References

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108.23 A recurrence relation derived graphically

The n-th triangular number, T_n, is defined as $\frac{1}{2}n(n + 1)$ for integer $n > 0$. We shall find a recurrence relation which gives setS of three consecutive triangle numbers summing to a triangle number. The first two examples are:

$$T_1 + T_2 + T_3 = T_4, \qquad T_8 + T_9 + T_{10} = T_{16}.$$

Let

$$T_k + T_{k+1} + T_{k+2} = T_l. \tag{1}$$

We shall use the fact that the suffix to the symbol for a triangular number is the side length of its figurate representation. This will enable us to transform a two-dimensional problem to one of a single dimension.

In Figure 1 we have drawn a triangle for T_l containing T_k, T_{k+1}, T_{k+2}. For equality, it must be the case that the regions of overlap sum to the central uncovered region. We can find the dimensions of these regions from the figure, leading to the reduced equation

$$T_{2k+1-l} + T_{2k+2-l} + T_{2k+3-l} = T_{2l-3k-4}. \tag{2}$$

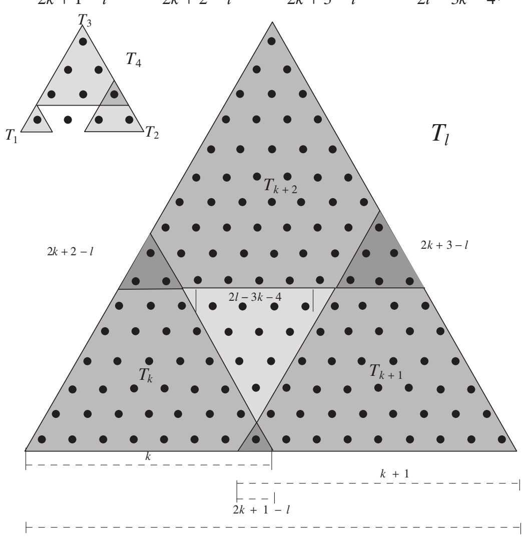

FIGURE 1

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Entering new values in the suffix to a term in (2), we obtain the old value as the suffix to the corresponding term in [1]:

$$k_n = 2k_{n+1} + 1 - l_{n+1}, \qquad l_n = 2l_{n+1} - 3k_{n+1} - 4,$$

whence we derive:

$$k_{n+1} = l_n + 2k_n + 2, \qquad l_{n+1} = 2l_n + 3k_n + 5.$$

With starting values $k_0 = 1$, $l_0 = 4$, derived from the inset diagram, this is the relation we require.

Let T_i^* denote the triangle representing T_i. For our count to be valid, we must ensure that the point sets $T_k^* \cap T_{k+1}^*$, $T_{k+1}^* \cap T_{k+2}^*$, $T_{k+2}^* \cap T_k^*$, are non-empty except in the special case $k = 1$. It is sufficient to show this for the smallest intersection, $T_k^* \cap T_{k+1}^*$. We require $2k + 1 - l > 0$, i.e. $l \not\geqslant 2k + 1$. $l \geqslant 2k + 1 \Leftrightarrow T_l \geqslant T_{2k+1} \Leftrightarrow T_k + T_{k+1} + T_{k+2} \geqslant T_{2k+1}$ (from (1)) $\Leftrightarrow k^2 - 3k - 6 \leqslant 0$. Since $k > 0$, the last condition is equivalent to $k < 5$. In the two smallest cases, cited at the head of the Note, the only k value below 8 is 1, our special case. In general therefore, $l \not\geqslant 2k + 1$, as required.

The referee pointed out that we have not established that our method gives *all* possible pairs k, l satisfying (1). Application of the following alternative method might give some psychological reassurance but would not strengthen the case that either method gave all pairs.

Using the identity $T_k + T_{k+1} + T_{k+2} = 3T_{k+1} + 1 = T_l$, we can find consecutive values of the same parameter, or the value of one parameter from that of the other of the same generation, since: l_n and k_n satisfy the equation

$$l_n(l_n + 1) = 3k_n(k_n + 3) + 8,$$

k_n and k_{n+1} satisfy the equation

$$k_{n+1}^2 + k_n^2 = 4k_nk_{n+1} + 3(k_{n+1} + k_n) + 6,$$

l_n and l_{n+1} satisfy the equation

$$T_{l_{n+1}-1} + T_{l_n - 1} = 2(l_{n+1}l_n - 1).$$

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108.24 Covering a triangular number with heptagonal numbers

Proposition: Where T_n and X_n denote the triangular number of n sides and the heptagonal number of n sides, respectively, for $n \in \mathbb{N}$, the following identity holds:

$$5X_n + 1 = T_{5n-2}.$$