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A NOTE ON BLASCHKE PRODUCTS WITH ZEROES IN A NONTANGENTIAL REGION

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ABSTRACT. We show that if B is a Blaschke product with nontangential zero set $\{z_k\}$ and $0 < p < 1$, $1/2 < \alpha p < 1$, then the condition $\sup_{0<r<1}(1 - r)\, M_p(r, D^{1+\alpha}B) < \infty$ is equivalent to the condition $\{ (1 - |z_k|)^{(1/p)-\alpha} k^{\alpha}\} \in \ell^{\infty}$.

1. **Introduction.** Let f be holomorphic in the open unit disc U (abbreviated $f \in H(U)$). For any p, $0 < p \leqq \infty$, we define

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \quad 0 \leqq r < 1, 0 < p < \infty$$

$$M_{\infty}(r, f) = \sup_{\theta} |f(re^{i\theta})|, \quad 0 \leqq r < 1.$$

The Hardy space H^p, $0 < p \leqq \infty$ consists of all functions $f \in H(U)$ for which

$$\|f\|_p = \sup_{0 \leqq r < 1} M_p(r, f)$$

is finite.

If $f(z) = \sum a_k z^k$ is holomorphic in U and $\alpha > 0$, following Flett ([4]), we define the fractional derivative of order α by

$$(D^{\alpha}f)(z) = \sum (k + 1)^{\alpha} a_k z^k$$

If $0 < p < \infty$ and $\alpha > 0$, then a function $f \in H(U)$ is said to belong to the space $\Lambda^{p,\alpha}$ if

$$\|f\|_{p,\alpha} = \sup_{0<r<1} (1 - r) M_p(r, D^{1+\alpha}f) < \infty.$$

If $\{z_k\}$ is a sequence of complex numbers in U for which $\sum(1 - |z_k|) < \infty$, then the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}$$

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converges uniformly on compact subsets of U and has $\{z_k\}$ as its zero set. See [3].

In this note we deal with Blaschke products whose zeroes $\{z_k\}$ lie in a fixed nontangential region $G = \{z \in U: |1 - z| < C(1 - |z|)\}$ for some $C > 1$. The class of all such infinite Blaschke products will be denoted by $\mathscr{B}$.

In [6], Theorem 3, J. Verbitski proved that if $B \in \mathscr{B}$ with zeroes $\{z_k\}$ then the condition $B \in \Lambda^{p,\alpha}$ is equivalent to the condition $\{(1 - |z_k|)^{(1/p)-\alpha}k^{\alpha}\} \in \ell^{\infty}$, under the assumptions $1 \leqq p < \infty$, $1/2 < \alpha p < 1$. In this note we extend this result to the case $0 < p < 1$.

In [7] J. Verbitski showed that if $1 \leqq p < \infty$ then $\mathscr{B} \subset \Lambda^{p,1/2p}$. We improve this by showing that

$$H^{\infty} \cap \Lambda^{p,1/2p} \subset H^{\infty} \cap \Lambda^{q,1/2q}, \quad \text{for all } p \leqq q,$$

and $\mathscr{B} \subset \Lambda^{p,1/2p}$ for all $0 < p < \infty$.

2. As stated in the Introduction we want to extend the results of [6] and [7] to the case $0 < p < 1$. Here and elsewhere the quantity $1 - |z_k|$ will be denoted by d_k.

THEOREM 2.1. *Let* $B \in \mathscr{B}$, *with zeroes* $\{z_k\}$. *If* $0 < p < 1$ *and* $1/2 < \alpha p < 1$ *then* $\{d_k^{(1/p)-\alpha}k^{\alpha}\} \in \ell^{\infty}$ *if and only if* $B \in \Lambda^{p,\alpha}$.

PROOF. Let n be the positive integer such that $1/(n+1) \leqq p < 1/(n)$. Note that $\alpha < 1/p \leqq 1/(1 - np)$ and hence the condition $M_p(r, D^{1+\alpha}B) = 0((1 - r)^{-1})$ is equivalent to the condition $M_p(r, D^{1+\alpha np}B) = 0((1 - r)^{-1-\alpha np+\alpha})$ ([4], Theorem 6). Since B is bounded.

$$|(D^{1+\alpha np}B)(z)| \leqq C(1 - |z|)^{-1-\alpha np}.$$

Thus,

$$\begin{aligned} M_{1/n}(r, D^{1+\alpha np}B) &= \left(\frac{1}{2\pi}\int_0^{2\pi} |(D^{1+\alpha np}B)(re^{it})|^{1/n}dt\right)^n \\ &\leqq C(1 - r)^{-(1+\alpha np)(1-np)}M_p^{np}(r, D^{1+\alpha np}B) \\ &\leqq C(1 - r)^{-1} \end{aligned}$$

Then the successive application of the above argument proves that

$$M_1(r, D^{1+\alpha p}B) = 0((1 - r)^{-1}).$$

By a result of J. Verbitski cited in the introduction

$$d_k = 0(k^{\alpha p/(\alpha p-1)}).$$

To prove the converse, we may suppose that

$$d_k \leqq k^{\alpha p/(\alpha p-1)}, \quad \text{for } k = 1, 2, \ldots .$$

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Let $n \geqq 2$ be the positive integer such that $1/n < p \leqq 1/(n-1)$. An easy calculation shows that

$$M_p(r, D^n B) \leqq CM_p(r, B^{(n)}).$$

In [4] it is proved that the condition

$$M_p(r, D^n B) = 0((1-r)^{\alpha - n})$$

is equivalent to

$$M_p(r, D^{1+\alpha} B) = 0((1-r)^{-1})$$

(Note that $\alpha < 1/p < n$). So the theorem will be proved if we show that

$$M_p(r, B^{(n)})^p = 0((1-r)^{\alpha p - np}).$$

Since B is bounded,

$$|B^{(n)}(z)| = 0((1-|z|)^{-n}). \tag{2.1}$$

From Lemma 3.4 of [5] and Lemma 3 of [1] (see also [2]) it follows that

$$|B^{(n)}(z)| \leqq C \sum \prod_{j=2}^{n+1} (g_j(z))^{\alpha_j} \tag{2.2}$$

where the sum is over the (finite) set of all n-tuples $(\alpha_2, \ldots, \alpha_{n+1})$ of non-negative integers such that

$$\sum_{j=1}^{n} j\alpha_{j+1} = n$$

and

$$g_j(z) = \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{|1-|z_k|z||^j}.$$

(Here we have used the fact that $\{z_k\} \subset G$.) Using (2.1) and (2.2) we find that

$$\begin{aligned} M_p^p(r, B^{(n)}) &\leqq C \int_0^{2\pi} \min\left\{(1-r)^{-np}, \sum \prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p}\right\} dt \\ &\leqq C \int_{|t| \leqq (1-r)^{\alpha p}} (1-r)^{-np} dt \\ &\quad + C \int_{|t| > (1-r)^{\alpha p}} \left(\sum \prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p}\right) dt \\ &\leqq C(1-r)^{\alpha p - np} + C \sum \int_{|t| > (1-r)^{\alpha \cdot p}} \left(\prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p}\right) dt. \end{aligned} \tag{2.3}$$

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We will show that each term of the finite sum on the right hand side of (2.3) is also $0((1-r)^{\alpha p-np})$. For $j=2,3,\ldots,n+1$ we let $\beta_j=n/(j-1)\alpha_j$. Then $\sum_{j=2}^{n+1} 1/\beta_j=1$ so it follows from Holder's inequality that

$$\int_{|t|>(1-r)^{\alpha p}}\left(\prod_{j=2}^{n+1}(g_j(re^{it}))^{\alpha_j p}\right)dt \leqq \prod_{j=2}^{n+1}\left(\int_{|t|>(1-r)^{\alpha p}}(g_j(re^{it}))^{\alpha_j\beta_j p}dt\right)^{1/\beta_j}. \tag{2.4}$$

Next we estimate the factors of the product on the right hand side of (2.4).

$$\begin{aligned}(2.5)\quad &\left(\int_{|t|>(1-r)^{\alpha p}}(g_j(re^{it}))^{\alpha_j\beta_j p}dt\right)^{1/\beta_j}\\ &\leqq C\left(\int_{|t|>(1-r)^{\alpha p}}\left(\sum_{k=1}^{\infty}\frac{d_k}{(d_k^2+t^2)^{j/2}}\right)^{\alpha_j\beta_j p}dt\right)^{1/\beta_j}\\ &\leqq C\left(\int_{t>(1-r)^{\alpha p}}\left(\sum_{E_j(t)}\frac{d_k}{(d_k^2+t^2)^{j/2}}+\sum_{E_j'(t)}\frac{d_k}{(d_k^2+t^2)^{j/2}}\right)^{\alpha_j\beta_j p}dt\right)^{1/\beta_j}\end{aligned}$$

where

$$E_j(t)=\{k\colon k \text{ is an integer}, \sqrt{j-1}\leqq tk^{\alpha p/(1-\alpha p)}\}$$

and

$$E_j'(t)=\{k\colon k \text{ is an integer}, \sqrt{j-1}> tk^{\alpha p/(1-\alpha p)}\}.$$

To deal with the sum over $E_j(t)$ we note that the function $f_j(x)=x/(x^2+t^2)^{j/2}$ is increasing in the interval $[0, t/\sqrt{j-1}]$.

$$\begin{aligned}(2.6)\quad \sum_{E_j(t)}\frac{d_k}{(d_k^2+t^2)^{j/2}} &\leqq \sum_{E_j(t)}k^{\alpha p/(\alpha p-1)}(k^{2\alpha p/(\alpha p-1)}+t^2)^{-j/2}\leqq t^{-j}\sum_{E_j(t)}k^{\alpha p/(\alpha p-1)}\\ &=t^{-j}\sum_{k=n_j}^{\infty}k^{\alpha p/(\alpha p-1)}\leqq Ct^{-j}n_j^{(1-2\alpha p)/(1-\alpha p)},\end{aligned}$$

where $n_j=\min E_j(t)$. (Here we have used the fact that $\alpha p>1/2$). From $\sqrt{j-1}\leqq tn_j^{\alpha p/(1-\alpha p)}$ we see that $n_j^{(1-2\alpha p)/(1-\alpha p)}\leqq Ct^{(2\alpha p-1)/\alpha p}$. Combining this with (2.6) we obtain

$$\sum_{E_j(t)}\frac{d_k}{(d_k^2+t^2)^{j/2}}\leqq Ct^{2-j-(1/\alpha p)}. \tag{2.7}$$

Now we deal with the sum over $E_j'(t)$.

$$\begin{aligned}(2.8)\quad \sum_{E_j'(t)}\frac{d_k}{(d_k^2+t^2)^{j/2}} &\leqq \max_{0<x<\infty}f_j(x)\sum_{E_j'(t)}1\\ &\leqq Ct^{1-j}\sum_{E_j'(t)}1\leqq Ct^{2-j-1/(\alpha p)}.\end{aligned}$$

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Now we substitute (2.7) and (2.8) into (2.5) to get

$$(2.9)\qquad \left(\int_{|t|>(1-r)^{\alpha p}} (g_j(re^{it}))^{\alpha_j\beta_j p}\,dt\right)^{1/\beta_j}$$
$$\leqq C\left(\int_{t>(1-r)^{\alpha p}} t^{(2-j-(1/\alpha p))\alpha_j\beta_j p}\,dt\right)^{1/\beta_j}$$
$$\leqq C(1-r)^{\alpha p[(2-j-(1/\alpha p))\alpha_j\beta_j p+1]\beta_j^{-1}}.$$

Using (2.9) and (2.4) we find that

$$\int_{|t|>(1-r)^{\alpha p}} \left(\prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p}\right)dt$$
$$\leqq C(1-r)^{\alpha p\cdot\sum_{j=2}^{n+1}[(2-j-(1/\alpha p))\alpha_j\beta_j p+1]\beta_j^{-1}}$$
$$\leqq C(1-r)^{\alpha p-p\cdot\sum_{j=2}^{n+1}[1+(j-2)\alpha p]\alpha_j}$$
$$\leqq C(1-r)^{\alpha p-p\cdot\sum_{j=2}^{n+1}(j-1)\alpha_j} = C(1-r)^{\alpha p-np}.$$

This finishes the proof of the theorem.

LEMMA 2.2. *If* $0 < p \leqq q < \infty$, *then* $H^\infty \cap \Lambda^{p,1/2p} \subset H^\infty \cap \Lambda^{q,1/2q}$.

PROOF. Suppose $f \in H^\infty \cap \Lambda^{p,1/2p}$. First assume $(1/2p) - (1/2q) < 1$. Since f is bounded,

$$(2.10)\qquad |(D^{1+(1/2q)}f)(z)| \leqq C(1-|z|)^{-1-(1/2q)}.$$

From $f \in \Lambda^{p,1/2p}$ it follows

$$(2.11)\qquad M_p(r, D^{1+(1/2q)}f) = 0((1-r)^{-1+(1/2p)-(1/2q)}) \quad ([4], \text{Theorem } 6).$$

Using (2.10) and (2.11) we find that

$$M_q^q(r, D^{1+(1/2q)}f) \leqq C(1-r)^{(-1-(1/2q))(q-p)}M_p^p(r, D^{1+(1/2q)}f)$$
$$\leqq C(1-r)^{-q},$$

i.e., $f \in \Lambda^{q,1/2q}$. Then the successive application of the above argument proves the lemma.

THEOREM 2.3. $\mathscr{B} \subset \Lambda^{p,1/2p}$, *for all* $0 < p < \infty$.

PROOF. Let $B \in \mathscr{B}$. In view of Lemma 2.2 it is sufficient to show that

$$(2.12)\qquad M_{p_n}(r, D^{1+(1/2p_n)}B) = 0((1-r)^{-1})$$

for a sequence p_n going to zero. We take $p_n = 1/2(n-1)$, $n \geqq 3$.

Then (2.12) becomes $M_{p_n}(r, D^nB) = 0((1-r)^{-1})$. So the theorem will be proved if we show that

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$$M_{p_n}(r, B^{(n)}) = 0((1 - r)^{-1}). \tag{2.13}$$

The proof is similar to that of Theorem 2.1.

$$\begin{aligned} M_{p_n}^{p_n}(r, B^{(n)}) &\leqq C \int_{|t| \leqq \sqrt{1-r}} (1 - r)^{-np_n} dt \\ &+ C \int_{|t| > \sqrt{1-r}} \left(\Sigma \prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p_n} \right) dt \\ &\leqq C(1 - r)^{-p_n} + C \Sigma \int_{|t| > \sqrt{1-r}} \left(\prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p_n} \right) dt. \end{aligned} \tag{2.14}$$

We estimate

$$\begin{aligned} g_j(re^{it}) &\leqq C \sum_{k=1}^{\infty} \frac{d_k}{(d_k^2 + t^2)^{j/2}} \\ &\leqq C\left(t^{-j} \left(\sum_{d_k \leqq t} d_k \right) + \sum_{d_k > t} d_k^{1-j} \right) \leqq Ct^{-j}. \end{aligned}$$

Using this we find that

$$\begin{aligned} &\int_{|t| > \sqrt{1-r}} \left(\prod_{j=2}^{n+1} (g_j(re^{it}))^{\alpha_j p_n} \right) dt \\ &\leqq C \int_{t > \sqrt{1-r}} t^{-(\Sigma_{j=2}^{n+1} j\alpha_j) p_n} dt \\ &\leqq C \int_{t > \sqrt{1-r}} t^{-2np_n} dt \leqq C(1 - r)^{-p_n}. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) we obtain (2.13).

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