Canad. Math Bull. Vol. **18** (2), 1975

ON THE STRUCTURE OF QUOTIENT RINGS WHICH ARE *QFX* RINGS

BY
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The object of this paper is to consider the relationships between matrix rings and rings having classical quotient rings which are quasi-Frobenius X (QFX) rings. The main result of this paper is Theorem 12, which shows that if S is a ring with a QFX right classical quotient ring T, then T is isomorphic to a direct sum of a finite number of matrix rings over local rings U_i, while S is almost a direct sum of matrix rings over rings C_i, the U_i being right classical quotient rings of the C_i.

QF rings and their generalizations and QF quotient rings have received much attention in recent years. Important results concerning these types of rings have been given by Jans [7], Faith and Walker [4], and Dieudonne [1]. Feller characterized and gave important properties of QFX rings in [5]. Further work on X rings has recently been completed by Zaks [11].

Much of the work on QF rings has been stimulated by results which were obtained for semisimple Artinian rings and semi-prime rings. In particular, Goldie's work [6] which characterized rings with semisimple Artinian quotient rings led to several characterizations of rings with QF quotient rings. The research for this paper was stimulated by the Faith-Utumi results [2, 3] regarding quotient rings which were semisimple Artinian quotient rings.

Notation. In order to conserve space, we introduce the following notation. The ring of all $n \times n$ matrices over the ring C will be denoted by C_n. We will denote that the ring R contains a subring isomorphic to V by writing $V \lessapprox R$. We use $T = \mathfrak{Q}_1(S)$ to denote that T is the right classical quotient ring of S where S has an identity and use $T = \mathfrak{Q}(S)$ when it is unknown if S has an identity. If $T = \mathfrak{Q}(S)$, we denote that T is also QF, an X ring or a QFX ring by writing $T = QF\mathfrak{Q}(S)$, $T = X\mathfrak{Q}(S)$, and $T = QFX\mathfrak{Q}(S)$, respectively.

1. Definition. A ring S is said to be a regular order in a ring T if S is a subring of T and for each $t \in T$ with $t \neq 0$ there exists a regular element $s \in S$ such that $ts \in S$ with $ts \neq 0$.

Note that every right classical quotient ring is a regular order.

2. Proposition [8, *p.* 116]. *Let S be a regular order in $T = U_n$, where U is local. Then there exists a ring C such that C is a regular order in U and $C_n \lessapprox S \subset U_n = T$.*

Received by the editors December 5, 1973.

https://doi.org/10.4153/CMB-1975-040-4 Published online by Cambridge University Press

From this proposition we immediately get a result which will be of interest to our study.

3. PROPOSITION. *Let* $T=\mathfrak{Q}(S)$, *where* $T=U_n$ *is right Artinian and* U *is local. Then* U *is right Artinian and there exists a ring* C *such that* $U=\mathfrak{Q}(C)$ *and* $C_n \lesssim S \subset U_n=T$.

Proof. Since T is right Artinian, U is right Artinian. Every right classical quotient ring is a regular order, so we may apply Prop. 2 to obtain a ring C such that C is a regular order in U and $C_n \sim S \subset U_n=T$. Since U is right Artinian and C is a regular order in U, every regular element of C has an inverse in U, and thus $U=\mathfrak{Q}(C)$.

In the following definition the superscript "r" denotes the right annihilator while the symbol "$\simeq$" denotes isomorphism.

4. DEFINITION. A ring W is said to be an X ring provided it satisfies the following condition for each pair of distinct primitive idempotents e_i and e_j: if $e_iW\simeq e_jW$, $a\in e_iW$ and $a^r \cap e_jW\neq 0$, then $ae_jW=0$.

5. PROPOSITION [5, *p*. 22]. *Let* T *be a* QFX *ring. Then* $T=\coprod_{i=1}^n A_i$, *where each* A_i *is a two-sided ideal of* T *which is indecomposable into two-sided ideals of* T *and which is a total matrix ring over a local ring. Each ideal* A_i *in this decomposition is said to be a block of the ring* T.

6. THEOREM. $T=QFX\mathfrak{Q}(S)$ *where* T *has one block if and only if there exist rings* C *and* U *with* U *local such that* $U=QF\mathfrak{Q}(C)$ *and* $C_n \lesssim S\subset U_n=T$.

Proof. Let $T=QFX\mathfrak{Q}(S)$ where T has one block. By Prop. 3 there exists a ring U such that $T=U_n$, where U is local. Since T is QF, U is QF. By Prop. 3 there exists a ring C such that $U=\mathfrak{Q}(C)$ and $C_n \lesssim S\subset U_n=T$.

Now let $C_n \lesssim S\subset U_n$ where $U=QF\mathfrak{Q}(C)$ and U is local. Then $U_n=\mathfrak{Q}(C_n)$ and hence $U_n=\mathfrak{Q}(S)$. Since U is QF, U_n is QF; U_n is an X ring by [5, p. 23].

This theorem shows that S contains a ring isomorphic to a total matrix ring. We next show that S is contained in a total matrix ring D_n which is close to S in the sense that D_n is isomorphic to a subring of T generated by S and the inverse of a regular element of S.

7. PROPOSITION. *Let* $T=QFX\mathfrak{Q}(S)$ *where* T *has one block. Then* $T=U_n$, *where* U *is local, and there exist a regular element* c *in* S *and a ring* D *such that* $S[c^{-1}]=D_n$, *where* $U=QF\mathfrak{Q}(D)$ *and* $S[c^{-1}]$ *is the subring of* T *generated by* S *and* c^{-1}. *Furthermore, if* R *is any ring such that* $S[c^{-1}]\subset R\subset T$, *then* R *is also a total matrix ring* E_n, *where* $U=\mathfrak{Q}(E)$.

Proof. By Prop. 3, there exists a ring U such that $T=U_n$, where U is a local ring. Since T is QF, U if QF. Let e_{ij}, $i,j=1,2,\ldots,n$, be the element of T with the identity of U in the (i, j) position and 0's elsewhere. Since $e_{ij} \in T$ there exists a regular element $c \in S$ such that $e_{ij}c \in S$. Let $S'=S[c^{-1}]$, the subring of T generated by S and c^{-1}. Since $c \in S$, $1 \in S'$. Now $S'=D_n$ for some ring D. To see this, let D be the collection of all elements of U which appear in the $(1, 1)$ position of some matrix in S'. Since $e_{ij} \in S'$, it is easy to verify that D is a subring of U. Now let $(a_{ij}) \in S'$. Since $e_{1i}(a_{ij})e_{j1} \in S'$, we have $a_{ij} \in D$ and thus $(a_{ij}) \in D_n$. This shows that $S' \subset D_n$. Now let $x \in D_n$ where $x=\sum e_{ij}d_{ij}$. For each (i, j) there exists a matrix $\alpha_{ij} \in S'$ such that d_{ij} is in the $(1, 1)$ position of α_{ij}. Then $x=\sum e_{ij}e_{i1}\alpha_{ij}e_{1j} \in S'$ and we have shown that $D_n \subset S'$. We now have $S \subset S'=D_n \subset U_n=T$.

By hypothesis, $T=\mathfrak{Q}(S)$, so $T=\mathfrak{Q}_1(S')$ and $U_n=\mathfrak{Q}_1(D_n)$. From this last equality we get $U=\mathfrak{Q}_1(D)$. To see this, let a be regular in D. Then $\sum e_{ii}a$ is regular in D_n and has inverse $\sum e_{ij}b_{ij}$ in U_n. Since $(\sum e_{ii}a)(\sum e_{ij}b_{ij})=1$ we must have $b_{ij}=0$ for $i\neq j$ and $b_{ii}=b$ for each i. Thus we have $cb=1$ and c has a two-sided inverse b in U.

Now let (c_{ij}) be any regular element of U_n. The matrix (c_{ij}) has inverse (b_{ij}) in U_n. Then $1=(\sum_k c_{ik}b_{kj})=(\sum_k b_{ik}c_{kj})$, which yield the equations $1_U=\sum_k c_{ik}b_{ki}=\sum_k b_{ik}c_{ki}$ for each i, where 1_U is the identity of U. Since U is local, the last equations show that there must be at least one regular element of U in every row and every column of (c_{ij}).

Let $u \in U$. Then $e_{11}u \in U_n$ and is of the form $(a_{ij})(c_{ij})^{-1}$, where $a_{ij}, c_{ij} \in D \subset U$. Then $(e_{11}u)(c_{ij})=(a_{ij})$, i.e., $uc_{1j}=a_{1j}$ for each j. Since (c_{ij}) is regular, at least one c_{1k} is regular in U and thus $u=a_{1k}c_{1j}$. We have shown that $U=\mathfrak{Q}_1(D)$.

By the same reasoning it is easy to verify that if $S' \subset R \subset T$, then R is also a complete matrix ring.

We now give an example illustrating the matrix rings of Theorem 6 and Prop. 7. Notice in this example that the ring S actually contains a total matrix ring whose classical quotient ring is $\mathfrak{Q}_1(S)$.

8. Example. Let Z be the ring of integers and R the ring of rational numbers. Let

$$S=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \,\middle|\, \begin{array}{l} a, b, c, d \in Z \text{ with } b, c \text{ both even and} \\ \text{with } a, d \text{ both odd or both even}\end{array}\right\}.$$

Then $\mathfrak{Q}_1(S) \subset R_2 \equiv T$ In this case we have $(Z_4)_2 \subset S \subset D_2 \subset R_2$, where $Z_4=\{4z \mid z \in Z\}$ and $D=\{z/2^k \mid z, k \in Z\}$. Notice also that D_2 is the ring generated by S and the inverse of $\begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix}$.

One of the questions raised by the preceding result is whether the result holds for other types of quotient rings. After defining the complete ring of quotients of a ring, we give an example which shows that the result does not hold in a case T is just the complete ring of quotients of S, even though T is a matrix ring over a field.

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9. DEFINITION. Let $\hat{S}$ be the injective hull of the ring S considered as a right S-module. Let $H = \operatorname{Hom}_S(\hat{S}, \hat{S})$ be the ring of endomorphisms of $\hat{S}$, where we write the endomorphisms on the left of their arguments. Let $Q^*(S) = \operatorname{Hom}_H(\hat{S}, \hat{S})$ be the ring of H-endomorphisms of $\hat{S}$, where the endomorphisms are written on the right. We call $Q^*(S)$ the complete ring of quotients of S.

10. EXAMPLE. Let Z denote the ring of integers and R the ring of rational numbers. Let

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} a, b, c, d \in Z \\ a+c = b+d \end{matrix} \right\}.$$

Then $T \equiv R_2$ is the complete ring of quotients of S but is not a classical quotient ring of S. Further, $S[c^{-1}]$ is not a total matrix ring for any regular element $c \in S$. This follows since all elements (a_{ij}) of $S[c^{-1}]$ must satisfy the same property as the elements of S, that is, $a_{11} + a_{21} = a_{12} + a_{22}$.

We now extend the results of Theorem 6 and Proposition 7 to QFX rings of more than one block.

11. PROPOSITION. *Let $\mathfrak{Q}(S) = T = \coprod_{i=1}^{n} H_i$, where the H_i are two-sided ideals of T. Then S contains a direct sum $R = \coprod_{i=1}^{n} R_i$ of two-sided ideals R_i of S such that $H_i = \mathfrak{Q}(R_i)$ and $T = \mathfrak{Q}(R)$.*

Proof. Let $1 = \sum_{i=1}^{n} e_i$ be a representation of $1 \in T$ in terms of central idempotents in T, where $e_i \in H_i$ and $H_i = e_i T$. Let $e_i = a_i c_i^{-1}$ and define $R_i = e_i S \cap S$. Then $H_i = \mathfrak{Q}(R_i)$. To show this, let $e_i q \in H_i$, where $q = bd^{-1}$. Then $e_i q = e_i b d^{-1} = e_i b e_i d^{-1} = (e_i c_i \bar{b} e_i c_i)(e_i d \bar{c} c_i)^{-1}$, where the last inverse is in $e_i H$ and $c_i^{-1} b = b c^{-1}$. Now $e_i c_i \bar{b} e_i c_i$ and $e_i d \bar{c} c_i = d \bar{c} e_i c_i$ are in R_i. It is easily verified that every regular element of R_i has an inverse in H_i. Thus $H_i = \mathfrak{Q}(R_i)$. Obviously, $T = \mathfrak{Q}(R)$, where $R = \coprod R_i$.

12. THEOREM. *Let $T = QFX\mathfrak{Q}(S)$, where T has a decomposition into m blocks. Then there exist a regular element $c \in S$ and rings C^i, D^i and U^i, $i = 1, 2, \ldots, m$, such that U^i is local and $\coprod (C^i)_{n_i} \subsetneq Sc \coprod (D^i)_{n_i} = S[c^{-1}] \subset \coprod (U^i)_{n_i} = T$, where $U^i = QF\mathfrak{Q}_1(D^i) = \mathfrak{Q}(C^i)$.*

Proof. By Prop. 5, T is a direct sum of a finite number of two-sided ideals H_i, each of which is a total matrix ring over a local ring U_i. Thus $T = \coprod (U^i)_{n_i}$. Let $H_i = e_i T$, where e_i is a central idempotent. Since T is QF and each H_i has an identity, it is easy to verify that H_i is QF, and hence U^i is QF. By Prop. 11, there exist rings $R_i = e_i S \cap S$ such that $H_i = \mathfrak{Q}(R_i)$. Applying Prop. 3 we find rings C^i such that $(C^i)_{n_i} \subsetneq R_i$ where $U^i = \mathfrak{Q}(C^i)$. Now let d_{ij}^k, $k = 1, 2, \ldots, m$, $i, j = 1, 2, \ldots, n_k$, be the element of $(U^k)_{n_k}$ with the identity of U^i in the (i, j) position and 0's elsewhere. Since $(U^k)_{n_k} = \mathfrak{Q}(C^k)_{n_k}$, there exists a regular element c_k in (C^k) such that $d_{ij}^k c_k$ and $e_k c_k \in (C^k)_{n_k}$. Now $c = \sum c_k$ is regular in S. Let $S' = S[c^{-1}]$, the subring of T generated by S and c^{-1}. By Prop. 11, $e_k T = \mathfrak{Q}(e_i S'\ S')$. Now

https://doi.org/10.4153/CMB-1975-040-4 Published online by Cambridge University Press

$S' = \coprod (e_k S' \cap S')$ since d_{ij}^k and $e_k \in e_k S' \cap S'$. By reasoning similar to that of the proof of Prop. 7 we find that $e_k S' \cap S'$ is a matrix ring $(D^k)_{n_k}$, where $U^k = \mathfrak{Q}_1(D_k)$. Thus $S' = \coprod (D^k)_{n_i}$ and the proof is complete.

ACKNOWLEDGEMENT. This paper contains a portion of the author's dissertation at the University of Wisconsin–Milwaukee, where the research was conducted under the direction of Professor Edmund H. Feller. The manuscript was prepared while the author was a recipient of a National Science Foundation Research Participation Award at the University of Oklahoma.

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