked that, in view of the definition of the function K, the partial differential equation in the theorem leads to

$$X_0(S) + K(x_1, \ldots, x_n; X_1(S), \ldots, X_n(S); t) = 0 ,$$

and its complete integral then provides through (25) and (26) the integrals of the equations of motion in the form (19). This result for the case of generalised coordinates and momenta has been stated in [4].

Thus, the modified Hamilton-Jacobi Theorem for integrating canonical equations of motion of holonomic systems with constraint multipliers leads to the solution which contains more constants of integration than are necessary to determine the motion. In fact, of the $2n$ constants of integration a_p, b_p only $2(n-m)$ will be arbitrary. The general solution will contain $2(n-m)$ arbitrary constants which are to be determined from the initial conditions of the problem.

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Department of Mathematics,
University of Islamabad,
Islamabad, Pakistan.