APPENDICES

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SOLUTION OF THE PROBLEM OF STABILITY OF A STELLAR SYSTEM WITH EMDEN'S DENSITY LAW AND A SPHERICAL DISTRIBUTION OF VELOCITIES

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ABSTRACT. Applying a criterion previously derived by the author, the stability of stellar systems with an isotropic distribution velocity distribution and Emden's polytropic density law is demonstrated for the exponent $n = 3/2$.

§I.

Emden's law connecting density and potential has the following form [1]:

$$\nu = c(U - H)^n. \tag{1}$$

In this case, $\nu = \nu(r)$ is the stellar density of a spherical system; $U = U(r)$ is the potential; H is its value at the boundary of the system; c and n are constants. By varying the exponent n, it is possible to obtain models which are more or less concentrated toward the center. In the present paper we shall assume that the phase density is a function of the energy integral only, *i.e.*, the velocity diagram is spherically symmetric at any distance from the center. In this case, in order to obtain Emden's law, it is necessary to give the phase density

$$\Psi = c_1(U - \frac{v^2}{2} - H)^{n-\frac{3}{2}}, \tag{2}$$

where c_1 is a certain new constant. This can be verified easily by integration with respect to the velocities. Furthermore, expression (2) is the only one corresponding to (1) under our assumptions, as can be seen from the theory of integral equations [2].

In what follows, we shall assume that $n \geq 3/2$. Otherwise, equation (2) gives a very artificial form of the velocity diagram; moreover, the usual definition of Lyapunov stability is not suitable here.

We shall prove the following theorems concerning stability with respect to the regular forces in stellar systems in which the phase density is a decreasing function of the energy integral:

1. For the stability of a stellar system, generally only perturbations preserving the spherical symmetry of the system can be dangerous (see Section IV).

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2. A sufficient condition for the stability of a stellar system is that $d^3\nu/du^3 \geq 0$ for all values of r (see Sections II and III). In particular, this condition encompasses Emden's law with $n > 2$.
3. We shall give below a proof that is valid for Emden's law with $n < 3$ only. Thus, the points 2 and 3 overlap.

Now, let $n < 3$. Consider the stability criterion given in [3]:

$$-\int\int \frac{q^2}{(dF/dE)}\, d\vec{r}d\vec{v} - mG \int\int\int\int \frac{q(\vec{r},\vec{v}) \cdot q(\vec{r}_1,\vec{v}_1)}{|\vec{r}-\vec{r}_1|}\, d\vec{r}d\vec{r}_1 d\vec{v}d\vec{v}_1, \tag{3}$$

where $\Psi = f(E)$ is the phase density. We have to prove that this expression is positive for any function q which can be represented in the form

$$q = \vec{v}\, grad_r f + grad_r U\, grad_v f, \tag{4}$$

where f is a differentiable function of the phase coordinates. As in the following formulae, if no limits are indicated, the integrals in equation (3) extend over the whole physical and phase volume occupied by the system.

The positivity of expression (3) can be proved with a very weak limitation imposed on q, namely,

$$\int\int q\, d\vec{r}d\vec{v} = 0. \tag{5}$$

By use of the Cauchy inequality, we have

$$\int \frac{q^2}{|dF/dE|} d\vec{v} \geq \left(\int q d\vec{v}\right)^2 \Big/ \int |dF/dE| d\vec{v}. \tag{6}$$

The absolute value sign is necessary here since $F(E)$ is a decreasing function of E.

Let us prove an auxiliary equality. Write the relation between the stellar density and the phase density as

$$\nu = \int \Psi(\tfrac{1}{2}\vec{v}^2 - U)\, d\vec{v}. \tag{7}$$

Differentiate equation (7) with respect to U

$$\frac{d\nu}{dU} = -\int \Psi'(\tfrac{1}{2}\vec{v}^2 - U)\, d\vec{v},$$

or

$$\int |df/dE| d\vec{v} = \frac{d\nu}{dU}.$$

Taking into consideration condition (6), we see that the stability will be proved if we succeed in demonstrating the positivity of the expression which is easily obtained from (3) and that is not greater than the latter:

$$\int \frac{p^2}{(d\nu/dU)} d\vec{r} - mG \int\int \frac{p(\vec{r}) \cdot p(\vec{r}_1)}{|\vec{r}-\vec{r}_1|}\, d\vec{r}d\vec{r}_1, \tag{8}$$

where

$$p(\vec{r}) = \int q\, d\vec{v}. \tag{9}$$

Let us limit our proof, for the time being, to the case of perturbations of the kind of radial pulsations which do not disturb the spherical symmetry of the system. The stability for perturbations of a more general form will be demonstrated later on.

Thus, let $p = p(r)$. If we disregard the coefficient $-mG$, the second term in expression (8) represents twice the energy of the field created by the "charge" distributed in space with a density $p(r)$. However, it is known [4] that the energy of such a field can be represented in a different form, namely,

$$\frac{1}{8\pi} \int |grad\, W_p|^2 d\vec{r} = \frac{1}{2} \int_0^\infty \left(\frac{dW_p}{dr}\right)^2 dr, \tag{10}$$

where W_p is the potential that corresponds, mathematically, to the charge p. It is easy to find that $dW_p/dr = 4\pi/r^2\tau$, where

$$\tau(r) = \int_0^r r^2 p\, dr. \tag{11}$$

The first term of (8) can be expressed in terms of the same function of τ. Indeed, if we differentiate equation (11), we obtain

$$p = \frac{1}{r^2}\frac{d\tau}{dr}. \tag{12}$$

Substituting these results in (8), we obtain

$$4\pi \int \left[\left(\frac{d\tau}{dr}\right)^2 / r^2 \frac{d\nu}{dU}\right] dr - 16\pi^2 mG \int \frac{\tau^2}{r^2}\, dr. \tag{13}$$

From (5) and (11) we have the boundary condition that τ becomes zero on the boundary of the system and outside it. From (11) it follows also that $\tau(0) = 0$, this zero being of the third order.

We have reached a variation problem typical for oscillatory phenomena. We are looking for the minimum of

$$\int \left[\left(\frac{d\tau}{dr}\right)^2 / r^2 \frac{d\nu}{dU}\right] dr \Big/ 4\pi mG \int \frac{\tau^2}{r^2}\, dr = \lambda,$$

and we have to prove that $\lambda > 1$. By the ordinary rules, the solution of this problem is given by the differential equation

$$\frac{d}{dr}\left(\frac{d\tau}{dr} / r^2 \frac{d\nu}{dU}\right) + 4\pi mG\lambda \frac{\tau}{r^2} = 0, \tag{14}$$

where λ turns out to be the first eigenvalue for the above mentioned boundary conditions $\tau(0) = \tau(b) = 0$, with b equal to the radius of the system.

Let us take the comparison function $\tau_0(r)$, which satisfies the initial condition $\tau_0(0) = 0$ and the differential equation

$$\frac{d}{dr}\left(\frac{d\tau_0}{dr}/r^2\frac{d\nu}{dU}\right) + 4\pi mG\frac{\tau_0}{r^2} = 0. \tag{15}$$

In the case of Emden's law, τ_0 can be found in an explicit form; it is equal to

$$\tau_0 = (n-3)r^2\frac{dU}{dr} + 4\pi mG(n-1)r^3\nu, \tag{16}$$

which can be verified easily by direct substitution in (15), taking into account the Poisson equation.

Let us now multiply (15) by τ, and (14) by τ_0; let us subtract one from the other and integrate from 0 to b. We obtain

$$\int_0^b\left[\tau_0\frac{d}{dr}\left(\frac{d\tau}{dr}/r^2\frac{d\nu}{dU}\right) - \tau\frac{d}{dr}\left(\frac{d\tau_0}{dr}/r^2\frac{d\nu}{dU}\right)\right]dr + 4\pi mG(\lambda-1)\int_0^b\frac{\tau\tau_0}{r^2}\,dr = 0.$$

We apply partial integration, and take into account that τ_0 has a zero of the third order for $r = 0$. Then it follows from (16):

$$\left(\tau_0\frac{d\tau}{dr}/r^2\frac{d\nu}{dU}\right)_{r=b} + 4\pi mG(\lambda-1)\int_0^b\frac{\tau\tau_0}{r^2}\,dr = 0. \tag{17}$$

However, inside the range $(0,b)$ τ is positive as the first eigenfunction, so that $(d\tau/dr)_{r=b} < 0$. As for τ_0, it is positive because of (16) (this is where the assumption $n < 3$ is required). On account of (2), we have $d\nu/dU > 0$. From (17) we then obtain $\lambda > 1$, which was required to prove stability.

§II.

The derivation of the stability criterion changes little if, instead of a system with a spherically symmetrical distribution of velocities, we consider a system of spherical form in which the phase density is a function of the energy and of the area integral in the orbital plane, and which is subject only to perturbations which would preserve spherical symmetry, (pulsations). Instead of dF/dE one must now write $\partial F/\partial E$.

Now, let us assign a certain scalar quantity V_s to each star. We shall discuss the distribution of this variable in detail below. Let us assume that V_s has the dimension of velocity although it does not vary with the motion of the star and, in turn, it does not affect its motion. Let us construct a phase density $\check{\Psi}(\vec{r},\vec{v},V_S)$, which depends on V_s as on a parameter. The choice of this fictitious phase density is, to a large degree, arbitrary, but in the integration with respect to V_s it must give the true phase density,

$$\int_{-\infty}^{+\infty}\check{\Psi}(\vec{r},\vec{v},V_S)\,dV_S = \Psi(\vec{r},\vec{v}). \tag{18}$$

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Since we are considering a stationary case, we can express $\check{\Psi}$ in terms of the integrals of motion, and write $\check{\Psi}(E, I, V_S)$ instead of $\check{\Psi}(\vec{r}, \vec{v}, V_S)$. Here $E = \Phi + \frac{1}{2}\vec{v}^2$ is the energy integral,* and

$$I = (\vec{r} \times \vec{v})^2 = (rV_t)^2$$

is the square of the angular momentum, where V_t is the tangential velocity.

Let us further assume that $\check{\Psi}$ has the form

$$\check{\Psi} = \check{F}(\check{E}, I), \tag{19}$$

where $\check{E} = E + \frac{1}{2}V_S^2$. Substituting in (18), we obtain

$$\Psi = \int_{-\infty}^{+\infty} \check{F}(E + \tfrac{1}{2}V_S^2), I)\, dV_S.$$

If Ψ is a sufficiently smooth function of E, this integral equation will determine a function $\check{F}(\check{E}, I)$, which is decreasing with respect to $\check{E}$.

With respect to the fictitious phase density $\check{\Psi}$, the stability problem is formulated in the same way as with respect to the true Ψ. It is obvious that the stability of $\check{\psi}$ implies the stability of Ψ. We thus arrive at the functional

$$\begin{aligned} &-\int\int\int \frac{q^2}{(d\check{F}/d\check{E})}\, d\vec{r}d\vec{v}dV_S \\ &-mG\int\int\int\int\int\int \frac{q(\vec{r}, \vec{v}, V_s)\cdot q(\vec{r}_1, \vec{v}_1, V_s)}{|\vec{r} - \vec{r}_1|} d\vec{r}d\vec{r}_1 d\vec{v}d\vec{v}_1 dV_s dV_{s1}, \end{aligned} \tag{20}$$

with

$$(q = \vec{v}\, grad_r f - grad_r \phi\, grad_v f).$$

Let us divide the whole phase space into "layers" with almost identical values of I within the limits of each layer. Essentially, this means a classification of stars with respect to I, which is an integral of motion even in a pulsating gravitational field; this cannot be said of E. We shall indicate the number of the layer by the subscript index $q = \sum_i q_i$, where q_i differs from zero only for $I_i < (rV_t)^2 < I_i + \Delta I_i$. We shall represent $\check{F}$ in exactly the same way:

$$\check{F} = \sum_i F_i(\check{E}),$$

where F_i differs from zero for $I_i < (rV_t)^2 < I_i + \Delta I_i$.

It is understood that I_i and ΔI_i depend on i only. The functional (20) takes the form

$$\begin{aligned} &-\sum_i \int\int\int \frac{q_i^2}{(dF_i d\check{E})} d\vec{r}d\vec{v}dV_s \\ &-mG\int\int\int\int\int\int \frac{\sum_i q_i(\vec{r}, \vec{v}, V_s)\cdot \sum_i q_i(\vec{r}_1, \vec{v}_1, V_{s1})}{|\vec{r} - \vec{r}_1|} d\vec{r}d\vec{v}dV_s d\vec{r}_1 d\vec{v}_1 dV_{s1}. \end{aligned} \tag{21}$$

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* The potential energy Φ equal to minus the potential is introduced in place of the potential U since in this way the formulae become more symmetrical.

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Let us substitute, for each q_i, the condition of orthogonality to an arbitrary function of $\check{E}$. This condition is weaker than is required. Indeed,

$$\sum_i \int\int\int q_i\sigma_i(\check{E})\, d\vec{r}d\vec{v}dV_s = \int\int\int q\sigma(\check{E}, I)\, d\vec{r}d\vec{v}dV_s$$
$$= \int\int\int (\vec{v}\,grad_r f - grad_r\Phi\, grad_v f)\, \sigma(\check{E}, I)\, d\vec{r}d\vec{v}dV_s$$
$$= \int\int\int [\vec{v} grad_r(f\sigma) - grad_r\Phi\, grad_v(f\sigma)] d\vec{r}d\vec{v}dV_s,$$

since the terms with $grad_r\sigma$ and $grad_v\sigma$ cancel. If one transforms to area integrals, it becomes clear that the last integral equals zero.

In turn, each q_i is now expanded into two terms in such a manner that both of them conserve orthogonality to the arbitrary functions of $\check{E}$, and, moreover, have the following properties:

$$q_i = Q_i + P_i,$$
$$Q_i = \frac{dF_i}{d\check{E}}\, (A_i(\check{E}) + B_i(r)), \tag{22}$$

$$\int\int P_i d\vec{v}dV_s = 0. \tag{23}$$

First, we must prove that such an expansion is always possible; secondly, we have to find the functions $A_i(\check{E})$ and $B_i(r)$. In order that Q_i is orthogonal to an arbitrary function of $\check{E}$, it suffices to require orthogonality of Q_i to the functions of the form 1 for $\check{E} < h$ and 0 for $\check{E} > h$; h is an arbitrary parameter that may have values between the limiting values of $\check{E}$.

We integrate first with respect to v and V_s:

$$\int\int Q_i\, dv dV_S = 2\pi \int\int Q_i V_t\, dV_t dV_r dV_S,$$

where V_r is the radial and V_t is the tangential velocity. The range of variation of V_t in which Q_i differs from zero can be easily found from the equalities

$$V_t^2 = \frac{I}{r^2}, \qquad 2V_t\Delta V_t = \frac{\Delta I}{r^2}.$$

Consequently,

$$\int\int_{\check{E}<h} Q_i\, d\vec{v}dV_S = \frac{\pi\Delta I}{r^2} \int\int Q_i\, dV_i dV_S.$$

We introduce "polar" coordinates, with $V^2 = V_r^2 + V_S^2$ and $\tan\theta = V_S/V_r$; also, $\check{E} = \frac{1}{2}V^2 + \Phi + I_t/2r^2$. Then

$$\int\int_{\check{E}<h} Q_i\, d\vec{v}dV_s = \frac{2\pi^2\Delta I_i}{r^2} \int_{\Phi_1}^{h} \frac{dF_i}{d\check{E}}\, (A_i(\check{E}) + B_i(r)) d\check{E}, \tag{24}$$

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where

$$\Phi_1 = \Phi + \frac{I_i}{2r^2}. \tag{25}$$

Physically, Φ_1 denotes the energy of a star that possesses the angular momentum $\sqrt{I}$ and that has its apocenter or pericenter at a distance r. Therefore, in particular, the regions of phase space where $\Phi_1 > 0$ are not occupied by stars in a stationary system. Φ_1 has the following properties:

$$\text{for} \quad r \to 0 \qquad \Phi_1 \to +\infty,$$
$$\text{for} \quad r \to \infty \qquad \Phi_1 \to 0,$$
$$r^2 \frac{d\Phi_1}{dr} = r^2 \frac{d\Phi}{dr} - \frac{I_i}{r},$$

which is obtained by differentiation of (25). Making use of the Poisson equation, we see that $r^2 d\Phi_1/dr$ is an increasing function. Consequently, Φ_1 has only one minimum at a certain point $r = r_0$.

Equation (24) may be written in the following form:

$$\int\int_{\check{E}<h} Q_i \, d\vec{v} dV_S = \frac{2\pi^2 \Delta I_i}{r^2} [M_i(h) - M_i(\Phi_1) + B_i(r)(F_i(h) - F_i(\Phi_1))], \tag{26}$$

where

$$M_i(\Phi_1) = \int_0^{\Phi_1} \frac{dF_1}{d\check{E}} A_i(\check{E}) \, d\check{E}. \tag{27}$$

Finally, we integrate (26) with respect to r, and equate the result to zero:

$$\int_{r_p}^{r_a} [M_i(h) - M_i(\Phi_1) + B_i(r)(F_i(h) - F_i(\Phi_1))] \, dr = 0. \tag{28}$$

Here r_a is the large root, and r_p is the small root of the equation $\Phi_1(r) = h$. We shall call these quantities reciprocal. We differentiate (28) with respect to h, taking into account that a substitution of r_a or r_p for r in the integrand changes the quantity Φ_1 into h, and the whole integrand into zero. Therefore, in (28), only the expression under the integral sign is differentiated with respect to h. We obtain

$$\int_{r_p}^{r_a} (F_i'(h) A_i(h) + B_i(r) F_i'(h)) \, dr = 0,$$

or

$$(r_a - r_p) A_i(h) + \int_{r_p}^{r_a} B_i(r) \, dr = 0. \tag{29}$$

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As for condition (23), it can be written in the form

$$\int\int Q_i \, dv dV_S = p_i, \tag{30}$$

where $p_i = \int\int q_i \, d\vec{v} dV_S$.

The integration in (30) is carried out as above (just as in (24) and (26)), and we obtain

$$p_i = -\frac{2\pi^2 \Delta I_i}{r^2}(M_i(\Phi_i) + B_i(r)F_i(\Phi_i)). \tag{31}$$

The feasibility of the required expansion will be proved if we succeed in solving the equations (29) and (31) for the functions A and B.

For this purpose, let us write (29) in a somewhat different form, taking into consideration that r_a can be chosen arbitrarily in the region $r > r_0$. Let us denote r_a by r, and the reciprocity relation by the sign $\sim$, so that $\Phi_1(\tilde{r})$ is always equal to $\Phi_1(r)$. Equation (29) then takes the form

$$(r - \tilde{r})A_i(\Phi_1) + \int_{\tilde{r}}^{r} B_i(r) \, dr = 0,$$

and the same result is obtained for $r < r_0$. Thus,

$$(r - r_0)A_i(\Phi_1) + \int_{r_0}^{r} B_i(r) \, dr = (\tilde{r} - r_0)A_i(\Phi_1) + \int_{r_0}^{r} B_i(r) \, dr. \tag{32}$$

Since (32) means that the expression on the left side does not change when r is changed to $\tilde{r}$, this expression represents a unique function of Φ_1:

$$(r - r_0)A_i(\Phi_1) + \int_{r_0}^{r} B_i(r) \, dr = \xi(\Phi_1).$$

We differentiate ξ with respect to r, and we isolate $B_i(r)$:

$$B_i(r) = -\frac{d}{dr}[(r - r_0)A_i(\Phi_1)] + \xi'(\Phi_1)\frac{d\Phi_1}{dr}. \tag{33}$$

Substituting (33) in (31), we obtain

$$M_i(\Phi_1) - F_i(\Phi_1)\frac{d}{dr}[(r - r_0)A_i(\Phi_1)] = -\frac{r^2 p_i}{2\pi^2 \Delta I_i} - F_i(\Phi_1)\xi'(\Phi_1)\frac{d\Phi_1}{dr}. \tag{34}$$

The lefthand side of (34) can be written in the form of a total derivative. Indeed, if one recalls the definition (27) of $M_i(\Phi_1)$, then

$$\frac{d}{dr}[(r - r_0)M_i(\Phi_1) - (r - r_0)F_i(\Phi_1)A_i(\Phi_1)]$$
$$= M_i(\Phi_1) - F_i(\Phi_1)\frac{d}{dr}[(r - r_0)A_i(\Phi_1)].$$

We integrate (34) from r_0 to r, introducing the function

$$\tau_i(r) = \int_0^r r^2 p_i \, dr, \tag{35}$$

so that

$$(r - r_0)(M_i(\Phi_1) - F_i(\Phi_1)A_i(\Phi_1)) = \\ = -\frac{\tau_i(r) - \tau_i(r_0)}{2\pi^2 \Delta I_i} - \int_{r_0}^{r} F_i(\Phi_1)\xi'(\Phi_1)\frac{d\Phi_1}{dr}\, dr. \tag{36}$$

Substituting, in (36), $\tilde{r}$ for r, we obtain

$$(\tilde{r} - r_0)(M_i(\Phi_1) - F_i(\Phi_1)A_i(\Phi_1)) \\ = -\frac{\tau_i(\tilde{r}) - \tau_i(r_0)}{2\pi^2 \Delta I_i} - - \int_{r_0}^{\tilde{r}} F_i(\Phi_1)\xi'(\Phi_1)\frac{d\Phi_1}{dr}\, dr. \tag{37}$$

The integral at the end of (36) permits replacement of the integration variable

$$\int_{r_0}^{r} F_i(\Phi_1)xi'(\Phi_1)\frac{d\Phi_1}{dr} dr = \int_{\Phi_1(r_0)}^{\Phi_1(r)} F_i(t)\xi'(t)dt = \int_{r_0}^{\tilde{r}} F_i(\Phi_1)\xi'(\Phi_1)\frac{d\Phi_1}{dr}\, dr.$$

We subtract (37) from (36), thus eliminating the function $\xi(\Phi_1)$:

$$M_i(\Phi_1) - F_i(\Phi_1)A_i(\Phi_1) = -\frac{\tau_i(r) - \tau_i(\tilde{r})}{2\pi^2 \Delta I_i(r - \tilde{r})}. \tag{38}$$

From (27) we find

$$A_i(\Phi_1) = \frac{M_i'(\Phi_1)}{F_i'(\Phi_1)}.$$

Substituting the righthand side of this equation in (38), we multiply by

$$\frac{F_i'(\Phi_1)}{[F_i(\Phi_1)]^2} \cdot \frac{d\Phi_1}{dr}$$

and we integrate from r_0 to r. We obtain

$$\frac{M_i(\Phi_1)}{F_i(\Phi_1)} = \frac{1}{2\pi^2 \Delta I_i} \int_{r_0}^{r} \frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}} \frac{F_i(\Phi_1)}{[F_i(\Phi_1)]^2} \frac{d\phi_1}{dr}\, dr + c \tag{39}$$

where c is a constant.

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Eliminating $M_i(\Phi_1)$ from (31) and (39), we obtain

$$B_i(r) = -\frac{r^2 p_i}{2\pi^2 \Delta I_i F_i(\Phi_1)} - \frac{1}{2\pi^2 \Delta I_i} \int_{r_0}^{r} \frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}} \frac{F'(\Phi_1)}{(F_i(\Phi_1))^2} \frac{d\Phi_1}{dr} dr - c. \tag{40}$$

From (38) we also obtain

$$A_i(\Phi_1) = \frac{1}{2\pi^2 \Delta I_i} \int_{r_0}^{r} \frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}} \frac{F_i'(\Phi_1)}{(F_i(\Phi_1))^2} \frac{d\Phi_1}{dr} + c + \frac{\tau_i(r) - \tau_i(\tilde{r})}{2\pi^2 \Delta I_i (r - \tilde{r}) F_i(\Phi_1)}. \tag{41}$$

Thus, the possibility of the required expansion is proved.

§III.

Let us return to the functional (21). Since we have written $q_i = P_i + Q_i$, the terms of the first summation in (21) may be given in the form

$$\int\int\int \frac{q_i^2}{(dF_i/d\check{E})} d\vec{r} d\vec{v} dV_s = \int\int\int \frac{P_i^2 + 2P_i Q_i + Q_i^2}{(dF_i/d\check{E})} d\vec{r} d\vec{v} dV_s$$
$$= \int\int\int \frac{P_i^2 + Q_i^2}{(dF_i/d\check{E})} d\vec{r} d\vec{v} dV_s + 2\int\int\int (A_i(\check{E}) + B_i(r)) P_i \, d\vec{r} d\vec{v} dV_s.$$

The integral $\int\int\int A_i(\check{E}) P_i \, d\vec{r} d\vec{v} dV_S$ becomes zero due to the orthogonality property of P_i, and the integral $\int\int\int B_i(r) P_i \, d\vec{r} d\vec{v} dV_S$ becomes zero on account of (23). Thus the expression

$$-\sum_i \int\int\int \frac{Q_i^2}{(dF_i/d\check{E})} d\vec{r} d\vec{v} dV_s - mG \int\int\int\int\int\int \frac{\sum Q_i(\vec{r}, \vec{v}, V_S) \cdot \sum Q_i(\vec{r}_1, \vec{v}_1, V_{s1})}{|\vec{r} - \vec{r}_1|} d\vec{r} d\vec{r}_1 d\vec{v} d\vec{v}_1 dV_s dV_{s1}, \tag{42}$$

is not greater than (21).

Let us substitute the expression (22) for Q_i in equation (42). We obtain

$$\int\int\int \frac{Q_i^2}{(dF_i/d\check{E}} d\vec{r} d\vec{v} dV_s = \int\int\int Q_i (A_i(\check{E}) + B_i(r)) \, d\vec{r} d\vec{v} dV_s$$
$$= \int\int\int Q_i B_i(r) \, d\vec{r} d\vec{v} dV_s = \int\int\int q_i B_i(r) \, d\vec{r} d\vec{v} dV_s$$
$$= \int p_i B_i(r) d\vec{r} = 4\pi \int_{a_1}^{b_1} r^2 p_i B_i(r) \, dr,$$

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again making use of (23) and of the property of orthogonality. Substituting (40), we obtain

$$\int\int\int \frac{Q_1^2}{(dF_i/d\check{E})}\, d\vec{r}d\vec{v}dV_s = -\frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i} \frac{r^1 p_i^2}{F_i(\Phi_1)}\, dr + 4\pi\int_{a_1}^{b_i} r^2 p_i H_i(r) dr. \tag{43}$$

Here, $H_i(r)$ is the remaining integral

$$H_i(r) = \frac{1}{2\pi^2\Delta I_i}\int_{r_0}^{r} \frac{\tau_i(r)-\tau_i(\tilde{r})}{r-\tilde{r}}\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\, dr + c. \tag{44}$$

The interval (a_i, b_i) marks the region of the physical space in which stars with angular momentum $\sqrt{I_i}$ can move (with $b_i = \tilde{a}_i$). It is clear from the definition (35) of $\tau_i(r)$ that it must satisfy the boundary conditions $\tau_i = 0$ for $r \geq b_i$ and $r \leq a_i$.

Let us carry out the partial integration of the second integral on the right side of (43):

$$\int_{a_i}^{b_i} r^2 p_i H_i(r)\, dr = \int_{a_i}^{b_i} \tau_i'(r) H_i(r)\, dr = -\int_{a_i}^{b_i} \tau_i(r) H_i'(r)\, dr.$$

We substitute the last result in (43), taking into consideration (44). As for the last integral in (42), it is transformed as the field energy, as explained above. We finally obtain from (42)

$$\begin{aligned}\sum_i \frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i} \frac{[\tau_i'(r)]^2}{F_i(\Phi_1)}\, dr - 16\pi^2 mG\int_0^\infty \frac{[\sum_i \tau_i(r)]^2}{r^2}\, dr \\ + \sum_i \frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i} \tau_i(r)\frac{\tau_i(r)-\tau_i(\tilde{r})}{r-\tilde{r}}\frac{d}{dr}\Big(\Big(\frac{1}{F_i(\Phi_1)}\Big)\, dr.\end{aligned} \tag{45}$$

Let us prove the auxiliary inequality

$$\int_{a_i}^{b_i} \tau_i(r)\frac{\tau_i(r)-\tau_i(\tilde{r})}{r-\tilde{r}}\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\, dr - \int_{a_i}^{b_i} \frac{[\tau_i(r)]^2}{r}\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\, dr > 0. \tag{46}$$

For this purpose, let us inspect the integral

$$\int_{a_i}^{b_i} \frac{\tau_i(r)-\tau_i(\tilde{r})}{r-\tilde{r}}\,\frac{r\tau_i(\tilde{r})-\tilde{r}\tau_i(r)}{r-\tilde{r}}\,\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\, dr. \tag{47}$$

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The fractions $[\tau_i(r) - \tau_i(\tilde{r})]/(r - \tilde{r})$ and $[r\tau_i(\tilde{r}) - \tilde{r}\tau_i(r)]/(r - \tilde{r})$ remain unchanged when $\tilde{r}$ replaces r. Consequently, they are single-valued functions of Φ_1. The integral (47) is equal to zero, because we have the derivative of a function of Φ_1 under the integral sign, while $\tau_i(a_i) = \tau_i(b_i) = 0$. Let us add (47) to the difference considered in (46), and let us take into account that

$$\tau_i(r)\frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}} - \frac{[\tau_i(r)]^2}{r} + \frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}}\,\frac{r\tau_i(\tilde{r}) - \tilde{r}\tau_i(r)}{r - \tilde{r}} = -\frac{[r\tau_i(\tilde{r}) - \tilde{r}\tau_i(r)]^2}{r(r - \tilde{r})^2}.$$

Then we obtain

$$\int_{a_i}^{b_i}\Big[\tau_i(r)\frac{\tau_i(r) - \tau_i(\tilde{r})}{r - \tilde{r}} - \frac{[\tau_i(r)]^2}{r}\Big]\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\,dr = \int_{a_i}^{b_i}\frac{[r\tau_i(\tilde{r}) - \tilde{r}\tau_i(r)]^2}{r(r - \tilde{r})^2}\,\frac{F_i'(\Phi_1)}{[F_i(\Phi_1)]^3}\,\frac{d\Phi_1}{dr}\,dr. \tag{48}$$

We write the integral $\int_{a_i}^{b_i}$ in the form of a sum, $\int_{r_0}^{b_i} + \int_{a_i}^{r_0}$, and we replace the integration variable r by $\tilde{r}$ in the first one, taking into consideration that $d\Phi_1 = (d\Phi_1/dr)dr = (d\Phi_1/d\tilde{r})d\tilde{r}$. Instead of (48) we now have

$$\int_{a_i}^{r_0}\left(\frac{r\tau_i(\tilde{r}) - \tilde{r}\tau_i(r)}{r - \tilde{r}}\right)^2\Big(\frac{1}{r} - \frac{1}{\tilde{r}}\Big)\frac{F_i'(\Phi_1)}{[F_i(\Phi_1)]^2}\,\frac{d\Phi_1}{dr}\,dr \geq 0$$

taking into account that $F_i'(\Phi_1) < 0$, and that for $a_i < r < r_0$, we have $\tilde{r} > r$ and $d\Phi_1/dr < 0$. Thus, the inequality (46) has been proved.

According to a well-known algebraic inequality,

$$\Big(\sum_i \tau_i\Big)^2 < \sum_i \nu_i \sum_i \frac{\tau_i^2}{\nu_i}, \qquad \text{or} \qquad \Big(\sum_i \tau_i\Big)^2 < \nu \sum_i \frac{\tau_i^2}{\nu_i}, \tag{49}$$

which we shall use for the transformation of the last integral in formula (45). The inequalities (46) and (49) show that it suffices, for the stability proof, to prove the positiveness of the functional

$$\sum_i\Big[\frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i}\frac{[\tau_i'(r)]^2}{F_i(\Phi_1)}\,dr + \frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i}\frac{[\tau_i(r)]^2}{r}\,\frac{d}{dr}\Big(\frac{1}{F_i(\Phi_1)}\Big)\,dr - 16\pi^2 mG\int_{a_i}^{b_i}\frac{\nu(\tau_i[r])^3}{r^2\nu_i}\,dr\Big]. \tag{50}$$

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Let us prove the positiveness of each square bracket in (50). For this purpose, we shall estimate the minimum

$$min \frac{\frac{2}{\pi \Delta I_i}\int_{a_i}^{b_i} \frac{[\tau_i(r)]^2}{F_i(\Phi_1)}\,dr + \frac{2}{\pi\Delta I_i}\int_{a_i}^{b_i} \frac{[\tau_i(r)]^2}{r}\frac{d}{dr}\left(\frac{1}{F_i(\Phi_1)}\right)dr}{16\pi^2 mG \int_{a_i}^{b_i} \frac{\nu(\tau_i[r])^3}{r^2\nu_i}\,dr} = \lambda.$$

We find again the classical variation problem from the theory of oscillations.
The corresponding differential equation is

$$\frac{2}{\pi\Delta I_i}\frac{d}{dr}\left(\frac{\tau_i'(r)}{F_i(\Phi_1)}\right) - \frac{2}{\pi\Delta I_i}\frac{\tau_i(r)}{r}\frac{d}{dr}\left(\frac{1}{F_i(\Phi_1)}\right) + 16\pi^2 mG\lambda\frac{\nu}{r^2\nu_i}\tau_i(r) = 0, \quad (51)$$

where λ is the first eigenvalue. Let us form the comparison function $N_i = r^3\nu_i$. It is easy to express ν_i in terms of F_i through integration with respect to the velocities and with respect to V_S, just as in the derivation of (24):

$$\begin{aligned} \nu_i &= \frac{2\pi^2\Delta I_i}{r^2}\int_{\Phi_1}^{0} F_i(E)\,dE, \\ N_i &= 2\pi^2\Delta I_i r\int_{\Phi_1}^{0} F_i(E)\,dE. \end{aligned} \quad (52)$$

The ν_i and, consequently, the N_i, become zero at the spatial boundaries of the ith layer. It is easy to find the differential equation satisfied by N_i. We have

$$\frac{dN_i}{dr} = 2\pi^2\Delta I_i\int_{\Phi_1}^{0} F_i(E)\,dE - 2\pi^2\Delta I_i r\frac{d\Phi_1}{dr}F_i(\Phi_1),$$

$$\frac{d}{dr}\left(\frac{N_i'(r)}{F_i(\Phi_1)}\right) = r^2\nu_i\frac{d}{dr}\left(\frac{1}{F_i(\Phi_1)}\right) + \frac{(d(r^2\nu_i)/dr}{F_i(\Phi_1)} - 2\pi^2\Delta I_i(r\frac{d^2\Phi_1}{dr^2} + \frac{d\Phi_1}{dr}),$$

but from (52) it follows that

$$\frac{1}{F_i(\Phi_1)}\frac{d}{dr}(r^2\nu_i) = -2\pi^2\Delta I_i\frac{d\Phi_1}{dr}.$$

Thus, we have

$$\begin{aligned} &\frac{d}{dr}\left(\frac{N_i'(r)}{F_i(\Phi_1)}\right) - \frac{N_i(r)}{r}\frac{d}{dr}\left(\frac{1}{F_i(\Phi_1)}\right) \\ &\quad = -2\pi^2\Delta I_i(r\frac{d\Phi_1}{dr^2} + 2\frac{d\Phi_1}{dr}) = -2\pi^2\Delta I_i(4\pi mGr\nu + \frac{I_i}{r^4}), \end{aligned}$$

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and from this we obtain the desired equation

$$\frac{2}{\pi \Delta I_i} \frac{d}{dr}\left(\frac{N_i'(r)}{F_i(\Phi_1)}\right) - \frac{2}{\pi \Delta I_i}\frac{N_i(r)}{r}\frac{d}{dr}\left(\frac{1}{F_i(\Phi_1)}\right) + 16\pi^2 mG \frac{\nu}{r^2 \nu_i} N_i(r) = -\frac{4\pi I_i}{r^4}. \tag{53}$$

Let us multiply (51) by N_i and (53) by τ_i, then subtract one from the other and integrate from a_i to b_i. After partial integration the boundary terms disappear on account of $\tau_i(b_i) = \tau_i(a_i) = 0$ and $N_i(b_i) = N_i(a_i) = 0$. We obtain

$$16\pi^2 mG(\lambda - 1) \int_{a_i}^{b_i} \frac{\nu}{r^2 \nu_i} \tau_i N_i \, dr = 4\pi I_i \int_{a_i}^{b_i} \frac{\tau_i}{r^4} \, dr,$$

hence $\lambda > 1$, and this concludes the stability proof under the assumptions specified above.

Let us note that the character of the dependence of the phase density on I is unessential, because it is always possible to introduce a sufficiently detailed classification of stars by the kinetic momentum, in order to make it possible to neglect the change of I inside each separate layer.

As an example of an application of the preceding discussions, let us consider the simplest case, when the phase density is a function of the energy integral only. Then the fictitious phase density must be a function of $\check{E}$. Let us consider what limitations on the stellar density are imposed by our condition that $\check{\Psi} = \check{F}(\check{E})$ must be a decreasing function. Let us integrate it with respect to the velocities and with respect to V_S:

$$\begin{aligned} \nu &= \int\int \check{F}(\check{E}) d\vec{v} dV_s \\ &= \int\int\int\int \check{F}\left(\Phi + \tfrac{1}{2}(u^2 + v^2 + w^2 + V_s^2)\right) du dv dw dV_s \\ &= \int_0^\infty \check{F}(\Phi + \tfrac{1}{2}W^2)\sigma(W)\, dW. \end{aligned}$$

Here $\sigma(W)$ is the surface of the four–dimensional hypersphere $u^2 + v^2 + w^2 + V_S^2 = W^2$. It is well known that $\sigma(W) = 2\pi^2 W^3$. Therefore,

$$\nu = 2\pi^2 \int_0^\infty \check{F}(\Phi + \tfrac{1}{2}W^2) W^3 \, dW.$$

We differentiate this equality twice with respect to Φ:

$$\frac{d\nu}{d\Phi} = -4\pi^2 \int_\Phi^\infty \check{F}(t)\, dt, \qquad \frac{d^2\nu}{d\Phi^2} = 4\pi^2 \check{F}(\Phi).$$

Thus, the condition of decreasing $\check{F}$ is equivalent to the condition $d^3\nu/d\Phi^3 \leq 0$, which is satisfied for the Emden laws with $n > 2$. Actually,

$$\nu = c(\Phi_0 - \Phi)^n, \qquad \frac{d^3\nu}{d\Phi^3} = -cn(n-1)(n-2)(\Phi_0 - \Phi)^{n-3} < 0.$$

This proves the stability of the Emden laws with $n > 2$ for a spherically symmetrical distribution of velocities with respect to spherically symmetrical pulsations.

§IV.

Let us now consider perturbations that violate the spherical symmetry of the system. We assume that $dF/dE < 0$. Any perturbation may always be made into a spherically symmetrical one by averaging over all orientations. We subtract this average and we obtain the supplementary density p, with the property

$$\int p\, d\omega = 0. \tag{54}$$

Here, $d\omega$ is an element of the unit sphere.

Let us expand p into a series of spherical functions

$$p = \sum p_n^{(m)} Y_n^{(m)}(\theta, \phi). \tag{55}$$

The spherical function with the index $n = 0$ (it is equal to a constant) is not included in the series (55) on account of (54). As is known from theory [5], the potential corresponding to the density distribution $p_n^{(m)}(r) Y_n^{(m)}(\theta, \phi)$ is proportional to the same spherical function $Y_n^{(m)}(\theta, \phi)$.

On account of the known property of orthogonality of spherical functions in the functional characterizing the stability [we take it in the form (8)], all the terms containing products of two different spherical functions disappear. It is therefore possible to substitute, in (8), each term of the series (47) separately, and that is what we are going to do.

As has been explained already, the functional (8) may be presented in the form

$$\begin{aligned} &\int\int \frac{r^2 p^2}{(d\nu/dU)}\, dr d\omega - \frac{mG}{4\pi} \int\int |gradW_p|^2\, dr d\omega \\ &= \int \frac{r^2 p^2}{(d\nu/dU)}\, dr d\omega - \frac{mG}{4\pi} \int\int [(\frac{\partial W_p}{\partial r})^2 + |grad_1 W_p|^2] r^2\, dr d\omega. \end{aligned} \tag{56}$$

Here, $grad_1 W_p$ is the projection of $grad\, W_p$ on a plane perpendicular to the radius. But, as has been stated above, $W_p = w_n^{(m)}(r) Y_n^{(m)}(\theta, \phi)$, so that

$$\int\int [(\frac{\partial W_p}{dr})^2 + |grad_1 W_p|^2] r^2\, d\omega dr = \int_0^\infty [r^2 (\frac{d\omega_n^{(m)}}{dr})^2 + n(n+1)[\omega_n^{(m)}]^2]\, dr.$$

We assume here that the spherical functions are normalized such that

$$\int\int [Y_n^{(m)}(\theta,\phi)]^2\,d\omega = 1. \tag{57}$$

Then the theory of spherical functions gives [6]

$$\int\int |grad Y_n^{(m)}(\theta,\phi)|^2\,d\omega = n(n+1). \tag{58}$$

By partial integration it is easy to obtain

$$\int_0^\infty \left[r^2\left(\frac{dm_n^{(m)}}{dr}\right)^2 + n(n+1)[\omega_n^{(m)}]^2\right] dr = \int_0^\infty \left[r\frac{d\omega_n^{(m)}}{dr} + (n+1)\omega_n^{(m)}\right]^2 dr.$$

Let us now use Poisson's equation to find the connection between $w_n^{(m)}$ and $p_n^{(m)}$.

$$4\pi\rho = \Delta W_p,$$

$$4\pi p_n^{(m)} Y_n^{(m)}(\theta,\phi) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dw_n^{(m)}}{dr}\right)Y_n^{(m)}(\theta,\phi) - \frac{n(n+1)}{r^2}w_n^{(m)}(r)Y_n^{(m)}(\theta,\phi). \tag{59}$$

Let us introduce the auxiliary function

$$\tau_n^{(m)}(r) = \frac{d\omega_n^{(m)}}{dr} + \frac{n+1}{r}\omega_n^{(m)}. \tag{60}$$

Then

$$\int\int \left[\left(\frac{\partial\omega_p}{\partial r}\right)^3 + |grad_1\omega_p|^3\right] dr\,d\omega = \int_0^\infty r^2[\tau_n^{(m)}]^2\,dr. \tag{61}$$

On the basis of (59) and (60) we have

$$4\pi p_n^{(m)} = \frac{dw_n^{(m)}}{dr^2} + \frac{2}{r}\frac{dw_n^{(m)}}{dr} - \frac{n(n+1)}{r^2}w_n^{(m)} = \frac{d\tau_n^{(m)}}{dr} - \frac{n-1}{r}\tau_n^{(m)}. \tag{62}$$

We express (56) through τ with the aid of (61), (62) and of the normalization condition (57); we obtain*

$$\frac{1}{(4\pi)^2}\int_0^\infty \frac{[r d\tau/dr - (n-1)\tau]^2}{(d\nu/dU)}\,dr - \frac{mG}{4\pi}\int_0^\infty r^2\tau^2\,dr. \tag{63}$$

* To shorten the notation we shall omit the subscript on τ in the remainder of this article

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Let us note that τ vanishes outside the system and on its boundary. Indeed, in the outer space W_p coincides, accurately, except for a constant factor, with the harmonic function $r^{-n-1}Y_n^{(m)}(\theta, \phi)$. It then follows from (60) that $\tau = 0$. It is possible, therefore, in (63), to put b, the radius of the system, as the upper boundary. For $r = 0$ continuity is required of $\tau(r)$, because at the center of the system W_p becomes zero.

We are back at the evaluation problem of a certain minimum

$$min \int_0^b \frac{[r d\tau/dr - (n-1)\tau]^2}{(d\nu/dU} dr / 4\pi m G \int_0^b r^2 \tau^2 \, dr = \lambda. \tag{64}$$

Let us write the differential equation corresponding to (64)

$$\frac{d}{dr}\Big(\frac{r^2 d\tau/dr}{(d\nu/dU)}\Big) - (n-1)\frac{d}{dr}\Big(\frac{r}{(d\nu/dU)}\Big) = -\frac{(n-1)^2}{(d\nu/dU)}\tau + 4\pi m G \lambda r^2 \tau = 0, \tag{65}$$

where λ is the first eigenvalue.

Let us take $N = r^{n-1}\nu$ as the comparison function. Let us construct a differential equation for it analogous to (65). We have

$$\begin{aligned}
\frac{dN}{dr} &= (n-1)r^{n-2}\nu + r^{n-1}\frac{d\nu}{dU}\frac{dU}{dr},\\
\frac{d}{dr}\Big(\frac{r^2(dN/dr)}{(d\nu/dU)}\Big) &= (n-1)r^{n-1}\nu \frac{d}{dr}\Big(\frac{r}{(d\nu/dU)} + (n-1)^2 \frac{r^{n-1}\nu}{(d\nu/dU)}\\
&\quad + (n-1)r^n \frac{dU}{dr} + \frac{d}{dr}\Big(r^{n+1}\frac{dU}{dr}\Big),\\
\frac{d}{dr}\Big(\frac{r^2(dN/dr)}{(d\nu/dU)}\Big) &- (n-1)\frac{d}{dr}\Big(\frac{r}{(d\nu/dU)}\Big)N + 4\pi m G r^2 N - \frac{(n-1)^2}{(d\nu/dU)}N\\
&= 2nr^n \frac{dU}{dr} + r^{n+1}\frac{d^2U}{dr^2} + 4\pi m G r^{n+1}\nu.
\end{aligned}$$

Eliminating d^2U/dr^2 with the aid of Poisson's equation, we obtain the desired equation

$$\begin{aligned}
\frac{d}{dr}\Big(\frac{r^2(dN/dr)}{(d\nu/dU)}\Big) &- (n-1)\frac{d}{dr}\Big(\frac{r}{(d\nu/dU)}\Big)N\\
&+ 4\pi m G r^2 N - \frac{(n-1)^2 N}{(d\nu/dU)} = 2(n-1)r^n \frac{dU}{dr}.
\end{aligned} \tag{66}$$

We multiply (65) by N, and (66) by τ, we subtract one from the other, and we integrate from 0 to b, taking into consideration that $N(b) = 0$. Then we obtain

$$4\pi m G(\lambda - 1)\int_0^b r^2 \tau N \, dr = -2(n-1)\int_0^b r^n \frac{dU}{dr}\tau \, dr.$$

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The right side is not negative, because $dU/dr < 0$ and n is a positive integer. Therefore, for $n > 1$, we have $\lambda > 1$, and for $n = 1$, we have $\lambda = 1$. In the latter case it would be possible to show that instability occurs, because (66) vanishes for the function τ, accurate to a constant factor. However, then

$$\tau = N = \nu, \qquad p_1^{(m)} = \frac{1}{4\pi}\frac{d\nu}{dr}.$$

Multiplying $\tau_i^{(m)}$ by the spherical function which, for $n = 1$, equals x/r, y/r, or z/r, we obtain

$$p = c\frac{x}{r}\frac{d\nu}{dr} = c\frac{\partial\nu}{\partial x},$$

etc. It is clear that we are dealing simply with a displacement of the system as a whole. Thus, perturbations of a "higher order" are not dangerous for the stability of the system, when $F(E)$ is a decreasing function. In particular, this conclusion is applicable to a system with Emden's density law, if $n \geq \frac{3}{2}$.

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