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ELEMENTARY OPERATORS ON $\mathcal{J}$-SUBSPACE LATTICE ALGEBRAS

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The abstract concept of an elementary operator was recently introduced and studied by other authors. In this paper, we describe the general form of elementary operators between standard subalgebras of $\mathcal{J}$-subspace lattice algebras. The result can apply to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras.

Throughout, if X is a Banach space by $\mathcal{B}(X)$ we mean the algebra of all bounded linear operators on X. The topological dual of X is denoted by X^*. For $x \in X$ and $f \in X^*$, the operator $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in X$, which has rank one if and only if both x and f are nonzero. For any non-empty subset $L \subseteq X$, $L^\perp$ stands for its annihilator, that is $L^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$.

Let $\mathcal{L}$ be a *subspace lattice* on a Banach space X, that is, a family of (closed) subspaces of X satisfying

(i) (0), $X \in \mathcal{L}$ and

(ii) $\cap_\gamma L_\gamma \in \mathcal{L}$, $\vee_\gamma L_\gamma \in \mathcal{L}$,

for every family $\{L_\gamma\}_\Gamma$ of elements of $\mathcal{L}$, where $\vee_\Gamma L_\gamma$ denotes the closed linear span of $\cup_\Gamma L_\gamma$. The associated *subspace lattice algebra* $\operatorname{Alg}\mathcal{L}$ is the set of all operators in $\mathcal{B}(X)$ which leave every subspace in $\mathcal{L}$ invariant. It is easy to see that $\operatorname{Alg}\mathcal{L}$ is a unital weakly closed operator algebra. Put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq (0) \text{ and } K_- \neq X\},$$

where $K_- = \vee\{L \in \mathcal{L} : K \not\subseteq L\}$. Call $\mathcal{L}$ a *$\mathcal{J}$-subspace lattice* if

(i) $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$,

(ii) $\cap\{K_- : K \in \mathcal{J}(\mathcal{L})\} = (0)$,

(iii) $K \vee K_- = X$ for every $K \in \mathcal{J}(\mathcal{L})$,

(iv) $K \cap K_- = (0)$ for every $K \in \mathcal{J}(\mathcal{L})$.

The class of $\mathcal{J}$-subspace lattices was defined in [**14**] and subsequently discussed in [**9, 10**]. The simplest example of a $\mathcal{J}$-subspace lattice is any pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, X\}$, where K, L and M are subspaces of a Banach space X such

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that $K \vee L = X$, $K \cap M = (0)$ and $L \subset M$. In this case, $K_- = M$, $L_- = K$ and $\mathcal{J}(\mathcal{P}) = \{K, L\}$. For further discussions of pentagon subspace lattices may see [**4, 6**]. Another important member of the class of $\mathcal{J}$-subspace lattices is atomic Boolean subspace lattices. The reference [**10**] tells us that every commutative $\mathcal{J}$-subspace lattice on a Hilbert space is an atomic Boolean subspace lattice. However, most $\mathcal{J}$-subspace lattices on a Hilbert space are non-commutative. Therefore, $\mathcal{J}$-subspace lattices and $\mathcal{J}$-subspace lattice algebras deserve some attention.

For a subspace lattice $\mathcal{L}$, the relevance of $\mathcal{J}(\mathcal{L})$ is due to the following lemma, from which we can see that every $\mathcal{J}$-subspace lattice algebra $\operatorname{Alg}\mathcal{L}$, where $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, is rich in rank one operators.

Lemma 1. (Longstaff [**8**].) *If $\mathcal{L}$ is a subspace lattice on a Banach space X, then the rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$ if and only if there exists some $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^{\perp}$, where $K_-^{\perp}$ means $(K_-)^{\perp}$.*

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be algebras over the same field. In the recent papers [**1, 2, 13**], the authors introduced and studied an abstract concept of elementary operators between $\mathcal{A}_1$ and $\mathcal{A}_2$. They considered an ordered pair (M, M^*) where $M : \mathcal{A}_1 \to \mathcal{A}_2$ and $M^* : \mathcal{A}_2 \to \mathcal{A}_1$ are linear mappings such that

$$(1) \qquad \begin{cases} M\big(xM^*(y)z\big) = M(x)yM(z), \\ M^*\big(yM(x)u\big) = M^*(y)xM^*(u) \end{cases}$$

for all $x, z \in \mathcal{A}_1, y, u \in \mathcal{A}_2$. Following those references, such a pair (M, M^*) is called an *elementary operator* of $\mathcal{A}_1$ into $\mathcal{A}_2$ (of length one). For $a, b \in \mathcal{A}_1$, denote by $M_{a,b}$ the two-sided multiplication given by $M_{a,b}(x) = axb$, $x \in \mathcal{A}_1$. Then $(M_{a,b}, M_{b,a})$ is an elementary operator of $\mathcal{A}_1$ into itself. The same is true for every double centraliser of a faithful algebra $\mathcal{A}_1$ (see [**1, 3**]). Further, if ϕ is an algebraic isomorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, then (ϕ, ϕ^{-1}) is also an elementary operator of $\mathcal{A}_1$ into $\mathcal{A}_2$.

Let X be a Banach space. Usually, a subalgebra $\mathcal{A} \subseteq \mathcal{B}(X)$ is called a *standard operator algebra on* X if it contains all finite rank operators in $\mathcal{B}(X)$. For convenience, for a subspace lattice $\mathcal{L}$ on X, we similarly call a subalgebra $\mathcal{A} \subseteq \operatorname{Alg}\mathcal{L}$ an *standard subalgebra of* $\operatorname{Alg}\mathcal{L}$ if it contains all finite rank operators in $\operatorname{Alg}\mathcal{L}$. In our previous papers [**7, 11, 12**], we studied derivations, isomorphisms, Jordan derivations and Jordan isomorphisms between standard subalgebras of $\mathcal{J}$-subspace lattice algebras. Here we turn our attention to elementary operators. The papers [**1, 2**] describe the general form of elementary operators on some concrete algebras which include polynomial algebras, finite dimensional central simple algebras, standard operator algebras and some special function algebras. Also, the paper [**13**] characterises surjective mappings (no linearity is assumed) between standard operator algebras having the property appearing in (1); in particular, such mappings are proved to be automatically additive. Note that one of the main results from [**1**] is the following.

THEOREM 1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are standard operator algebras on Banach space X_1 and X_2 respectively, and the pair (M, M^*) is an elementary operator of $\mathcal{A}_1$ into $\mathcal{A}_2$. Then there exist bounded linear operators $T : X_1 \to X_2$ and $S : X_2 \to X_1$ such that $M(A) = TAS, A \in \mathcal{A}_1$, and $M^*(B) = SBT, B \in \mathcal{A}_2$.*

The aim of this note is to extend this result to the case where $\mathcal{A}_1$, $\mathcal{A}_2$ are standard subalgebras of $\mathcal{J}$-subspace lattice algebras and M, M^* are linear surjections. We shall also discuss the continuity of M and M^* as well as the case where no surjectivity of M and M^* is assumed.

The following lemma is taken from [**14**], which is just [**5**, Corollary 3.8] and [**4**, Lemma 3.1] when $\mathcal{L}$ is an atomic Boolean subspace lattice and a pentagon subspace lattice, respectively.

LEMMA 2. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X and suppose that $T \in \operatorname{Alg}\mathcal{L}$ is nonzero. Then T has rank one if and only if whenever $ATB = 0$ with $A, B \in \operatorname{Alg}\mathcal{L}$, then either $AT = 0$ or $TB = 0$.*

Now let us state our main result, which can apply to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras.

THEOREM 2. *Let $\mathcal{L}_i$ be a $\mathcal{J}$-subspace lattice on a Banach space X_i and $\mathcal{A}_i$ be a standard subalgebra of $\operatorname{Alg}\mathcal{L}_i$, where $i = 1, 2$. Suppose that the pair (M, M^*) is an elementary operator of $\mathcal{A}_1$ into $\mathcal{A}_2$, that is, $M : \mathcal{A}_1 \to \mathcal{A}_2$ and $M^* : \mathcal{A}_2 \to \mathcal{A}_1$ are linear mappings satisfying*

$$(2) \qquad \begin{cases} M\big(AM^*(B)C\big) = M(A)BM(C), \\ M^*\big(EM(D)F\big) = M^*(E)DM^*(F) \end{cases}$$

for all $A, C, D \in \mathcal{A}_1$ and $B, E, F \in \mathcal{A}_2$. If in addition, both M and M^* are surjective, then there exist two densely defined, closed, injective linear mappings $T : \mathcal{D}(T) \subseteq X_1 \to X_2$ and $S : \mathcal{D}(S) \subseteq X_2 \to X_1$ with dense ranges, and with $A\mathcal{R}(S) \subseteq \mathcal{D}(T)$ and $B\mathcal{R}(T) \subseteq \mathcal{D}(S)$ for each $A \in \mathcal{A}_1$ and each $B \in \mathcal{A}_2$, such that

$$M(A)y = TASy \quad \text{and} \quad M^*(B)x = SBTx$$

hold for all $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$, $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(S)$. Here the notation $\mathcal{D}(\cdot)$ and $\mathcal{R}(\cdot)$ denote the domain, and respectively the range of a mapping.

For clarity of exposition, we shall organise the proof in a series of lemmas.

LEMMA 3. *M and M^* are bijective.*

PROOF: To see that M is injective, let $M(A) = 0$ for some $A \in \mathcal{A}_1$. Let $K \in \mathcal{J}(\mathcal{L}_1)$ be arbitrary. Suppose that there is $x \in K$ such that $Ax \neq 0$. For any nonzero $f \in K_-^\perp$, by Lemma 1, $x \otimes f, Ax \otimes f \in \mathcal{A}_1$. Noting that the surjectivity of M^*, we can write $M^*(B) = Ax \otimes f$ and $M^*(D) = x \otimes f$ for some $B, D \in \mathcal{A}_2$. We thus by (2) have that $f(Ax)Ax \otimes f = M^*(B)AM^*(D) = M^*\big(BM(A)D\big) = 0$, and so $f(Ax) = 0$. Hence

$Ax \in K \cap K_- = (0)$, a contradiction. This shows that $A(K) = (0)$. Since $\bigvee\{K : K \in \mathcal{J}(\mathcal{L}_1)\} = X_1$, it follows that $A = 0$. The proof of injectivity of M^* goes similarly, completing the proof. $\square$

In the following, we shall say that a mapping $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ preserves rank one operators in both directions if for any $T \in \mathcal{A}_1$, the operator $\Phi(T)$ is of rank one if and only if T is of rank one.

Lemma 4. *M and M^* preserve rank one operators in both directions.*

Proof: Let $T \in \mathcal{A}_1$ be arbitrary. Suppose first that T is of rank one and let $B, D \in \mathcal{A}_2$ with $BM(T)D = 0$. Then $M^*(B)TM^*(D) = M^*(BM(T)D) = 0$ by (2). Applying Lemma 2, we get that either $M^*(B)T = 0$ or $TM^*(D) = 0$. In the case that $M^*(B)T = 0$, let $L \in \mathcal{J}(\mathcal{L}_2)$ be arbitrary and choose nonzero $y \in L$. For any $g \in L_-^\perp$, then $y \otimes g \in \mathcal{A}_2$ and there exists $A \in \mathcal{A}_1$ such that $M(A) = y \otimes g$ since M is surjective. So $y \otimes (BM(T))^* g = M(A)BM(T) = M(AM^*(B)T) = 0$, where $(BM(T))^*$ is the adjoint of $BM(T)$. It follows that $(BM(T))^* g = 0$ which implies that $(BM(T))^*(L_-^\perp) = (0)$. Since $\cap\{L_- : L \in \mathcal{J}(\mathcal{L}_2)\} = (0)$, it is easily seen that the linear span of $\cup\{L_-^\perp : L \in \mathcal{J}(\mathcal{L}_2)\}$ is weak* dense in X_2^*. Thus $(BM(T))^* = 0$ and then $BM(T) = 0$. If $TM^*(D) = 0$ we can similarly obtain that $M(T)D = 0$. Making use of Lemma 2 again, it follows that $M(T)$ is also of rank one.

For the reverse implication, suppose that $M(T)$ is of rank one. Observe that the pair $(M^{-1}, M^{*^{-1}})$ is an elementary operator of $\mathcal{A}_2$ into $\mathcal{A}_1$, that is, the linear mappings $M^{-1} : \mathcal{A}_2 \to \mathcal{A}_1$ and $M^{*^{-1}} : \mathcal{A}_1 \to \mathcal{A}_2$ satisfy

$$(3) \qquad \begin{cases} M^{-1}(EM^{*-1}(D)F) = M^{-1}(E)DM^{-1}(F), \\ M^{*^{-1}}\big(AM^{-1}(B)C\big) = M^{*^{-1}}(A)BM^{*^{-1}}(C) \end{cases}$$

for all $A, C, D \in \mathcal{A}_1$ and $B, E, F \in \mathcal{A}_2$. Then we must have that M^{-1} maps every rank one operator of $\mathcal{A}_2$ to a rank one operator of $\mathcal{A}_1$. So T is of rank one.

The statement that M^* preserves rank one operators in both directions can be proved in a similar way. The proof is complete. $\square$

In what follows, if $\mathcal{L}$ is a subspace lattice and $K \in \mathcal{J}(\mathcal{L})$, we write $\mathcal{F}(K)$ for the set $\{x \otimes f : x \in K, f \in K_-^\perp\}$. For a $\mathcal{J}$-subspace lattice $\mathcal{L}$, the following basic properties are clear and will get repeated use.

(i) $K \subseteq L_-$ for any $K, L \in \mathcal{J}(\mathcal{L})$ with $K \neq L$;

(ii) if the rank one operator $x \otimes f \in \operatorname{Alg}\mathcal{L}$, then there exists a unique $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$.

Lemma 5. *Let $K \in \mathcal{J}(\mathcal{L}_1)$. Then there exists a unique $\widehat{K} \in \mathcal{J}(\mathcal{L}_2)$ such that $M(\mathcal{F}(K)) = \mathcal{F}(\widehat{K})$ and $M^*(\mathcal{F}(\widehat{K})) = \mathcal{F}(K)$. In particular, the mapping $K \to \widehat{K}$ from $\mathcal{J}(\mathcal{L}_1)$ into $\mathcal{J}(\mathcal{L}_2)$ is bijective.*

Proof: Noting that $K \cap K_- = (0)$, choose fixed nonzero elements $x_K \in K$ and $f_K \in K_-^\perp$ with $f_K(x_K) = 1$. By Lemmas 1 and 4, there exist an element, say $\widehat{K}$, in

$\mathcal{J}(\mathcal{L}_2)$, $y_K \in \widehat{K}$ and $g_K \in \widehat{K}_-^\perp$ such that, $M(x_K \otimes f_K) = y_K \otimes g_K \neq 0$. We want to prove that $M(\mathcal{F}(K)) = \mathcal{F}(\widehat{K})$. Let $x \in K$ and $f \in K_-^\perp$ be arbitrary nonzero elements; then $x \otimes f, x \otimes f_K, x_K \otimes f \in \mathcal{A}_1$. Write

$$\begin{aligned} M(x \otimes f) &= y \otimes g, \quad y \in L,\ g \in L_-^\perp,\ L \in \mathcal{J}(\mathcal{L}_2), \\ M(x \otimes f_K) &= y_1 \otimes g_1, \quad y_1 \in L_1,\ g_1 \in L_{1-}^\perp,\ L_1 \in \mathcal{J}(\mathcal{L}_2), \\ M(x_K \otimes f) &= y_2 \otimes g_2, \quad y_2 \in L_2,\ g_2 \in L_{2-}^\perp,\ L_2 \in \mathcal{J}(\mathcal{L}_2). \end{aligned}$$

Moreover, since M^* is surjective and preserves rank one operators in both directions, we also have $M^*(y_0 \otimes g_0) = x_K \otimes f_K$ for some nonzero $y_0 \in L_0$ and $g_0 \in L_{0-}^\perp$, where $L_0 \in \mathcal{J}(\mathcal{L}_2)$. Applying (2) we obtain that

$$\begin{aligned} y \otimes g &= M(x \otimes f_K \cdot M^*(y_0 \otimes g_0) \cdot x_K \otimes f) \\ &= M(x \otimes f_K) \cdot y_0 \otimes g_0 \cdot M(x_K \otimes f) \\ &= g_1(y_0) g_0(y_2) y_1 \otimes g_2. \end{aligned}$$

It follows that $L = L_1 = L_2 = L_0$ since each of y, y_1, y_2, g, g_1 and g_2 is nonzero. In the above equations, substituting x_K and f_K for x and f respectively, then $L_0 = \widehat{K}$ and so $L = \widehat{K}$. This proves that $M(\mathcal{F}(K)) \subseteq \mathcal{F}(\widehat{K})$. For the reverse inclusion, because of (3), we can similarly obtain an $K' \in \mathcal{J}(\mathcal{L}_1)$ such that $M^{-1}(\mathcal{F}(\widehat{K})) \subseteq \mathcal{F}(K')$. As $x_K \otimes f_K = M^{-1}(y_K \otimes g_K)$, we have $K' = K$. Thus $\mathcal{F}(\widehat{K}) \subseteq M(\mathcal{F}(K))$.

Next we shall prove that $M^*(\mathcal{F}(\widehat{K})) = \mathcal{F}(K)$. By the symmetry of M and M^*, we can get that $M^*(\mathcal{F}(\widehat{K})) = \mathcal{F}(K'')$ for some $K'' \in \mathcal{J}(\mathcal{L}_1)$. It suffices to show that $K'' = K$. Choose $z \in \widehat{K}$ and $h \in \widehat{K}_-^\perp$ with $h(z) = 1$. Since $M(\mathcal{F}(K)) = \mathcal{F}(\widehat{K})$, there exist $u \in K$ and $l \in K_-^\perp$ such that $M(u \otimes l) = z \otimes h$. Suppose to the contrary that $K'' \neq K$. Noticing that $M^*(z \otimes h)$ is an operator of rank one in $\mathcal{F}(K'')$, we have

$$\begin{aligned} z \otimes h &= M(u \otimes l) \cdot z \otimes h \cdot M(u \otimes l) \\ &= M(u \otimes l \cdot M^*(z \otimes h) \cdot u \otimes l) \\ &= M(0) = 0, \end{aligned}$$

which is a contradiction.

The uniqueness of $\widehat{K}$ is obvious.

It remains to prove the last statement. Suppose $L \in \mathcal{J}(\mathcal{L}_2)$; let $y \in L$ and $g \in L_-^\perp$ be nonzero. Then by Lemma 4 there exist $x \in K$ and $f \in K_-^\perp$ for some $K \in \mathcal{J}(\mathcal{L}_1)$ such that $M(x \otimes f) = y \otimes g$. Therefore $y \otimes g \in \mathcal{F}(\widehat{K}) \cap \mathcal{F}(L)$ and so $L = \widehat{K}$. This proves the surjectivity. The injectivity is clear. We are done. ∎

LEMMA 6. *Let $x \in K$, $f \in K_-^\perp$, $y \in L$ and $g \in L_-^\perp$, where $K \in \mathcal{J}(\mathcal{L}_1)$ and $L \in \mathcal{J}(\mathcal{L}_2)$. Then $g(M(x \otimes f)y) = 1$ if and only if $f(M^*(y \otimes g)x) = 1$.*

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PROOF: Since

$$\begin{aligned} g(M(x\otimes f)y)M(x\otimes f)y &= M(x\otimes f)\cdot y\otimes g\cdot M(x\otimes f)y \\ &= M(x\otimes f\cdot M^*(y\otimes g)\cdot x\otimes f)y \\ &= f(M^*(y\otimes g)x)M(x\otimes f)y, \end{aligned}$$

the "only if" part follows. The "if" part can be obtained similarly. ∎

LEMMA 7. *For every $K\in\mathcal{J}(\mathcal{L}_1)$, there exist bijective linear mappings $T_K : K \to \widehat{K}$ and $S_K : \widehat{K}\to K$ such that,*

$$M(A)y = T_K A S_K y \quad \text{and} \quad M^*(B)x = S_K B T_K x \tag{4}$$

for all $A\in\mathcal{A}_1$, $B\in\mathcal{A}_2$, $x\in K$ and $y\in\widehat{K}$.

PROOF: For $K\in\mathcal{J}(\mathcal{L}_1)$, fix two nonzero elements $x_K\in K$ and $f_K\in K_-^\perp$. By Lemma 5, $M(x_K\otimes f_K)\in\mathcal{F}(\widehat{K})$ being nonzero. In fact, it is easy to see that $M(x_K\otimes f_K)\neq 0$ on $\widehat{K}$. So there exist $y_K\in\widehat{K}$ and $g_K\in\widehat{K}_-^\perp$ such that $g_K(M(x_K\otimes f_K)y_K)=1$. Define linear mappings T_K and S_K by

$$\begin{aligned} T_K x &= M(x\otimes f_K)y_K, \quad x\in K, \\ S_K y &= M^*(y\otimes g_K)x_K, \quad y\in\widehat{K}. \end{aligned}$$

Clearly, $T_K(K)\subseteq\widehat{K}$ and $S_K(\widehat{K})\subseteq K$. Let $y\in\widehat{K}$ and $A\in\mathcal{A}_1$. Then

$$\begin{aligned} M(A)y &= g_K(M(x_K\otimes f_K)y_K)M(A)y \\ &= M(A)\cdot y\otimes g_K\cdot M(x_K\otimes f_K)y_K \\ &= M(AM^*(y\otimes g_K)x_K\otimes f_K)y_K \\ &= M(AS_Ky\otimes f_K)y_K \\ &= T_KAS_Ky. \end{aligned}$$

This proves that the first equality is true in (4). To prove the second equality, let $x\in K$ and $B\in\mathcal{A}_2$. By Lemma 6, $f_K(M^*(y_K\otimes g_K)x_K)=1$. Thus

$$\begin{aligned} M^*(B)x &= f_K(M^*(y_K\otimes g_K)x_K)M^*(B)x \\ &= M^*(B)\cdot x\otimes f_K\cdot M^*(y_K\otimes g_K)x_K \\ &= M^*(BM(x\otimes f_K)y_K\otimes g_K)x_K \\ &= M^*(BT_Kx\otimes g_K)x_K \\ &= S_KBT_Kx. \end{aligned}$$

Now we want to prove that T_K is bijective. Let us first assume that $T_Kx=0$ for some $x\in K$. Fix a nonzero $x_0\in K$ and let $f\in K_-^\perp$ be arbitrary. Then there is $B\in\mathcal{A}_2$ such that $M^*(B)=x_0\otimes f$. So $f(x)x_0=M^*(B)x=S_KBT_Kx=0$, and moreover

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$f(x) = 0$. This implies that $x \in K \cap K_- = (0)$. Hence T_K is injective. In order to prove that T_K is surjective, let $y \in \widehat{K}$ be nonzero. Pick $g \in \widehat{K}_-^{\perp}$ with $g(y) = 1$. By Lemma 5 we can write $M(x \otimes f) = y \otimes g$ for some $x \in K$ and $f \in K_-^{\perp}$. We thus have $y = M(x \otimes f)y = T_K \cdot x \otimes f \cdot S_K y = f(S_K y)T_K x$. This yields the surjectivity of T_K.

Similarly we can prove that S_K is bijective. The proof is complete. ∎

In the remainder of this paper, for every $K \in \mathcal{J}(\mathcal{L}_1)$, T_K and S_K will denote the linear mappings as constructed in Lemma 7. Obviously, they depend on the choices of x_K, f_K, y_K and g_K. So it will be assumed that those choices have been made for each $K \in \mathcal{J}(\mathcal{L}_1)$.

For a subspace lattice $\mathcal{L}$, it will be convenient to denote by $\langle \mathcal{J}(\mathcal{L}) \rangle$ the (not necessarily closed) linear span of $\cup\{K : K \in \mathcal{J}(\mathcal{L})\}$. Suppose that $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice and that $K_1, \cdots, K_n$ are distinct elements of $\mathcal{J}(\mathcal{L})$. If $x_i \in K_i$ such that $\sum_{i=1}^{n} x_i = 0$, then each $x_i \in K_i \cap (\vee_{j \neq i} K_j) \subseteq K_i \cap K_{i-} = (0)$. Thus every $x \in \langle \mathcal{J}(\mathcal{L}) \rangle$ has a representation as follows: $x = \sum_{i=1}^{n} x_i$ with $x_i \in K_i$, $1 \leqslant i \leqslant n$, where $K_1, \cdots, K_n$ are distinct elements of $\mathcal{J}(\mathcal{L})$. If x is nonzero and each x_i is required to be nonzero, this representation is unique up to permutations of $x_1, \cdots, x_n$. In addition, Lemma 5 tells us that the mapping $K \to \widehat{K}$ from $\mathcal{J}(\mathcal{L}_1)$ onto $\mathcal{J}(\mathcal{L}_2)$ is bijective. Therefore, the following linear mappings T_0 and S_0 are well-defined.

LEMMA 8. *Define $T_0 : \langle \mathcal{J}(\mathcal{L}_1) \rangle \to \langle \mathcal{J}(\mathcal{L}_2) \rangle$ by $T_0 x = \sum_{i=1}^{m} T_{K_i} x_i$, where $x = \sum_{i=1}^{m} x_i$ with $x_i \in K_i$, $1 \leqslant i \leqslant n$, and $K_1, \cdots, K_m$ being distinct elements of $\mathcal{J}(\mathcal{L}_1)$; and define $S_0 : \langle \mathcal{J}(\mathcal{L}_2) \rangle \to \langle \mathcal{J}(\mathcal{L}_1) \rangle$ by $S_0 y = \sum_{j=1}^{n} S_{K_j} y_j$, where $y = \sum_{j=1}^{n} y_j$ with $y_j \in \widehat{K_j}$, $1 \leqslant j \leqslant n$, and $K_1, \cdots, K_n$ being distinct elements of $\mathcal{J}(\mathcal{L}_1)$. Then T_0 and S_0 are bijective and satisfy*

$$M(A)y = T_0 A S_0 y \quad \text{and} \quad M^*(B)x = S_0 B T_0 x$$

for all $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$, $x \in \langle \mathcal{J}(\mathcal{L}_1) \rangle$ and $y \in \langle \mathcal{J}(\mathcal{L}_2) \rangle$.

PROOF: By Lemma 7, it is easily proved that T_0 and S_0 are bijective.

Let $A \in \mathcal{A}_1$ and let $y \in \langle \mathcal{J}(\mathcal{L}_2) \rangle$ being of the form described in this lemma. Applying (4) we compute

$$\begin{aligned} M(A)y = \sum_{j=1}^{n} M(A)y_j = \sum_{j=1}^{n} T_{K_j} A S_{K_j} y_j = T_0\Big(\sum_{j=1}^{n} A S_{K_j} y_j\Big) \\ = T_0 A\Big(\sum_{j=1}^{n} S_{K_j} y_j\Big) = T_0 A S_0 y. \end{aligned}$$

The other equality can be proved similarly, completing the proof. ∎

Now we are in a position to prove our main result.

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PROOF OF THEOREM 2: Suppose $G(T_0) = \left\{(x, T_0x) : x \in \langle\mathcal{J}(\mathcal{L}_1)\rangle\right\}$ and $G(S_0) = \left\{(y, S_0y) : y \in \langle\mathcal{J}(\mathcal{L}_2)\rangle\right\}$, which are the graphs of T_0 and S_0 respectively. To define T and S, we let

$$\mathcal{D}_1 = \left\{x \in X_1 : (x,y) \in \overline{G(T_0)} \text{ for some } y \in X_2\right\},$$

$$\mathcal{D}_2 = \left\{y \in X_2 : (y,x) \in \overline{G(S_0)} \text{ for some } x \in X_1\right\}$$

where $\overline{G(T_0)}$ and $\overline{G(S_0)}$ denote the norm closures. Obviously, they are linear manifolds and $\langle\mathcal{J}(\mathcal{L}_1)\rangle \subseteq \mathcal{D}_1$, $\langle\mathcal{J}(\mathcal{L}_2)\rangle \subseteq \mathcal{D}_2$. Since $\langle\mathcal{J}(\mathcal{L}_1)\rangle$ and $\langle\mathcal{J}(\mathcal{L}_2)\rangle$ are dense in X_1 and X_2 respectively, the same must be true for $\mathcal{D}_1$ and $\mathcal{D}_2$.

For every $x \in \mathcal{D}_1$, we shall prove that there exists a unique $y \in X_2$ such that $(x,y) \in \overline{G(T_0)}$. For this purpose, assume that $(x,y_1),(x,y_2) \in \overline{G(T_0)}$ with $y_1, y_2 \in X_2$. Then $(0, y_1 - y_2) \in \overline{G(T_0)}$. So there is a sequence $\{x_n\}_1^\infty$ of elements in $\langle\mathcal{J}(\mathcal{L}_1)\rangle$, such that $x_n \to 0$ and $T_0x_n \to y_1 - y_2$. Let $L \in \mathcal{J}(\mathcal{L}_2)$ be arbitrary, and pick a nonzero $y \in L$. Then for any $g \in L_-^\perp$, we have $M^*(y \otimes g)x_n \to 0$. On the other hand, by Lemma 8, $M^*(y \otimes g)x_n = S_0 \cdot y \otimes g \cdot T_0x_n = g(T_0x_n)S_0y \to g(y_1 - y_2)S_0y$. Noting that $S_0y \neq 0$, it follows that $g(y_1 - y_2) = 0$. Hence $y_1 - y_2 \in L_-$; moreover, $y_1 - y_2 \in \bigcap\left\{L_- : L \in \mathcal{J}(\mathcal{L}_2)\right\} = (0)$. Thus $y_1 = y_2$, as desired.

Similarly, we can prove that for every $y \in \mathcal{D}_2$, there exists a unique $x \in X_1$ such that $(y,x) \in \overline{G(S_0)}$.

Define two mappings $T : \mathcal{D}(T) \subseteq X_1 \to X_2$ and $S : \mathcal{D}(S) \subseteq X_2 \to X_1$ in an obvious way, such that $G(T) = \overline{G(T_0)}$ and $G(S) = \overline{G(S_0)}$, where $\mathcal{D}(T) = \mathcal{D}_1$ and $\mathcal{D}(S) = \mathcal{D}_2$. It is easily seen that T and S are densely defined, closed and linear. Also, they are injective. For example, suppose that $Sy = 0$ for some $y \in \mathcal{D}(S)$; then $(y,0) \in \overline{G(S_0)}$. So there exists a sequence $\{y_n\}_1^\infty$ of elements in $\langle\mathcal{J}(\mathcal{L}_2)\rangle$, such that $y_n \to y$ and $S_0y_n \to 0$. For any $K \in \mathcal{J}(\mathcal{L}_1)$, pick a nonzero $z \in \widehat{K}$. For every $h \in \widehat{K}_-^\perp$, since $M\big(\mathcal{F}(K)\big) = \mathcal{F}(\widehat{K})$, there are $x \in K$ and $f \in K_-^\perp$ such that $M(x \otimes f) = z \otimes h$. We then have $M(x \otimes f)y_n \to M(x \otimes f)y$. But, by Lemma 8, $M(x \otimes f)y_n = T_0 \cdot x \otimes f \cdot S_0y_n = f(S_0y_n)T_0x \to 0$. Thus $h(y)z = M(x \otimes f)y = 0$, which implies $y \in \widehat{K}_-$. Since $\mathcal{J}(\mathcal{L}_2) = \left\{\widehat{K} : K \in \mathcal{J}(\mathcal{L}_1)\right\}$ by Lemma 5, we get $y = 0$. Similar arguments apply to T.

Since T extends T_0 and the range of T_0 is $\mathcal{J}(\mathcal{L}_2)$ by Lemma 8, it follows that $\mathcal{R}(T)$ is dense in X_2. Similarly, S_0 has dense range.

Now we shall prove that $A\mathcal{R}(S) \subseteq \mathcal{D}(T)$ and $M(A)y = TASy$, for every $A \in \mathcal{A}_1$, $y \in \mathcal{D}(S)$. Because of $(y, Sy) \in \overline{G(S_0)}$, we then choose a sequence $\{y'_n\}_1^\infty$ of elements in $\langle\mathcal{J}(\mathcal{L}_2)\rangle$ satisfying $y'_n \to y$ and $S_0y'_n \to Sy$. So $AS_0y'_n \to ASy$ and, by Lemma 8, $T_0AS_0y'_n = M(A)y'_n \to M(A)y$. Noting that $(AS_0y'_n, T_0AS_0y'_n) \in G(T_0)$ for each y'_n, we get that $(ASy, M(A)y) \in G(T)$. Therefore $ASy \in \mathcal{D}(T)$ and $M(A)y = TASy$. A similar argument shows that $B\mathcal{R}(T) \subseteq \mathcal{D}(S)$ and $M^*(B)x = SBTx$, for every $B \in \mathcal{A}_2$ and $x \in \mathcal{D}(T)$. This completes the proof. $\square$

Taking into account Theorem 1, a natural question is proposed as follows: In Theorem 2, if the condition that M and M^* are surjective is removed, what is the conclusion? To this question, we have observed that there are linear mappings $T : \langle \mathcal{J}(\mathcal{L}_1) \rangle \to \langle \mathcal{J}(\mathcal{L}_2) \rangle$ and $S : \langle \mathcal{J}(\mathcal{L}_2) \rangle \to \langle \mathcal{J}(\mathcal{L}_1) \rangle$ such that, $M(A)y = TASy$ and $M^*(B)x = SBTx$ for $A \in \mathcal{A}_1$, $B \in \mathcal{A}_2$, $x \in \langle \mathcal{J}(\mathcal{L}_1) \rangle$ and $y \in \langle \mathcal{J}(\mathcal{L}_2) \rangle$. Obviously T and S are densely defined. But, we cannot prove that they are also closed.

We shall conclude by considering the continuity of M and M^* in Theorem 2.

Proposition 1. *Under the assumptions of Theorem 2, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subalgebras, then M and M^* are (norm) continuous.*

Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed, by the closed graph theorem it suffices to show that M and M^* are closed operators.

For every $K \in \mathcal{J}(\mathcal{L}_1)$, we first prove that T_K and S_K are closed operators and hence continuous, where T_K and S_K are defined as in Lemma 7. Suppose $K \in \mathcal{J}(\mathcal{L}_1)$. Let $\{x_n\}_1^\infty$, x be in K and y in $\widehat{K}$ such that $x_n \to x$ and $T_K x_n \to y$. For any $g \in \widehat{K}_-^\perp$, $g\big(T_K(x_n - x)\big) \to g(y - T_K x)$. On the other hand, choose $z \in \widehat{K}$ with $S_K z \neq 0$ since $S_K : \widehat{K} \to K$ is bijective. It follows from (4) that

$$g\big(T_K(x_n - x)\big) S_K z = S_K \cdot z \otimes g \cdot T_K(x_n - x) = M^*(z \otimes g)(x_n - x) \to 0.$$

Hence $g(y - T_K x) = 0$, and so $y - T_K x \in \widehat{K} \cap \widehat{K}_- = (0)$. This shows that T_K is closed. By similar arguments, we can conclude that S_K is also closed.

Now suppose that $\{A_n\}_1^\infty$, A are in $\mathcal{A}_1$ and B in $\mathcal{A}_2$ satisfying $A_n \to A$ and $M(A_n) \to B$. Let $K \in \mathcal{J}(\mathcal{L}_1)$ and $y \in \widehat{K}$. Then applying Lemma 7, $M(A_n)y = T_K A_n S_K y \to T_K A S_K y = M(A)y$. But $M(A_n)y \to By$ holds also. So $M(A)y = By$ for all $y \in \widehat{K}$. Noting that Lemma 5, we have $M(A) = B$. Hence M is closed. Similarly, M^* is closed. This completes the proof. ∎

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