Glasgow Math. J. **42** (2000) 255–261. © Glasgow Mathematical Journal Trust 2000. Printed in the United Kingdom

ON THE EQUATION $x(x+d)\dots(x+(k-1)d) = by^2$

B. BRINDZA,* L. HAJDU**

Institute of Mathematics and Informatics, Lajos Kossuth University, H-4010 Debrecen, PO Box 12, Hungary

and I. Z. RUZSA***

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, PO Box 127, H-1364 Budapest, Hungary

(Received 11 August, 1998)

Abstract. In this paper we give a new bound for the solutions x of the title equation, provided that $k \geq 8$. This bound is polynomial in d. Moreover, under the same condition, a similar bound for the number of solutions in (x, k, y, l) is given.

1991 *Mathematics Subject Classification*. 11D41.

1. Introduction. It is an old problem to describe those finite arithmetic progressions for which the product of the terms yields a perfect power or an 'almost' perfect power. The first result in this direction is due to Erdős [**4**] and Rigge [**12**]. They independently proved that the equation

$$x(x+1)\dots(x+k-1) = y^l \tag{I}$$

for $l = 2$ has no solutions with $k \geq 2$ and $x \geq 1$; that is the product of two or more consecutive positive integers is never a perfect square. Later, Erdős and Selfridge in 1975 (cf. [**6**]) obtained a deep generalization of this: namely that equation (I) for $l \geq 2$ has no solutions with $k \geq 2$ and $x > 1$ or, in other words, the product of two or more consecutive positive integers is never a perfect power.

Another, closely related problem is to determine those binomial coefficients which are perfect powers. This problem was studied by Erdős (see [**5**]). He showed that the equation

$$\binom{x+k-1}{k} = y^l \tag{II}$$

in positive integers x, k, y, l with $x \geq k+1$, $l \geq 2$ has no solutions if $k \geq 4$. Győry [**8**] proved that the only solution of equation (II) with $k \geq 2, (l, k) \neq (2, 2)$ is $(x, k, y, l) = (48,\ 3,\ 140,\ 2)$. He settled the case $k = 3$ and pointed out that the case $k = 2$ is a consequence of a recent result of Darmon and Merel [**3**]. $k = l = 2$ must be excluded because in this case equation (II) clearly has infinitely many solutions.

*Research supported in part by the Hungarian Academy of Sciences and by Grant D 23992 from the Hungarian National Foundation for Scientific Research.

**Research supported in part by the Hungarian Academy of Sciences, by Grants 023800 and T 016 975 from the Hungarian National Foundation for Scientific Research, by the Pro Regione Foundation of the Hajdúsági Agráripari Rt and by the Universitas Foundation of the Kereskedelmi Bank Rt.

***Research supported in part by Grants 17433 and 25617 from the Hungarian National Foundation for Scientific Research.

For a positive integer n denote by $P(n)$ the greatest prime divisor of n. (We set $P(1)=1$.) A common generalization of equations (I) and (II) is

$$x(x+1)\dots(x+k-1) = by^l \text{ with } l, k \geq 2, P(b) \leq k. \tag{III}$$

Using a result of Sylvester [23], it turns out that if $P(y) \leq k$ for a given k, then (III) has only 'small' solutions, and these solutions can be easily determined (cf. [9]). On the other hand, if $P(y) > k$ then, apart from the case $k = l = 2$, equation (III) has only the solution $(x, k, b, y, l) = (48, 3, 6, 140, 2)$. This result is due to Győry [9] (case $k \leq 3$) and Saradha [14] (case $k \geq 4$).

For further related results on the previous equations and on the more general equation

$$x(x+d)\dots(x+(k-1)d) = by^l \text{ with } l \geq 2, k \geq 3, (x, d) = 1, P(b) \leq k \tag{IV}$$

in positive integers x, d, k, b, y, l, we refer to the works of Beukers, Shorey and Tijdeman [2], Győry [10], Marszalek [11], Saradha [14] and [15], Shorey [16] and [17], Shorey and Nesterenko [18] and [19], Shorey and Tijdeman [20], [21] and [22], and Tijdeman [24] and [25], and the references given there.

In the present paper we deal with the equation

$$x(x+d)\dots(x+(k-1)d) = by^2 \text{ with } k \geq 3, (x, d) = 1, P(b) \leq k, \tag{1}$$

where x, d, k, b, y are unknown. If $d = 1$, then suppose that $P(y) > k$ in (1). Among several other inequalities, Shorey and Tijdeman [20] proved that if $k \geq C_1$, where C_1 is an effectively computable absolute constant, then (1) implies that

$$x + (k-1)d \leq 17d^2k(\log k)^4;$$

see [20, Theorem 3]. Combining this result with the inequality

$$2^{\omega(d)} > C_2 \frac{k}{\log k}$$

(cf. [20, Theorem 1.(a)]), where C_2 is an effective computable absolute positive constant and $\omega(d)$ is the number of distinct prime divisors of d, one can see that x is bounded by $c_\varepsilon d^{2+\varepsilon}$, where c_ε is a constant depending only on ε, provided that k is large enough.

2. Results.

THEOREM 1. *If* (x, d, k, b, y) *is a solution of* (1) *with* $k \geq 8$ *and* $d > 1$, *then* $x < 4d^4(\log d)^4$.

REMARKS. 1. We may assume that $d \geq 23$, as (1) in the case $1 < d \leq 22$ was solved by Saradha [15], and the only solutions are $(x, d, k, b, y) = (2, 7, 3, 2, 12)$, $(18, 7, 3, 1, 120)$ and $(64, 17, 3, 2, 504)$.

2. Erdős conjectured that relation (IV) implies that k is bounded by an absolute constant. In the case $l = 2$, under the further assumption $x \geq 4d^4(\log d)^4$, our Theorem

gives an answer to this problem, with a good bound for k. For other results and references concerning this and related conjectures of Erdős see [**16**], [**17**] and [**10**].

3. Our argument makes it possible to assume only that $P(b) \le ck$, where c is "around" 1.1. The details are not worked out here.

4. Our theorem provides a method for determining all solutions of equation (1), with d fixed. The 'small' solutions can be found for example by a simple search, and 'large' solutions may occur only when $k \le 7$. However, in this case equation (1) can be reduced either to simultaneous Pell's equations, or to simple elliptic equations of the form $x'^3 + a'x' + b' = y'^2$, and these equations can be resolved easily. Using this approach, Filakovszky and Hajdu [**7**] resolved equation (1) for several values of d.

A combination of Theorem 1 with some recent results of Bennett [**1**] and Saradha [**15**] yields the following result.

Theorem 2. *Suppose that* $d > 1$. *Then equation* (1) *has at most* cd^6 *solutions in* (x, k, b, y)*, where c is an effectively computable absolute constant.*

3. Auxiliary results. Every factor of the left hand side of (1) can be uniquely written in the form

$$x + id = a_i x_i^2, \quad a_i \text{ is square-free}, \quad i = 0, \dots, k-1. \tag{2}$$

Lemma 1. *For every solution* (x, d, k, b, y) *of* (1) *for which the* a_i*'s in* (2) *are all different, we have* $k \le 7$.

Proof. First we prove that under the assumptions of the Lemma, $k < 210$ holds. Let p be a prime with $p \nmid d$, and write

$$A = \prod_{i=0}^{k-1} a_i, \quad B = \prod_{p \le k} p^{\lceil k/p \rceil}.$$

If $p > k$ then, for every $i = 0, \dots, k-1$, $p \nmid a_i$. On the other hand, if $p \le k$, then only every p-th number $x + id$ can be divisible by p; thus p has at most $\lceil k/p \rceil$ multiples among them; hence $A|B$. For every $i = 0, \dots, k-1$ let a_i' be the odd part a_i, and put

$$A' = \prod_{i=0}^{k-1} a_i', \quad B' = \prod_{3 \le p \le k} p^{\lceil k/p \rceil}.$$

Then we get $A'|B'$. Let $1 = h_1 < h_2 < \dots$ be the sequence of the odd square-free numbers. Since every h_j may occur as a_i' at most twice, we obtain

$$A' \ge h_1 h_2 \dots h_m h_1 h_2 \dots h_{m'},$$

where $m = [k/2]$ and $m' = k - m = [k/2]$.

Set $H(x) = \#\{i : h_i \le x\}$, and let $F(x) = [x] - [x/2]$; that is $F(x)$ is the number of the positive odd integers not greater than x. By a sieve formula we have

$$H(x) = \sum_{2 \nmid t} \mu(t) F(x/t^2) \le \sum_{t|15} \mu(t) F(x/t^2).$$

https://doi.org/10.1017/S0017089500020115 Published online by Cambridge University Press

Indeed, the right hand side expression enumerates those odd integers not exceeding x that are free of 3^2 and 5^2. The inequality $|F(x) - x/2| \leq 1/2, (x \geq 0)$ yields

$$\left|\sum_{t|15} \mu(t)F(x/t^2) - (32/75)x\right| \leq 2,$$

whence

$$H(x) \leq (32/75)x + 2.$$

In particular

$$H(h_j) = j \leq (32/75)h_j + 2,$$

and we get

$$h_j \geq (75/32)(j - 2).$$

Using this estimate for $j \geq 4$ together with $h_1 = 1$, $h_2 = 3$ and $h_3 = 5$, we obtain

$$h_1 \dots h_m \geq (75/32)^m (m-2)!.$$

A similar estimate is valid for $h_1 \dots h_{m'}$. By multiplying these inequalities, we have

$$A' > (75/32)^{m+m'} + (m-2)!(m'-2)!.$$

Using

$$(m-2)!(m'-2)! = (k-4)!\binom{k-4}{m-2}^{-1} > (k-4)!2^{-(k-4)} > (16/k^4)2^{-k}k!,$$

we conclude that

$$A' > (16/k^4)(75/64)^k k!.$$

Now we estimate B'. By Legendre's formula for the prime factorization of $k!$

$$\sum_{p \geq 3} [k/p] \log p = \log k! - S, \tag{3}$$

where

$$S = \sum_{p^t \in Q} [k/p^t] \log p \text{ with } Q = \{2\} \cup \{p^j : j \geq 2\}.$$

Let $Q_0 = Q \cap [1, 500]$. Then

$$S \geq \sum_{p^t \in Q_0} [k/p^t] \log p \geq k \sum_{p^t \in Q_0} \log p/p^t - \sum_{p^t \in Q_0} \log p \geq \alpha k - \beta, \tag{4}$$

where $\alpha = 1.046874$ and $\beta = 27.8$. The identity

$$\lceil k/p \rceil = 1 + [(k-1)/p]$$

implies that

$$B' \leq \prod_{p \leq k} p \prod p^{[(k-1)/p]}.$$

https://doi.org/10.1017/S0017089500020115 Published online by Cambridge University Press

For every k we have $\prod_{p\le k} p < e^{\gamma k}$, with $\gamma = 1.001102$ (cf. [**13**, Theorem 6]). Using this together with (3) and (4), we obtain

$$B' < e^{\beta-\alpha(k-1)+\gamma k}(k-1)!.$$

This inequality with $A' \le B'$ leads to

$$(16/k^4)(75/64)^k k! < e^{(\gamma-\alpha)k+\alpha+\beta}(k-1)!.$$

and, by a simple computation, we obtain $k < 210$.

Now suppose that $9 \le k \le 209$, and let $p \nmid d$ be a prime. It is clear that among the numbers $x+id$, $i=0,\dots,k-1$, there are at most

$$r(p) := \sum_{t=1}^{\left[\frac{\log k}{2\log p}\right]+1} (\lceil k/p^{2t-1}\rceil - [k/p^{2t}]),$$

ones, whose factorizations contain p on an odd exponent. However, after fixing k and calculating the values of $r(p)$, it turns out that it is impossible to construct k different a_i's from these primes. Indeed, let k be any integer satisfying $9 \le k \le 209$, and determine the value of $r(p)$ for every prime $p \nmid d$ not exceeding k. Among the primes involved, choose the smallest one p', for which $r(p') < 2^{\pi(p')-1}$ holds, where $\pi(x)$ denotes the number of positive primes not greater than x. Clearly, we can construct at most

$$2^{\pi(p')-1} + \sum_{p'\le p\le k} r(p)$$

different a_i's from our primes. However, a computation yields that the number of these a_i's is always less than k in this case, and we obtain $k \le 8$.

Now suppose that $k = 8$. In this case one can construct eight different a_i's. However, as it was pointed out by Professor Tijdeman, it is impossible to arrange them into a 'valid' order. This fact can be proved by a simple combinatorial argument. Hence the Lemma is proved. □

Note. As it was pointed out by N. Saradha, the assertion of Lemma 1 could also be derived by combining some formulas and arguments of [**15**].

We remark that the bound $k < 210$ obtained in the first part of the proof of the Lemma could be made sharper, but in view of the second part of the proof, it was not necessary.

Lemma 2. *Equation* (1) *with* $d > 1$ *implies that* $k < 4d(\log d)^2$.

Proof. The case $d \ge 23$ is just Theorem 3 in [**15**]. Furthermore, in [**15**] all the solutions of (1) with $1 < d \le 22$ are determined, and the Lemma follows. □

Lemma 3. *Let* a *and* b *be positive nonsquare integers and let* u, v *be nonzero integers with* $av \ne bu$. *Then the system of simultaneous Pell-equations*

https://doi.org/10.1017/S0017089500020115 Published online by Cambridge University Press

$$x^2 - az^2 = u, \quad y^2 - bz^2 = v$$

in positive integers (x, y, z) *has at most* $c2^{\min\{\omega(u),\omega(v)\}}\log(|u| + |v|)$ *solutions, where* c *is an effectively computable absolute constant and* $\omega(n)$ *denotes the number of the distinct prime factors of* n.

Proof of Lemma 3. This is the main result in [**1**]. □

4. Proofs of the theorems.

Proof of Theorem 1. Let (x, d, k, b, y) be any solution of (1) with $k \geq 8$. As mentioned above, we may suppose that $d \geq 23$. Using Lemma 1 we obtain that there exist $0 \leq j < i \leq k-1$ such that in (2) $a_i = a_j$. This yields

$$(k-1)d \geq (x + id) - (x + jd) = a_j(x_i^2 - x_j^2) \geq a_j(2x_j + 1),$$

and therefore

$$(k-1)^2 d^2/4 > x.$$

The last estimate together with Lemma 2 implies that

$$x < 4d^4(\log d)^4,$$

and the Theorem is proved. □

Proof of Theorem 2. First suppose that in (1) $k \geq 8$ holds. Then, by Theorem 1, we have $x < 4d^4(\log d)^4$. Furthermore, by Lemma 2, we have $k < 4d(\log d)^2$, and in this case Theorem 2 follows.

Now suppose that in (1) $k \leq 7$ holds. Using (2), for $i = 0, 1, 2$ we can write $x = a_0x_0^2$, $x + d = a_1x_1^2$, $x + 2d = a_2x_2^2$, and we obtain the simultaneous Pell-equations

$$a_1x_1^2 - a_0x_0^2 = d, \quad a_2x_2^2 - a_0x_0^2 = 2d. \tag{5}$$

Clearly, for the coefficients $a_i, i = 0, 1, 2$ we have $a_i | 2 \cdot 3 \cdot 5 \cdot 7$. Hence we have to consider at most 2^{12} simultaneous Pell-equations of the form (5). Thus to prove Theorem 2, it is sufficent to give an appropriate upper bound for the number of solutions of (5). Multiplying the equations of (5) by a_1 and a_2, respectively, and putting $y_0 = x_0, y_1 = a_1x_1$ and $y_2 = a_2x_2$, we obtain the system of equations

$$y_1^2 - a_0a_1y_0^2 = a_1d, \quad y_2^2 - a_0a_2y_0^2 = 2a_2d. \tag{6}$$

We may suppose that $a_0 \neq a_1$ and $a_0 \neq a_2$. Indeed, otherwise just as in the proof of Theorem 1, we obtain $x < 4d^4(\log d)^4$, which yields a much better bound for the number of solutions of (1) than stated. Now one can check easily that the assumptions of Lemma 3 are fulfilled, and we obtain that the number of solutions in (y_1, y_2, y_3) to (6) is less than $c2^{\omega(d)}$, where c is an effectively computable absolute constant, and the Theorem is proved. □

ACKNOWLEDGEMENTS. The authors are grateful to Professors K. Győry, J. Pintz, R. Tijdeman and the referee for their valuable remarks.

REFERENCES

1. M. A. Bennett, On the number of solutions of simultaneous Pell equations, *J. Reine Angew. Math.* **498** (1998), 173–199.

2. F. Beukers, T. N. Shorey and R. Tijdeman, Irreducibility of polynomials and arithmetic progressions with equal products of terms, in *Number Theory* (K. Győry, H. Iwaniec and J. Urbanowicz, eds.) (Walter de Gruyter, Berlin–New York), 11–26, 1999.

3. H. Darmon and L. Merel, Winding quotients and some variants of Fermat's Last Theorem, *J. Reine Angew. Math.* **490** (1997), 81–100.

4. P. Erdős, Note on products of consecutive integers, *J. London Math. Soc.* **14** (1939), 194–198.

5. P. Erdős, On a diophantine equation, *J. London Math. Soc.* **26** (1951), 176–178.

6. P. Erdős and J. L. Selfridge, Note on products of consecutive integers, *Illinois J. Math.* **19** (1975), 292–301.

7. P. Filakovszky and L. Hajdu, The resolution of the equation $x(x+d)\dots(x+(k-1)d)=by^2$ for fixed d, to appear.

8. K. Győry, On the diophantine equation $\binom{n}{k}=x^l$, *Acta Arith.* **80** (1997), 289–295.

9. K. Győry, On the diophantine equation $n(n+1)\dots(n+k-1)=bx^l$, *Acta Arith.* **83** (1998), 87–92.

10. K. Győry, Power values of products of consecutive integers and binomial coefficients, *Number Theory and its Applications* (Kluwer Acad. Publ., 1999), 145–156.

11. R. Marszalek, On the product of consecutive elements of an arithmetic progression, *Monatsh. Math.* **100** (1985), 215–222.

12. O. Rigge, Über ein diophantisches Problem in *9th Congress Math. Scand., Helsingfors 1938*. (Mercator, 1939), 155–160.

13. J. B. Rosser and L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$, *Math. Comp.* **29** (1975), 243–269.

14. N. Saradha, On perfect powers in products with terms from arithmetic progressions, *Acta Arith.* **82** (1997), 147–172.

15. N. Saradha, Squares in products with terms in an arithmetic progressions, *Acta Arith.* **86** (1998), 27–43.

16. T. N. Shorey, Perfect powers in products of arithmetical progressions with fixed initial term, *Indag. Math. N.S.,* **7** (1996), 521–525.

17. T. N. Shorey, Exponential diophantine equations involving products of consecutive integers and related equations, to appear.

18. T. N. Shorey and Yu. Nesterenko, Perfect powers in products of integers from a block of consecutive integers, *Acta Arith.* **49** (1987), 71–79.

19. T. N. Shorey and Yu. Nesterenko, Perfect powers in products of integers from a block of consecutive integers II, *Acta Arith.* **76** (1996), 191–198.

20. T. N. Shorey and R. Tijdeman, Perfect powers in product of terms in an arithmetical progression, *Compositio Math.* **75** (1990), 307–344.

21. T. N. Shorey and R. Tijdeman, Perfect powers in product of terms in an arithmetical progression II, *Compositio Math.* **82** (1992), 119–136.

22. T. N. Shorey and R. Tijdeman, Perfect powers in product of terms in an arithmetical progression III, *Acta Arith.* **61** (1992), 391–398.

23. J. J. Sylvester, On arithmetic series, *Messenger Math.* **21** (1892), 1–19 and 87–120.

24. R. Tijdeman, Diophantine equations and diophantine approximations, in *Number Theory and Applications* (R. A. Mollin, ed.) (Kluwer Acad. Publ., 1989), 215–243.

25. R. Tijdeman, Exponential diophantine equations 1986–1996 in *Number Theory: Diophantine, Computational and Algebraic Aspects* (K. Győry, A. Pethő and V. T. Sós, eds.), (Walter de Gruyter, Berlin–New York, 1998), 523–540.