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On Kleiman–Piene's question for Gauss maps

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Abstract

We study the product of a Fermat hypersurface $X_0^{p+1} + \cdots + X_n^{p+1} = 0 \subset \mathbf{P}^n$ with $n \geqslant 3$ and $\mathbf{P}^1$, embedded in $\mathbf{P}^{2n+1}$ by Segre embedding where $p > 0$ is the characteristic of the base field. This smooth variety is nonreflexive and has Gauss map which is an embedding. This gives a negative answer to the following Kleiman–Piene question in any positive characteristic: does the separability of the Gauss map imply reflexivity? The only known smooth examples, which give a negative answer, are given by Kaji in characteristic 2.

1. Introduction

Let $X \subset \mathbf{P}^N$ be a smooth projective variety of dimension n, CX be the conormal variety $\{(x, H) \in X \times \mathbf{P}^{N^*} \mid \mathbf{T}_x X \subset H\} \subset X \times \mathbf{P}^{N^*}$, where $\mathbf{T}_x X$ is the projective embedded tangent space at a point x, with the natural projection $p_2 : CX \to \mathbf{P}^{N^*}$, and let $X^* = p_2(CX)$ be its dual. The Gauss map γ on X is the morphism from X to the Grassmannian $\mathbf{G}(n, N) \cong \mathbf{G}^*(N-n-1, N)$ which assigns to a point $x \in X$ the projective tangent space $\mathbf{T}_x X \in \mathbf{G}(n, N)$, or its dual $(\mathbf{T}_x X)^* \in \mathbf{G}^*(N-n-1, N)$. Now we study $\gamma : X \to \mathbf{G}^*(N-n-1, N)$. We have the diagram

$$\begin{array}{ccccc} CX & \xrightarrow{\gamma'} & I_{\gamma(X)} & \longrightarrow & X^* \\ \downarrow & \square & \downarrow & & \\ X & \xrightarrow{\gamma} & \gamma(X) & & \end{array}$$

where $I_{\gamma(X)} = \{(E, H) \in \gamma(X) \times \mathbf{P}^{N^*} \mid H \in E\} \subset \gamma(X) \times \mathbf{P}^{N^*}$ and $\gamma'(x, H) = ((\mathbf{T}_x X)^*, H)$. Then, Kleiman and Piene raised the following question.

Question [KP91, pp. 108–109]. Is $I_{\gamma(X)} \to X^*$ separable?

If $\dim \gamma(X) = 1$, then it is known that $I_{\gamma(X)} \to X^*$ is always separable [Fuk05, Kaj92].

Nonreflexive projective varieties with separable Gauss maps give a negative answer. (X is called reflexive if $CX \to X^*$ is separable [Kle86].) In characteristic 2, the first such varieties were found by Kaji [Kaj03]. He studied Segre varieties (i.e. products of projective spaces embedded by Segre embeddings) and their duals. He proved that some odd-dimensional Segre varieties, for example $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, are not reflexive in characteristic 2 and have Gauss maps that are embeddings. If we do not need the smoothness of X, then the present author already found nonreflexive varieties with birational Gauss maps in any positive characteristic [Fuk].

In this paper we study the product of a Fermat hypersurface $X_0^{p+1} + \cdots + X_n^{p+1} = 0 \subset \mathbf{P}^n$ and $\mathbf{P}^1$, embedded in $\mathbf{P}^{2n+1}$ by Segre embedding where $p > 0$ is the characteristic of the base field. This smooth variety, call it X, has a Gauss map which is an embedding and inseparable morphism $I_{\gamma(X)} \to X^*$ when $n \geqslant 3$. Consequently, X is nonreflexive if $n \geqslant 3$. The author thinks that this

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is the first smooth example, in characteristic $p > 2$ or of even dimension, which gives a negative answer to the Kleiman–Piene's question.

We work over an algebraically closed field K of characteristic $p > 0$. Varieties are integral algebraic schemes over K. Here $[v] \in \mathbf{P}^N$ denotes the point of $\mathbf{P}^N$ corresponding to the equivalence class of $v \in \mathbf{A}^{N+1} \setminus 0$.

2. Product of a Fermat hypersurface and the projective line

Let $Y \subset \mathbf{P}^n$ with $n \geqslant 3$ be a Fermat hypersurface given by $X_0^{p+1} + \cdots + X_n^{p+1} = 0$, and $X = Y \times \mathbf{P}^1 \subset \mathbf{P}^{2n+1}$ embedded by Segre embedding. Let $(1 : x_1 : \cdots : x_n)$ be an affine coordinates of Y and $(1 : u)$ be of $\mathbf{P}^1$, then X is (the closure of) the image of $f : Y \times \mathbf{P}^1 \to \mathbf{P}^{2n+1}$; $(1 : x_1 : \cdots : x_n) \times (1 : u) \mapsto (1 : x_1 : \cdots : x_{n-1} : u : x_n : x_1 u : \cdots : x_n u)$. We take $x_1, \dots, x_{n-1}$ as a system of local coordinates of Y (i.e. the function field $K(Y)$ is separable algebraic over $K(x_1, \dots, x_{n-1})$). The projective tangent space at $f(x_1, \dots, x_n, u)$ is spanned by the $n+1$ row vectors of the following matrices:

$$\begin{pmatrix} 1 & x_1 & \dots & x_{n-1} & u & x_n & x_1 u & \dots & x_{n-1}u & x_n u \\ 0 & 1 & \dots & 0 & 0 & \dfrac{\partial x_n}{\partial x_1} & u & \dots & 0 & \dfrac{\partial x_n}{\partial x_1}u \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dfrac{\partial x_n}{\partial x_{n-1}} & 0 & \dots & u & \dfrac{\partial x_n}{\partial x_{n-1}}u \\ 0 & 0 & \dots & 0 & 1 & 0 & x_1 & \dots & x_{n-1} & x_n \end{pmatrix} \sim \begin{pmatrix} I_{n+1} & A \end{pmatrix}$$

where I_{n+1} is the $(n+1) \times (n+1)$ unit matrix and

$$A = \begin{pmatrix} x_n - \displaystyle\sum_{j=1}^{n-1} \frac{\partial x_n}{\partial x_j} x_j & -x_1 u & \dots & -x_{n-1}u & -\displaystyle\sum_{j=1}^{n-1} \frac{\partial x_n}{\partial x_j} x_j u \\ \dfrac{\partial x_n}{\partial x_1} & u & \dots & 0 & \dfrac{\partial x_n}{\partial x_1}u \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial x_n}{\partial x_{n-1}} & 0 & \dots & u & \dfrac{\partial x_n}{\partial x_{n-1}}u \\ 0 & x_1 & \dots & x_{n-1} & x_n \end{pmatrix}.$$

We also have

$$\frac{\partial x_n}{\partial x_j} = -\frac{x_j^p}{x_n^p}$$

for $j = 1, \dots, n-1$. These imply that the Gauss map on X is an embedding. Calculation of the dual vector space shows that $\gamma(X) \subset \mathbf{G}^*(n, 2n+1)$ is locally represented by the matrix $B = \begin{pmatrix} -{}^{\mathrm{t}}A & I_{n+1} \end{pmatrix}$. Let ρ_i be the $(i+1)$th row vector of B for $0 \leqslant i \leqslant n$. By using a local trivialization, $I_{\gamma(X)} \to X^*$ is generically identified with the morphism $g : Y_0 \times \mathbf{A}^1 \times \mathbf{P}^n \to X^*$; $(x_1, \dots, x_n) \times (u) \times (t_0 : \cdots : t_n) \mapsto [t_0\rho_0 + \cdots + t_n\rho_n]$ where Y_0 is an affine locus of Y with $X_0 X_n \neq 0$. The affine lifting is $\hat{g} : Y_0 \times \mathbf{A}^1 \times \mathbf{A}^{n+1} \to \widehat{X^*}$; $(x_1, \dots, x_n) \times (u) \times (t_0, \dots, t_n) \mapsto t_0\rho_0 + \cdots + t_n\rho_n$ where $\widehat{X^*}$ is the affine cone of X^*. By easy computation, we have

$$\frac{\partial \hat{g}}{\partial x_j} = \left(t_j + t_n \frac{\partial x_n}{\partial x_j} \right) \mathbf{u}$$

where $\mathbf{u} = {}^{\mathrm{t}}(u, 0, \dots, 0, -1, 0, \dots, 0)$. This implies that the rank of the differential of g is $n+2$, and hence drops when $n \geqslant 3$. We can easily check that X^* is a hypersurface, hence $I_{\gamma(X)} \to X^*$ is inseparable when $n \geqslant 3$.

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