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# Non-Orientable Surfaces and Dehn Surgeries

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*Abstract.* Let $K$ be a knot in $S^3$. This paper is devoted to Dehn surgeries which create 3-manifolds containing a closed non-orientable surface $\hat{S}$. We look at the slope $p/q$ of the surgery, the Euler characteristic $\chi(\hat{S})$ of the surface and the intersection number $s$ between $\hat{S}$ and the core of the Dehn surgery. We prove that if $\chi(\hat{S}) \geq 15 - 3q$, then $s = 1$. Furthermore, if $s = 1$ then $q \leq 4 - 3\chi(\hat{S})$ or $K$ is cabled and $q \leq 8 - 5\chi(\hat{S})$. As consequence, if $K$ is hyperbolic and $\chi(\hat{S}) = -1$, then $q \leq 7$.

## 1 Introduction

Let $K$ be a non-trivial knot in $S^3$, $N(K)$ a regular neighborhood of $K$ and $X_K = S^3 - \operatorname{int} N(K)$ the exterior of $K$. The unoriented isotopy class of an essential simple closed curve on a torus is called its *slope*. We use $\mathbb{Q} \cup \{\frac{1}{0}\}$ as in [17] to parameterize the slopes on $\partial X_K$ with respect to a basis of $H_1(\partial X_K)$ formed by a meridian and a preferred longitude of $K$. Note that $\{\frac{1}{0}\}$ is the slope of a meridian of $K$.

Let $r$ be a slope on $\partial X_K$, and let $K(r)$ be the closed 3-manifold obtained by $r$-Dehn surgery on $K$. Thus $K(r) = X_K \cup J_r$, where $J_r$ is a solid torus, glued to $X_K$ along their boundaries in such a way that $r$ bounds a meridian disk in $J_r$. We denote by $K_r$ the core of the attached solid torus $J_r$ in $K(r)$. $K(r)$ is said to be *integral* when $r = p/q$ and $q = 1$. We may assume that $q$ is non-negative.

In general, knowing the number of intersections of a specific surface with the core of a surgery can be very useful.

Concerning the production of essential 2-spheres (those that do not bound a 3-ball) *the Cabling Conjecture* [5] says that only cabled knots in $S^3$ can produce essential 2-spheres by a Dehn surgery. This is equivalent to say that the minimal number of intersections of an essential sphere with the core of a surgery is 2.

Concerning the production of essential tori, Gordon and Luecke (see [11]) have shown that if a non-integral surgery on a hyperbolic knot creates an essential torus which meets the core of the surgery twice, then that in fact the knot has to be one of the infinite family of such knots described by Eudave-Muñoz (see [3]).

This paper is devoted to the study of the production of a closed non-orientable surface $\hat{S}$ by a $p/q$-Dehn surgery on a knot $K$ in $S^3$.

Recall that if $\chi(\hat{S}) \geq 0$, then $q = 1$. In fact if $\hat{S}$ is a Klein bottle, then by [9, Theorem 1.3], $q = 1$. Furthermore if $\chi(\hat{S}) = 1$, *i.e.* $\hat{S}$ is a projective plane, then $K(p/q)$ is reducible or $K(p/q)$ is homeomorphic to the real 3-dimensional projective

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space, but in both cases $q = 1$, by [13, Theorem 1] and the Cyclic Surgery Theorem [1].

Because of these remarks, throughout this paper $\hat{S}$ denotes a closed non-orientable surface in $K(p/q)$ with $\chi(\hat{S}) < 0$.

Recall that the genus of a closed non-orientable surface $\hat{S}$ is the maximal number of disjoint Mobius bands in $\hat{S}$. We denote by $g$ the genus of $\hat{S}$, so $\hat{S}$ is the connected sum of $g$ projective planes, and $\chi(\hat{S}) = 2 - g$. In the following, we assume that $g \geq 3$.

Before going further, we may note that if $K(p/q)$ contains a closed non-orientable surface, then $p$ is even and $q$ is odd, by [9, Lemma 6.2].

***Conjecture A*** *Only integral surgeries can produce a closed non-orientable surface of genus three.*

*Let $s$ be the minimal number of intersection between $K_{p/q}$ and $\hat{S}$. The main result is the following:*

***Theorem 1.1***

(i) *If $s \neq 1$, then $\chi(\hat{S}) \leq 14 - 3q$.*
(ii) *If $s = 1$, then $q \leq 4 - 3\chi(\hat{S})$ or $K$ is cabled and $q \leq 8 - 5\chi(\hat{S})$.*

***Corollary 1.2*** *Let $K$ be an hyperbolic knot in $S^3$. Suppose that $K(p/q)$ contains a closed non-orientable surface of genus three. Then $q \leq 7$.*

*Furthermore, if $s > 1$ then $q \leq 5$.*

**Proof** Since $\chi(\hat{S}) = -1$, and $K$ is not a cable knot, the result follows immediately from Theorem 1.1. ■

Let $K$ be a knot in $S^3$, *a non-orientable Seifert surface for $K$* is a connected compact non-orientable surface with boundary $K$. It is easy to see that any even integer can be the boundary slope of some non-orientable Seifert surface for $K$ (see 14). Hence $K(2n/1)$ contains a closed non-orientable surface $\hat{S}$ and $|K_{2n/1} \cap \hat{S}| = 1$.

Actually, if $p$ is even then, $K(p/q)$ always contain a closed non-orientable surface, which intersects the core of the surgery once. This is because a simple closed curve $\gamma$ of slope $p/q$ ($p$ even) on the boundary of the exterior $X$ of $K$ is null homologous in $X$, modulo 2; so $\gamma$ bounds a compact surface which is non-orientable (since $p$ is not 0).

Note also that it is proved in [2] that alternating knots have essential non-orientable Seifert surfaces; furthermore their boundary slopes are not necessary unique (see [15]).

***Question B*** *Does there exist a surgery that produces a closed non-orientable surface of genus three such that its core intersects the surface necessarily more than once?*

Here are a few words about the argument of the Theorem 1.1, and the organization of the paper.

Assume that there exists a $p/q$-Dehn surgery that produces a non-orientable closed surface $\hat{S}$. We may suppose that $\hat{S}$ is chosen (among all closed non-orientable surfaces in $K(p/q)$) to minimize the intersection number $s = \#|K_{p/q} \cap \hat{S}|$. Now, among all these *minimal* closed non-orientable surfaces, we choose $\hat{S}$ of minimal genus.

The proof of Theorem 1.1 is based on the combinatorics of the intersection graphs coming from a pair consisting of a level sphere $\hat{Q}$ in $S^3$, and the boundary $\hat{P}$ of a thin regular neighborhood of $\hat{S}$ in $K(p/q)$.

In Section 2, we recall the basic definitions of the intersection graphs. We consider $M$ a compact, connected and orientable 3-manifold whose boundary is a torus. Then we recall the basic definitions of a pair of intersection graphs that come from two orientable surfaces with boundaries, and properly embedded in $M$. Finally, we give an inequality relating the Euler characteristics of the corresponding surfaces and the number of vertices, *trivial loops* and *Scharlemann cycles* of the graphs. This inequality generalizes the inequality of C. Hayashi and K. Motegi [14].

In Section 3, we use the specific properties of the pair of intersection graphs associated to the punctured surfaces $\hat{Q} \cap X_K$ and $\hat{P} \cap X_K$, *i.e.* the surfaces ($\hat{Q}$ and $\hat{P}$ respectively) with the discs of the Dehn filling removed. We then improve the inequality of the previous section.

In the next section, we consider the case where $\hat{S}$ is pierced only once by the core of the Dehn surgery. Then we prove that if $K$ is a cabled knot then $q \leq 5g-2$, otherwise $q \leq 3g - 2$.

In Section 5, we consider the case where $\hat{S}$ is pierced more than once by the core of the Dehn surgery. Then we prove (using the inequality of Section 3) that $q < 5 + \frac{g}{s}$ where $s$ is the number of intersections between $\hat{S}$ and the core of the Dehn surgery.

Finally, the last section is devoted to the special case where the knots are composite knots, Conway knots or 2-bridge knots. We apply the previous results to obtain the following.

**Lemma 1.3** *Let $K$ be a knot in $S^3$, on which $p/q$ surgery produces a closed non-orientable surface of genus $g \geq 3$, with $s = 1$.*

*If $K$ is a composite knot, then $q \leq g - 1$.*

*If $K$ is a Conway knot or a 2-bridge knot, then $q \leq g$.*

**Lemma 1.4** *Let $K$ be a knot in $S^3$, on which $p/q$ surgery produces a closed non-orientable surface of genus $g \geq 3$, with $s \neq 1$.*

*If $K$ is a composite knot, then $q \leq \frac{g+1}{3}$.*

*If $K$ is a Conway knot or a 2-bridge knot, then $q \leq \frac{g+4}{3}$.*

**Corollary 1.5** *Let $K$ be a knot in $S^3$, on which $p/q$ surgery produces a closed non-orientable surface of genus $g = 3$.*

*If $K$ is a composite knot, then $q = 1$.*

*If $K$ is a Conway knot or a 2-bridge knot, then $q = 1$ or $q = 3$ and $s = 1$.*

## 2 Intersection Graphs

Let $M$ be a compact, connected and orientable 3-manifold whose boundary is a torus. Let $(F_i, \partial F_i) \subset (M, \partial M)$ $(i = 1, 2)$ be a compact surface, which is possibly non-orientable, compressible or $\partial$-compressible in $M$.

Suppose that $\partial F_i \cap \partial M \neq \varnothing$ and that the components of $\partial F_1 \cap \partial M$ and of $\partial F_2 \cap \partial M$ represent distinct slopes $r_1$ and $r_2$ on $\partial M$. We assume that $F_1$ and $F_2$ intersect transversally and that $\partial F_1$ and $\partial F_2$ intersect in the minimal number of points, so that each component of $\partial F_1$ and each of $\partial F_2$ intersect just $\Delta$ times in $\partial M$, where $\Delta = \Delta(r_1, r_2)$ is the minimal geometric intersection number between the slopes $r_1$ and $r_2$.

We obtain a surface $\hat{F}_i$ (denoted also by $F(r_i)$) when we cap off the components of $\partial F_i \cap \partial M$ with meridian discs of the $r_i$-Dehn filling. The intersections of $F_1$ and $F_2$ in $M$ give rise to a pair of labelled graphs, $G_1 \subset F(r_1)$ and $G_2 \subset F(r_2)$, in the usual way (see [7] for more details). We obtain these graphs as follows:

$G_1$ is the graph obtained by taking as fat vertices the meridian disks of the $r_1$-Dehn filling, which cap off the punctured surface $F_1$ to obtain $\hat{F}_1$, *i.e.* the components of $F(r_1) - \operatorname{Int} F_1$. Similarly the vertices of $G_2$ are the components $F(r_2) - \operatorname{Int} F_2$.

The edges of $G_1$ are the arc components of $F_1 \cap F_2$ in $F(r_1)$. Similarly, the edges of $G_2$ are the arc components of $F_1 \cap F_2$ in $F(r_2)$.

We number the components of $\partial F_1 : 1, 2, \ldots, n_1$ in the order in which they appear on $\partial M$. Similarly, we number the components of $\partial F_2 : 1, 2, \ldots, n_2$. That gives a numbering of the vertices of $G_1$ and $G_2$. Furthermore, it induces a labelling of the endpoints of edges in $G_1$ and $G_2$: each endpoint $\alpha$ of an edge of $G_1$ is in a component $C$ of $\partial F_1$, then the *label* of $\alpha$ is the number of the component of $\partial F_2$, which intersects $C$ in $\alpha$; similarly, we label the endpoints of edges in $G_2$ by the number of the corresponding components of $\partial F_1$.

On a vertex of $G_1$ (resp. $G_2$) one sees the labels 1 through $n_2$ repeated $\Delta$ times (resp. 1 through $n_1$ repeated $\Delta$ times) appearing in order around the vertex. We say that $(G_1, G_2)$ is *a pair of intersection graphs of type* $(\hat{F}_1, \hat{F}_2)$, or *associated to* $(F_1, F_2)$, where $F_1$ and $F_2$ are a pair of transverse surfaces properly embedded, and in general position in $M$.

Let $i \in \{1, 2\}$. When $F_i$ is orientable, we say that two vertices on $G_i$ are *parallel* if the ordering of the labels on each is clockwise or the ordering on each is anticlockwise, otherwise the vertices are called *antiparallel*. We use this definition only from Section 4.

A *cycle* is a subgraph homeomorphic to a circle, when vertices are considered as points. Let $x$ be a label of $G_i$. An *$x$-edge* is an edge with label $x$ at one endpoint.

An *$x$-cycle* of $G_i$ is a cycle $\Sigma$ of $x$-edges of $G_i$, such that (see Figure 1):

(i) $\Sigma$ bounds a disk $D$ in $\hat{F}_i$;
(ii) $\Sigma$ can be oriented such that the tail of each edge has the label $x$;
(iii) for all the vertices of $G_i$ in $\Sigma$, the labelling is in the same sense (clockwise or anticlockwise).

A *Scharlemann cycle* of $G_i$ is an $x$-cycle that bounds a disk face of $G_i$. The number of edges in a Scharlemann cycle $\Sigma$, is called the *length* of $\Sigma$. A pair of edges $\{e_1, e_2\}$ in

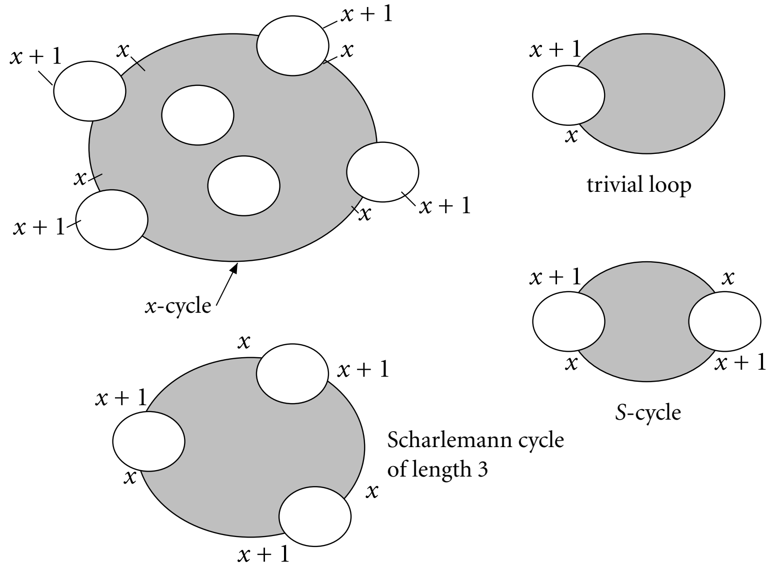


*Figure 1*: $x$-cycles.

$G_Q$ is called an *S-cycle* if it is a Scharlemann cycle of length 2. A Scharlemann cycle of length 1 is called *trivial loop*. In the following, all the Scharlemann cycles are assumed to be of length at least two; otherwise we will talk about trivial loops.

***Lemma 2.1*** *Let $M$ be an orientable and compact 3-manifold with a torus as boundary. Let $(G_1, G_2)$ be a pair of intersection graphs of type $(\hat{F}_1, \hat{F}_2)$, where $\hat{F}_1$ and $\hat{F}_2$ are orientable. For all $i \in \{1, 2\}$, in the graph $G_i$, we denote by:*

(i) *$n_i$ the number of vertices;*
(ii) *$t_i$ the number of trivial loops;*
(iii) *$m_i$ the number of Scharlemann cycles.*

*Then*

$$\Delta \leq 2 + \frac{t_1 - \chi(\hat{F}_1)}{n_1} + \frac{t_2 - \chi(\hat{F}_2)}{rn_2} + \frac{m_1}{n_1} + \frac{m_2}{n_2}.$$

**Proof** Let $i = 1$ or 2. Let $\varepsilon > 0$ and $\hat{F}_i \times [-\varepsilon, +\varepsilon]$ be a thin regular neighborhood of $\hat{F}_i$. Let $\tilde{F}_i = \hat{F}_i \times \{-\varepsilon\} \cup \hat{F}_i \times \{+\varepsilon\}$. Let $\{\sigma_1, \ldots, \sigma_{m_i}\}$ be the set of all the Scharlemann cycles of $G_i$ (possibly an empty set); and $D_k$ the disk face bounded by $\sigma_k$, for all $k \in \{1, \ldots, m_i\}$. Now, let $A_k$ be an annulus in $D_k \times [-\varepsilon, +\varepsilon]$, such that one boundary-component of $A_k$ lies in $A_k \times \{-\varepsilon\}$, and the other lies in $A_k \times \{+\varepsilon\}$. Let $A = A_1 \cup \cdots \cup A_{m_i}$. Let $\hat{R}_i$ be the resulting surface $\hat{R}_i = \tilde{F}_i^* \cup A$, where $\tilde{F}_i^*$ is the surface obtained by removing in $\tilde{F}_i$ the disks bounded by the boundary components of the annuli $A_k$ (in $D_k \times \{\pm\varepsilon\}$). If $m_i \neq 0$ then the genus of $\hat{R}_i$ is $2g(\hat{F}_i) + m_i - 1$, where $g(\hat{F}_i)$ denotes the genus of $\hat{F}_i$. Therefore $\chi(\hat{R}_i) = 2\big(1 - 2g(\hat{F}_i) - m_i + 1\big) = 2\chi(\hat{F}_i) - 2m_i$. As usual, we denote $\hat{R}_i \cap M$ by $R_i$.

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Let $S_i = R_i$ if $m_i \neq 0$; otherwise $S_i = F_i$ ($m_i = 0$). Let $(H_1, H_2)$ be the pair of intersection graphs associated to the pair of surfaces $(S_1, S_2)$. Now, we apply the Hayashi-Motegi inequality ([14, Theorem 2.1]). If $m_1 = m_2 = 0$ then we get immediately the required inequality. Next, assume that both $m_1$ and $m_2$ are $\neq 0$. By construction, in the graph $H_i$ (for $i = 1$ or 2) the number of vertices is $2n_i$, the number of of trivial loops is $2t_i$ and there is no Scharlemann cycle. Then

$$\Delta \leq 2 + \frac{2t_1 - \chi(\hat{R}_1)}{2n_1} + \frac{2t_2 - \chi(\hat{R}_2)}{2n_2},$$

which gives

$$\Delta \leq 2 + \frac{t_1 - \chi(\hat{F}_1)}{n_1} + \frac{t_2 - \chi(\hat{F}_2)}{n_2} + \frac{m_1}{n_1} + \frac{m_2}{n_2}.$$

Now, without loss of generality, we may assume that $m_1 = 0$ and that $m_2 \neq 0$. We obtain:

$$\Delta \leq 2 + \frac{t_1 - \chi(\hat{F}_1)}{n_1} + \frac{2t_2 - \chi(\hat{R}_2)}{2n_2},$$

which gives again the required inequality (since $m_1 = 0$). ■

We say that two edges in $G_i$ are *parallel* if they represent a cycle of length two, which bounds a disk (when vertices are considered as points) in the punctured surface $F_i$; for $i = 1$ or 2. When we represent each family of parallel edges in a graph $G_i$ by a single edge, we obtain the *reduced graph* of $G_i$.

In the following $M = X_K$, the exterior of a knot $K$ in $S^3$.

## 3 General Situation

Suppose that $K(p/q)$ contains a closed non-orientable surface $\hat{S}$, which is chosen to minimize the intersection number $s = \#|K_{p/q} \cap \hat{S}|$, where $K_{p/q}$ is the core of the surgery. Let $S = \hat{S} \cap X_K$.

Let $p_+$, $p_-$ be two points in $S^3$. We can write $S^3 - \{p_+, p_-\} = \hat{Q} \times ]-1, 1[$. By Gabai's Lemma (see [4, p. 491]) putting $K$ in thin position, we can find a 2-sphere $\hat{Q} = \hat{Q} \times \{i\}$ for some $i$ such that

(i) $\hat{Q}$ intersects $K$ transversely. Thus $Q = \hat{Q} \cap X_K$ is a properly embedded planar surface in $X_K$ such that each component of $\partial Q$ is a copy of the meridian of $K$;
(ii) $Q$ intersects $S$ transversely and no arc component of $Q \cap S$ is parallel in $Q$ to $\partial Q$ or parallel in $S$ to $\partial S$.

Now, let $\hat{P} = \partial N(\hat{S})$ be the boundary of a thin regular neighborhood of $\hat{S}$ in $K(p/q)$. Then $\hat{P}$ is an orientable surface, meeting $K_{p/q}$ in $n_1 = 2s$ points. Note that $\chi(\hat{P}) = 2\chi(\hat{S})$ and the genus of $\hat{P}$ is $g(\hat{P}) = g(\hat{S}) - 1$.

In the following, we denote by $g$ the genus of $\hat{S}$. Note that $\chi(\hat{S}) = 2 - g$.

Let $P = \hat{P} \cap X_K$. By the minimality of $\hat{S}$ and the choice of $Q$, no circle component of $P \cap Q$ bounds a disk in either $P$ or $Q$; moreover no arc-component of $P \cap Q$ is boundary-parallel in either $P$ or $Q$. We consider the pair of intersection graphs $(G_P, G_Q)$ associated to the pair of surfaces $(P, Q)$, and the pair of intersection graphs $(G_S, H_Q)$ associated to the pair of surfaces $(S, Q)$. The graph $G_P$ may be seen as the double cover of $G_S$, where its edges are also doubled (see [16, Figures 2.1, 2.3 and 2.4]).

Since the surface $\hat{P}$ is separating, we consider two sides in $K(p/q)$: the *black side* containing $\hat{S}$, and the *white side*, *i.e.*, $K(p/q) - N(\hat{S})$.

Therefore, the faces of $G_Q$ are divided into black and white faces. Furthermore, each black face of $G_Q$ is a disk-face of length two, corresponding to the regular neighborhood of an edge of $H_Q$.

***Lemma 3.1*** *The graphs $G_P$ and $G_S$ cannot contain a Scharlemann cycle.*

**Proof** Otherwise, $S^3$ would contain a non-trivial lens space as connected summand (see [1] for more details) which is impossible. ∎

***Lemma 3.2*** *The integer $s \geq 1$ is odd.*

**Proof** If $s$ is even, then we can obtain a closed non-orientable surface in $X_K$ by attaching suitable annuli in $\partial X_K$ to $S$ along $\partial S$, which is impossible (since $S^3$ does not contain closed non-orientable surface). ∎

Let $n_2$ be the number of vertices in $G_Q$, and $m$ be the number of Scharlemann cycles in $G_Q$. Since $\chi(\hat{P}) = 2\chi(\hat{S})$ and $\chi(\hat{Q}) = 2$, we obtain the following by Lemma 2.1. Recall that $\Delta(\frac{1}{0}, p/q) = q$.

***Lemma 3.3*** $q \leq 2 - \dfrac{\chi(\hat{S})}{s} + \dfrac{m}{n_2} - \dfrac{2}{n_2}.$

## 4 Intersection Number One

In this section, we suppose that $s = 1$. Thus, the graph $G_S$ has a single vertex and all its edges are cycles (when the vertex is considered as a point).

We divide the edges of $G_S$ into two kinds: *positive* and *negative edges*. When the vertex is considered as a point, the regular neighborhood on $\hat{S}$ of a positive edge is an annulus, while the regular neighborhood on $\hat{S}$ of a negative edge is a Möbius band. The positive edges join antiparallel vertices in $H_Q$, while the negative edges join parallel vertices in $H_Q$. Recall that the edges of $G_Q$ are the doubles of those in $H_Q$. Therefore, each negative edge gives rise to an $S$-cycle in $G_Q$. Let $\mathcal{P}$ and $\mathcal{N}$ be the number of positive and negative edges, respectively.

***Lemma 4.1*** *If $K$ is not a cable knot then $\mathcal{N} \leq g(n_2 - 1)$; otherwise $\mathcal{N} \leq g(2n_2 - 1)$.*

**Proof** Let $\mathcal{F}$ be a family of parallel negative edges in $G_S$, and $\#|\mathcal{F}|$ be its number of edges.

***Claim 4.2*** *If $K$ is not a cable knot then $\#|\mathcal{F}| < n_2$; otherwise $\#|\mathcal{F}| < 2n_2$.*

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**Proof** If $\#|\mathcal{F}| \geq n_2$ then $K$ is a cable knot, by [8, Section 5]. So, we assume that $K$ is cabled and that $\#|\mathcal{F}| \geq 2n_2$. The edges in $\mathcal{F}$ describe orbits in $H_Q$ (see [8, Section 5] for more details). Since $\#|\mathcal{F}| \geq 2n_2$, each orbit is represented at least twice. By an innermost argument, there are two edges $e, f$ in one orbit which are parallel in $H_Q$. But $e, f$ are already parallel edges in $G_S$; since we assume that $K(p/q)$ does not contain a projective plane (otherwise $q = 1$) this is impossible by [6, Lemma 2.1]. ■

Let $\Gamma$ be the *reduced graph* of $G_S$. Since $\hat{S}$ is the connected sum of $g$ projective planes, $\Gamma$ contains at most $g$ negative edges. This completes the proof of Lemma 4.1 ■

***Lemma 4.3*** $\mathcal{P} \leq \dfrac{(g-1)n_2}{2}$.

**Proof** Let $x$ be a label of $G_S$, and $\Gamma_x$ be the subgraph of $G_S$ consisting of all the positive edges with one label $x$. We denote by $E_x$ the number of edges of $\Gamma_x$.

***Claim 4.4*** *For all labels $x \in \{1, \dots, n_2\}$, the graph $\Gamma_x$ cannot contain a disk-face.*

**Proof** Assume for a contradiction that there exists a label $x$, such that $\Gamma_x$ contains a disk-face $D$. Then, its boundary is a cycle, whose edges are $x$-edges. Therefore, by [13, Lemma 2.2] the graph $G_S$ contains a Scharlemann cycle in $D$, in contradiction to Lemma 3.1. ■

***Claim 4.5*** *For all labels $x \in \{1, \dots, n_2\}$, $E_x \leq g - 1$, equivalently $\chi(\hat{S}) \leq 1 - E_x$.*

**Proof** By the Euler characteristic equality:

$$\chi(\hat{S}) = V - E + \sum_{f \text{ face of } \Gamma_x} \chi(f)$$

where $V = 1$ is the number of vertices and $E = E_x$ the number of edges of $\Gamma_x$. By the previous claim, $\chi(\hat{S}) \leq 1 - E_x$. Since $\chi(\hat{S}) = 2 - g$, $E_x \leq g - 1$. ■

An edge in $G_S$ which has the same label at its both endpoints is a negative edge (because it joins the same vertex in $H_Q$). Therefore, by the previous claim, each vertex in $H_Q$ is incident to at most $g - 1$ positive edges. Thus: $\mathcal{P} \leq \frac{1}{2}(g-1)n_2$, proving Lemma 4.3. ■

***Lemma 4.6*** *If $K$ is not a cable knot then $q \leq 3g - 2$; otherwise $q \leq 5g - 2$.*

**Proof** The number of edges of $G_S$ is $\frac{1}{2}qsn_2 = \frac{qn_2}{2} = \mathcal{P} + \mathcal{N}$. Using Lemmas 4.1 and 4.3, we obtain two inequalities according to whether $K$ is cabled or not:

(1) $\frac{qn_2}{2} \leq \frac{1}{2}(g-1)n_2 + g(2n_2 - 1)$, or
(2) $\frac{qn_2}{2} \leq \frac{1}{2}(g-1)n_2 + g(n_2 - 1)$;

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which gives

(1) $q \leq 5g - 1 - \frac{2g}{n_2}$, if $K$ is cabled, and
(2) $q \leq 3g - 1 - \frac{2g}{n_2}$, if $K$ is not cabled. ■

## 5 Large Intersection Number

In this section, we assume that $s > 1$, then $s \geq 3$, by Lemma 3.3. Consequently $n_1 = 2s \geq 6$.

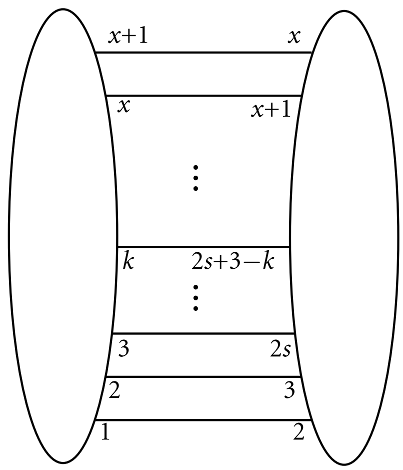


*Figure 2*: Family of parallel edges in $G_Q$.

***Lemma 5.1*** *No Scharlemann cycle in $G_Q$ bounds a white face. In particular every Scharlemann cycle is an S-cycle.*

**Proof** We argue by contradiction. If $G_Q$ contains a Scharlemann cycle which bounds a white disk-face, then there exists a corresponding Scharlemann cycle $\sigma$ in $H_Q$, which bounds a disk-face $f$. Let $x, x+1 \pmod{s}$ be the labels of $\sigma$ (since $s \geq 3$ we have $x \neq x+1$). Let $H$ be the annulus in $\partial X_K$ between the components labelled by $x$ and $x+1$ of $\partial S$. Let $S' = \hat{S}$ with the meridian disks labelled by $x$ and $x+1$ removed. Then we can use $f$ to compress $S' \cup H$ to obtain a new non-orientable closed surface with the same genus as $\hat{S}$, but intersecting the core of the surgery fewer times, in contradiction to the minimality of $\hat{S}$. ■

***Lemma 5.2*** $m \leq 3n_2 - 6$.

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**Proof** Let $\Gamma$ be the subgraph of $G_Q$ whose vertices are the vertices of $G_Q$ and whose edges are in one-one correspondence with the $S$-cycle in $G_Q$, each $S$-cycle being represented by a single edge in the obvious way.

***Claim 5.3*** *The graph $\Gamma$ does not contain a disk-face of length two.*

**Proof** Assume for a contradiction that there exist two edges in $\Gamma$, which cobound a disk-face $Z$. Let $\sigma$ and $\sigma'$ be the corresponding $S$-cycles. Without loss of generality, we may assume that $\{1, 2\}$ are the labels of $\sigma$ and $\{x, x + 1\}$ are those of $\sigma'$.

Then $G_Q \cap Z$ is a family of parallel edges, within no more $S$-cycle inside. Let $e$ be an edge in $G_Q \cap Z$. Therefore, if $k$ is the label at one endpoint of $e$ then $2s + 3 - k \pmod{2s}$ is the label at the other endpoint (see Figure 2).

Consequently, $x = 2s/2 + 1 = s + 1$, which is an even integer. Thus, $\sigma'$ is a white Scharlemann cycle, in contradiction to Lemma 5.1. ■

Recall that there is no trivial loop in $G_Q$, so all the disk-faces of $\Gamma$ have at least two sides. Let $F$ be the number of disk-faces of $\Gamma$. By the previous lemma, each disk-face has at least three sides. Since $m$ is the number of edges of $\Gamma$, we obtain $3F \leq 2m$. The Euler characteristic equality gives:

$$2 = \chi(\hat{Q}) = V - E + \sum_{f \text{ face of } \Gamma} \chi(f)$$

where $V = n_2$ is the number of vertices and $E = m$ the number of edges of $\Gamma$. Therefore $2 \leq n_2 - m + F$, with $F \leq 2m/3$; so $m \leq 3n_2 - 6$ finishing the proof of Lemma 5.2. ■

With Lemma 3.3, we obtain the following.

***Corollary 5.4*** $q \leq 5 - \dfrac{\chi(\hat{S})}{s} - \dfrac{8}{n_2}$, *and* $q < 5 + \dfrac{g}{s} - \dfrac{8}{n_2}$.

***Corollary 5.5*** $\chi(\hat{S}) \leq 14 - 3q$, *equivalently* $g \geq 3q - 12$.

**Proof** Since $s \geq 3$, and assuming that $\chi(\hat{S}) < 0$ (otherwise $q = 1$), the previous corollary gives $q < 5 - \frac{\chi(\hat{S})}{3}$. ■

***Corollary 5.6*** *If $g \leq 8$ (equivalently if $\chi(\hat{S}) \geq -6$) then $q \leq 5$.*

**Proof** The previous corollary gives that $q < 7$; but $q$ is odd. ■

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## 6 Composite and Conway Knots

A knot is said to be a *composite knot* if there exists a 2-sphere $\hat{Q}$ in $S^3$ such that $\hat{Q}$ intersects $K$ in 2 points and $Q = \hat{Q} \cap X_K$ is incompressible in $X_K$.

A knot is said to be *a Conway knot* if there exists a 2-sphere $\hat{Q}$ in $S^3$ such that $\hat{Q}$ intersects $K$ in 4 points and $Q = \hat{Q} \cap X_K$ is incompressible in $X_K$.

If $K$ is a composite knot we may assume that $n_2 = 2$, choosing for $\hat{Q}$ the incompressible 2-punctured sphere in $X_K$. Similarly, if $K$ is a Conway knot or a knot with two bridges, we may assume that $n_2 = 4$.

***Lemma 6.1*** *If $K$ is a composite knot then $q \leq 1 - \frac{\chi(\hat{S})}{s}$, equivalently $q \leq 1 + \frac{g-2}{s}$.*

**Proof** We may assume that $n_2 = 2$. Since $\hat{Q}$ is separating, and since the graph $G_Q$ cannot have a trivial loop, it cannot have a Scharlemann cycle either. Therefore, by Lemma 3.3, we obtain $q \leq 1 - \frac{\chi(\hat{S})}{s}$, equivalently $q \leq 1 + \frac{g-2}{s}$. ■

***Lemma 6.2*** *If $K$ is a Conway knot or a 2-bridge knot then $q \leq 2 - \frac{\chi(\hat{S})}{s}$, equivalently $q \leq 2 + \frac{g-2}{s}$.*

**Proof** We may assume that $n_2 = 4$. Since $\hat{Q}$ is separating, the possible Scharlemann cycles in $G_Q$ are of length two. By Claim 5.3 they cannot be parallel. Therefore $G_Q$ contains at most 2 Scharlemann cycles. Then, by Lemma 3.3, we obtain $q \leq 2 - \frac{\chi(\hat{S})}{s}$, equivalently $q \leq 2 + \frac{g-2}{s}$. ■

Recall that $s$ and $q$ are odd. Therefore Lemmas 6.1 and 6.2 prove Lemmas 1.3 and 1.4.

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