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ERRATUM

ERRATUM TO APPENDIX TO '2-ADIC INTEGRAL CANONICAL MODELS'

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The purpose of this erratum is to use the results of [BMS15, BMS18] to fill in a serious conceptual gap in the appendix to [KM16], which contained a purported proof of the Tate conjecture for K3 surfaces in characteristic 2. This gap was pointed out by Koshikawa, to whom I am very grateful. The suggestion that the results of Bhatt–Morrow–Scholze can be used to fill this gap is also due to him. Koshikawa also communicated to me his paper in progress with Ito and Ito [IIK], where a somewhat different fix is given, also using the compatibility of Kisin's functor in [Kis06] with crystalline cohomology. As such, the first complete proof of the Tate conjecture for K3 surfaces in characteristic 2 should be credited to Ito–Ito–Koshikawa.

The gap lies in the proof of [KM16, Proposition A.12]. In the paragraph below (A.12.1), it is claimed that the proof of [Mad16, Lemma 6.16(4)] applies to show that the map ι^{KS} in (A.12.1) factors through the open substack $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$, the so-called 'primitive locus', where the de Rham realization $\boldsymbol{f}_{\mathrm{dR}}$ of the tautological special endomorphism f generates a direct summand of $\boldsymbol{L}^{\diamond}_{\mathrm{dR}}$.

This, however, is not true when $p = 2$. In [Mad16] and [Mad15], the proof of the relevant result uses the smoothness of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ to reduce to showing the following assertion: If k is a perfect field of characteristic $p > 0$, $W = W(k)$ is its ring of Witt vectors, and s is a W-valued point of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$, then the de Rham realization $\boldsymbol{f}_{\mathrm{dR},\iota^{\mathrm{KS}}(s)}$ generates a direct summand of $\boldsymbol{L}^{\diamond}_{\mathrm{dR},\iota^{\mathrm{KS}}(s)}$.

To show this assertion, the proof appeals to [Mad16, Lemma 6.14]. However—and this is the key error—this lemma is misstated there and is *invalid* if $p = 2$ and the abelian scheme $\mathcal{A}_{s_0}$ does not have connected 2-divisible group. More precisely, suppose that $p = 2$ and that $\mathcal{A}$ is an abelian scheme over W with reduction $\mathcal{A}_0$ over k. Then the natural map

$$\mathrm{End}(\mathcal{A}) \otimes \mathbb{Z}_2 \to \mathrm{End}(\mathcal{A}_0) \otimes \mathbb{Z}_2$$

need not have saturated image (unlike when $p > 2$). This happens for instance when $\mathcal{A}_0$ is ordinary, and $\mathcal{A}$ is *not* the canonical lift, but is isogenous to it. (Such lifts do not exist when $p > 2$: they correspond to nontrivial torsion points on the deformation space of $\mathcal{A}_0$ equipped with its Serre–Tate formal group structure, and when $p > 2$, the fields of definition of such points are always ramified.) Therefore, one cannot deduce that the de Rham realization of a special endomorphism of $\mathcal{A}$ is primitive simply by knowing that it generates a direct summand of $\mathrm{End}(\mathcal{A}) \otimes \mathbb{Z}_2$.

To correct this, one must make use of the fact that the quasipolarized K3 surface $(\boldsymbol{\mathcal{X}}_s, \boldsymbol{\xi}_s)$ is *primitive*. When $p > 2$, this point was not directly relevant, except in ensuring that the moduli stack in question is smooth, and one was able to get away somewhat cheaply by appealing to the very general result of [**Mad16**, Lemma 6.14]. When $p = 2$, it appears essential to appeal to the integral p-adic Hodge theory of [**BMS18**] to relate the two notions of primitiveness at a smooth point: that of the polarization of $\boldsymbol{\xi}_s$ and of the special endomorphism $f_{\iota^{\mathrm{KS}}(s)}$.

There is one further issue, also pointed out by Koshikawa: at the end of the proof of [**KM16**, Proposition A.12], an appeal is made to a comparison isomorphism of Bloch and Kato [**BK86**] for ordinary varieties. This is done so that, for every ordinary point s, one obtains a canonical identification of F-crystals

$$\boldsymbol{L}_{\mathrm{dR},\iota^{\mathrm{KS}}(s)}(-1) \xrightarrow{\simeq} \boldsymbol{P}^2_{\mathrm{dR},s},$$

which underlies an identification of *F-isocrystals* obtained from the crystalline comparison isomorphism; see [**Mad15**, Lemma 5.9]

However, for this to be valid, one must also know that the Bloch–Kato isomorphism is compatible with the crystalline comparison isomorphism. While this should be true, it is not obvious or available in the literature. When $p > 2$, this can be alleviated by a direct appeal to the properties of the crystalline comparison isomorphism instead, but it is less clear that is immediately possible when $p = 2$. However, in the course of filling in the first gap, one in any case finds an alternate proof that appeals to the results of [**BMS18**]. This observation is due to Ito–Ito–Koshikawa [**IIK**].

1. A (correct) proof of [KM16, Proposition A.12]

The main thing to be shown is Lemma 1.11 below.

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1.1. We will need to recall the notation and results stated in [KM16, Section 1]. We will take $K = K_0 = W[1/p]$, Γ_K to be the absolute Galois group, $\varpi = p$, and $\mathcal{E}(u) = -u + p \in W[u]$. We then have the category of Breuil–Kisin modules over W (with respect to p), consisting of pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where $\mathfrak{M}$ is a finite-free $\mathfrak{S}$-module equipped with an isomorphism

$$\varphi^*\mathfrak{M}[\mathcal{E}(u)^{-1}] \xrightarrow{\sim} \mathfrak{M}[\mathcal{E}(u)^{-1}].$$

We also have the fully faithful tensor functor $\mathfrak{M}$ of Kisin [Kis10] described in [KM16, Section 1.3], which goes from the category $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}(\mathbb{Z}_p)$ of Γ_K-stable $\mathbb{Z}_p$-lattices in crystalline Γ_K-representations to the category of Breuil–Kisin modules over W (with respect to p). It is not exact, but the induced functor to vector bundles over $\mathrm{Spec}\,\mathfrak{S}\backslash\{(u, p)\}$ *is* exact. That is, for every short exact sequence

$$0 \to L_1 \to L_2 \to L_3 \to 0$$

in $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}(\mathbb{Z}_p)$, the complex

$$0 \to \mathfrak{M}(L_1) \to \mathfrak{M}(L_2) \to \mathfrak{M}(L_3) \to 0$$

has its cohomology supported at the maximal ideal of $\mathfrak{S}$.

1.2. We will actually need a bit more. Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of the localization $\mathfrak{S}_{(p)}$. Let $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$ be the category of pairs $(\mathcal{M}, \varphi_{\mathcal{M}})$ consisting of finitely generated $\mathcal{O}_{\mathcal{E}}$-modules equipped with an isomorphism

$$\varphi_{\mathcal{M}} : \varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}.$$

Let $K_\infty \subset K_0^{\mathrm{alg}}$ be the extension of K_0 generated by a compatible sequence of p-power roots of p, and let $\Gamma_{K_\infty} \subset \Gamma_K$ be the corresponding closed subgroup.

Then it is shown in [Fon90, A.1.2.4] that there is an exact, faithful tensor functor $\mathcal{M}$ from the category of continuous representations of Γ_{K_∞} on finitely generated $\mathbb{Z}_p$-modules to $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$. Moreover, if L belongs to $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}(\mathbb{Z}_p)$, then, by the very construction of $\mathfrak{M}(L)$, we have a canonical identification

$$\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}(L) \xrightarrow{\sim} \mathcal{M}(L|_{\Gamma_{K_\infty}})$$

in $\mathrm{Mod}^{\varphi}_{/\mathcal{O}_{\mathcal{E}}}$; see the proof of [Kis10, Theorem (1.2.1)].

LEMMA 1.3. *Suppose that we have inclusions*

$$\mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$$

of Breuil–Kisin modules, and let

$$\mathscr{M}_1, \mathscr{M}_2 \subset \mathscr{M}$$

be the embeddings in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$ *obtained via change of scalars. Then* $\mathfrak{M}_1 = \mathfrak{M}_2$ *if and only if* $\mathscr{M}_1 = \mathscr{M}_2$.

Proof. This is essentially a special case of [Kis06, Lemma (2.1.12)].

Only one direction requires proof. Assume therefore that $\mathscr{M}_1 = \mathscr{M}_2$. After twisting the Breuil–Kisin module (this amounts to multiplying the φ-semilinear endomorphism by powers of $\mathcal{E}(u)$) $\mathfrak{M}$ if necessary, we can assume that all the objects involved are *effective*, so that for $\mathfrak{N} = \mathfrak{M}, \mathfrak{M}_1, \mathfrak{M}_2$, we have

$$\varphi_{\mathfrak{N}} : \varphi^*\mathfrak{N} \to \mathfrak{N}.$$

Now, set

$$\mathfrak{M}' = (\mathfrak{M}_1 + \mathfrak{M}_2)[1/p] \cap (\mathscr{M}_1 + \mathscr{M}_2) \subset \mathfrak{M}.$$

This is finite-free over $\mathfrak{S}$ (for instance, since $\mathfrak{M}'/p\mathfrak{M}'$ is torsion-free and hence free over $k[\![u]\!]$), and is therefore itself a Breuil–Kisin submodule of $\mathfrak{M}$.

To finish, we appeal to [Kis06, Lemma (2.1.9)], which shows that the inclusions

$$\mathfrak{M}_1 \hookrightarrow \mathfrak{M}'; \quad \mathfrak{M}_2 \hookrightarrow \mathfrak{M}'$$

are both bijections. □

1.4. Let $\bar{s}$ be the K_0^{alg}-valued point induced by s. Denote the induced points of $\mathcal{Z}(2d)_{(p)}$ by s and $\bar{s}$, as well. Then, from the characteristic 0 theory, one finds a canonical isometry of crystalline Γ_K-representations

$$\alpha_{p,\bar{s}} : \boldsymbol{L}_{p,\bar{s}}(-1) \xrightarrow{\simeq} \boldsymbol{P}^2_{p,\bar{s}}.$$

There is also an isometric inclusion

$$\boldsymbol{L}_{p,\bar{s}} \hookrightarrow \boldsymbol{L}^{\diamond}_{p,\bar{s}}.$$

That is, $\boldsymbol{L}_{p,\bar{s}}$ (respectively $\boldsymbol{L}^{\diamond}_{p,\bar{s}}$) is the homological realization associated with a quadratic space L_d (respectively $L^{\diamond}$) over $\mathbb{Z}$, and the above isometric inclusion arises from a choice of isometric embedding $L_d \hookrightarrow L^{\diamond}$.

Let $\boldsymbol{\Lambda}_{p,\bar{s}} \subset \boldsymbol{L}^{\diamond}_{p,\bar{s}}$ be the subspace generated by the p-adic realization of the special endomorphism f_s: it is a direct summand and is trivial as a Γ_K-representation.

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By construction, $\boldsymbol{L}^{\diamond}_{p,\bar{s}}$ is the homological realization associated with the self-dual quadratic space $L^{\diamond}_{(p)}$ over $\mathbb{Z}_{(p)}$, and is thus equipped with a nondegenerate symmetric pairing. The direct summands $\boldsymbol{L}_{p,\bar{s}}$ and $\boldsymbol{\Lambda}_{p,\bar{s}}$ are orthogonal complements to each other under this pairing. Therefore, we obtain a canonical isomorphism of (trivial) Γ_K-modules

$$\lambda : \boldsymbol{\Lambda}^{\vee}_{p,\bar{s}}/\boldsymbol{\Lambda}_{p,\bar{s}} \xleftarrow{\simeq} \boldsymbol{L}^{\diamond}_{p,\bar{s}}/\big(\boldsymbol{L}_{p,\bar{s}} \oplus \boldsymbol{\Lambda}_{p,\bar{s}}\big) \xrightarrow{\simeq} \boldsymbol{L}^{\vee}_{p,\bar{s}}/\boldsymbol{L}_{p,\bar{s}}.$$

This is characterized by the property that an element of $\boldsymbol{L}^{\vee}_{p,\bar{s}} \oplus \boldsymbol{\Lambda}^{\vee}_{p,\bar{s}}$ belongs to $\boldsymbol{L}^{\diamond}_{p,\bar{s}}$ if and only if its image in

$$\boldsymbol{L}^{\vee}_{p,\bar{s}}/\boldsymbol{L}_{p,\bar{s}} \oplus \boldsymbol{\Lambda}^{\vee}_{p,\bar{s}}/\boldsymbol{\Lambda}_{p,\bar{s}}$$

is of the form $(\lambda(x), x)$.

1.5. Let $\boldsymbol{H}^2_{p,\bar{s}}(1)$ be the (twisted) degree 2 p-adic cohomology of the K3 surface $\boldsymbol{\mathcal{X}}_s$: This is also a Galois stable lattice in a crystalline representation, equipped with the Γ_K-submodule $\boldsymbol{\Delta}_{p,\bar{s}} = \langle \mathrm{ch}_p(\boldsymbol{\xi}_s) \rangle$ generated by the Chern class of the quasipolarization $\boldsymbol{\xi}_s$. By definition, we have

$$\boldsymbol{P}^2_{p,\bar{s}}(1) = \boldsymbol{\Delta}^{\perp}_{p,\bar{s}} \subset \boldsymbol{H}^2_{p,\bar{s}},$$

where the orthogonal complement is taken with respect to the Poincaré pairing. Just as above, we now obtain a canonical isomorphism of trivial Γ_K-modules

$$\delta : \boldsymbol{\Delta}^{\vee}_{p,\bar{s}}/\boldsymbol{\Delta}_{p,\bar{s}} \xrightarrow{\simeq} (\boldsymbol{P}^2_{p,\bar{s}}(1))^{\vee}/\boldsymbol{P}^2_{p,\bar{s}}(1) \xrightarrow[\simeq]{\alpha^{-1}_{p,\bar{s}}(1)} \boldsymbol{L}^{\vee}_{p,\bar{s}}/\boldsymbol{L}_{p,\bar{s}}.$$

1.6. Applying the functor $\mathfrak{M}$ to the maps of Galois representations above now gives us maps of Breuil–Kisin modules

$$\mathfrak{M}(\boldsymbol{P}^2_{p,\bar{s}}) \xrightarrow[\simeq]{\mathfrak{M}(\alpha_{p,\bar{s}})^{-1}} \mathfrak{M}(\boldsymbol{L}_{p,\bar{s}}(-1)) \hookrightarrow \mathfrak{M}(\boldsymbol{L}^{\diamond}_{p,\bar{s}}(-1)). \tag{1.6.1}$$

For simplicity, set

$$\mathfrak{L} = \mathfrak{M}(\boldsymbol{L}_{p,\bar{s}}); \quad \mathfrak{L}^{\diamond} = \mathfrak{M}(\boldsymbol{L}^{\diamond}_{p,\bar{s}}); \quad \mathfrak{L}_0 = \mathfrak{M}(\boldsymbol{\Lambda}_{p,\bar{s}});$$
$$\mathfrak{H}_0 = \mathfrak{M}(\boldsymbol{\Delta}_{p,\bar{s}}); \quad \mathfrak{H} = \mathfrak{M}(\boldsymbol{H}_{p,\bar{s}}(1)).$$

Note that, by the triviality of the Γ_K-representations $\boldsymbol{\Lambda}_{p,\bar{s}}$ and $\boldsymbol{\Delta}_{p,\bar{s}}$, we have canonical identifications of Breuil–Kisin modules

$$\mathfrak{L}_0 \xrightarrow{\simeq} \mathbf{1} \otimes_{\mathbb{Z}_p} \boldsymbol{\Lambda}_{p,\bar{s}}; \quad \mathfrak{H}_0 \xrightarrow{\simeq} \mathbf{1} \otimes_{\mathbb{Z}_p} \boldsymbol{\Delta}_{p,\bar{s}},$$

where $\mathbf{1}$ is the trivial Breuil–Kisin module.

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The Γ_K-equivariant pairings on the Galois representations considered above induce symmetric pairings on all Breuil–Kisin modules. These pairings are nondegenerate on $\mathfrak{L}^\diamond$ and $\mathfrak{H}$, and give us injective maps:

$$\mathfrak{L} \hookrightarrow \mathfrak{L}^\vee; \quad \mathfrak{L}_0 \hookrightarrow \mathfrak{L}_0^\vee; \quad \mathfrak{H}_0 \hookrightarrow \mathfrak{H}_0^\vee.$$

Set

$$\mathfrak{d}(\mathfrak{L}) = \mathfrak{L}^\vee/\mathfrak{L}; \quad \mathfrak{d}(\mathfrak{L}_0) = \mathfrak{L}_0^\vee/\mathfrak{L}_0; \quad \mathfrak{d}(\mathfrak{H}_0) = \mathfrak{H}_0^\vee/\mathfrak{H}_0.$$

The module $\mathfrak{d}(\mathfrak{L})$ is equipped with an isomorphism

$$\varphi_{\mathfrak{d}(\mathfrak{L})} : \varphi^*\mathfrak{d}(\mathfrak{L})[\mathcal{E}(u)^{-1}] \xrightarrow{\simeq} \mathfrak{d}(\mathfrak{L})[\mathcal{E}(u)^{-1}].$$

And, since $\mathfrak{L}_0$ and $\mathfrak{H}_0$ are trivial as Breuil–Kisin modules, we actually obtain canonical identifications

$$\mathfrak{S} \otimes_{\mathbb{Z}_p} \boldsymbol{\Lambda}_{p,\bar{s}}^\vee/\boldsymbol{\Lambda}_{p,\bar{s}} \xrightarrow{\simeq} \mathfrak{d}(\mathfrak{L}_0); \quad \mathfrak{S} \otimes_{\mathbb{Z}_p} \boldsymbol{\Delta}_{p,\bar{s}}^\vee/\boldsymbol{\Delta}_{p,\bar{s}} \xrightarrow{\simeq} \mathfrak{d}(\mathfrak{H}_0) \tag{1.6.2}$$

compatible with the natural φ-module structure on either side.

Consider the associated objects

$$\mathscr{D}(\mathfrak{L}) = \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{d}(\mathfrak{L}); \quad \mathscr{D}(\mathfrak{L}_0) = \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{d}(\mathfrak{L}_0); \quad \mathscr{D}(\mathfrak{H}_0) = \mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{d}(\mathfrak{H}_0)$$

in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$. By the exactness of Fontaine's functor and its compatibility with Kisin's, we actually have

$$\mathscr{D}(\mathfrak{L}) = \mathscr{M}(\boldsymbol{L}_{p,\bar{s}}^\vee/\boldsymbol{L}_{p,\bar{s}}); \quad \mathscr{D}(\mathfrak{L}_0) = \mathscr{M}(\boldsymbol{\Lambda}_{p,\bar{s}}^\vee/\boldsymbol{\Lambda}_{p,\bar{s}}); \quad \mathscr{D}(\mathfrak{H}_0) = \mathscr{M}(\boldsymbol{\Delta}_{p,\bar{s}}^\vee/\boldsymbol{\Delta}_{p,\bar{s}}).$$

Therefore, we obtain canonical isomorphisms

$$\mathscr{M}(\lambda) : \mathscr{D}(\mathfrak{L}_0) \xrightarrow{\simeq} \mathscr{D}(\mathfrak{L}); \quad \mathscr{M}(\delta) : \mathscr{D}(\mathfrak{H}_0) \xrightarrow{\simeq} \mathscr{D}(\mathfrak{L})$$

in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$.

The next result is the heart of the matter, and crucially uses the results of [**BMS18**].

LEMMA 1.7. *The map $\mathscr{M}(\lambda)$ induces an isomorphism of $\mathfrak{S}$-modules*

$$\mathfrak{M}(\lambda) : \mathfrak{d}(\mathfrak{L}_0) \xrightarrow{\simeq} \mathfrak{d}(\mathfrak{L}). \tag{1.7.1}$$

Proof. This is equivalent to showing that $\mathscr{M}(\delta)$ induces an isomorphism

$$\mathfrak{M}(\delta) : \mathfrak{d}(\mathfrak{H}_0) \xrightarrow{\simeq} \mathfrak{d}(\mathfrak{L}).$$

Indeed, the composition

$$\mathscr{M}(\delta)^{-1} \circ \mathscr{M}(\lambda) : \mathscr{D}(\mathfrak{L}_0) \xrightarrow{\simeq} \mathscr{D}(\mathfrak{H}_0)$$

is compatible with the φ-module structures on both sides, and therefore we find from equation (1.6.2) that it must necessarily carry $\mathfrak{d}(\mathfrak{L}_0)$ onto $\mathfrak{d}(\mathfrak{H}_0)$.

We claim that the submodule $\mathfrak{H}_0$ of $\mathfrak{H}$ is a direct summand.

Assume this for now. Then we also have $\mathfrak{L} = \mathfrak{H}_0^{\perp} \subset \mathfrak{H}$. Indeed, by functoriality of Kisin's functor, we have $\mathfrak{L} \subset \mathfrak{H}_0^{\perp}$, and so it is sufficient to show that $\mathfrak{L}$ is also a direct summand of $\mathfrak{H}$; or, equivalently, that the map $\mathfrak{H} \to \mathfrak{L}^{\vee}$ induced by the pairing on $\mathfrak{H}$ is surjective. However, the cohomology of the complex

$$0 \to \mathfrak{H}_0 \to \mathfrak{H} \to \mathfrak{L}^{\vee} \to 0$$

is supported at the maximal ideal of $\mathfrak{S}$. Therefore, the map $\mathfrak{H}/\mathfrak{H}_0 \to \mathfrak{L}^{\vee}$ of finite-free $\mathfrak{S}$-modules is an isomorphism at all codimension-1 primes, and is therefore itself an isomorphism.

From the previous paragraph and the nondegenerate pairing on $\mathfrak{H}$, we now obtain canonical isomorphisms

$$\mathfrak{d}(\mathfrak{L}) = \mathfrak{L}^{\vee}/\mathfrak{L} \xleftarrow{\simeq} \mathfrak{H}/(\mathfrak{L} \oplus \mathfrak{H}_0) \xrightarrow{\simeq} \mathfrak{H}_0^{\vee}/\mathfrak{H}_0 = \mathfrak{d}(\mathfrak{H}).$$

Thus, once it is known that $\mathfrak{H}_0$ is a direct summand of $\mathfrak{H}$, we will be done. For this, since the cokernel of this map is finite-free of rank 21 at all nonmaximal primes, it is sufficient to see that

$$W \otimes_{\mathfrak{S}, u \mapsto 0} \mathfrak{H}_0 \hookrightarrow W \otimes_{\mathfrak{S}, u \mapsto 0} \mathfrak{H}$$

maps onto a direct summand. Equivalently, we have to check the same for the map

$$W \otimes_{\mathfrak{S}, u \mapsto 0} \varphi^* \mathfrak{H}_0 \hookrightarrow W \otimes_{\mathfrak{S}, u \mapsto 0} \varphi^* \mathfrak{H}. \tag{1.7.2}$$

By property [**KM16**, (1.3.4)] and the compatibility of Chern classes with the crystalline comparison isomorphism, the map in question can be identified, after inverting p, with the inclusion

$$\langle \mathrm{ch}_{\mathrm{cris}}(\boldsymbol{\xi}_s) \rangle [1/p] \hookrightarrow \boldsymbol{H}^2_{\mathrm{cris},s}(1)[1/p], \tag{1.7.3}$$

where $\langle \mathrm{ch}_{\mathrm{cris}}(\boldsymbol{\xi}_s) \rangle \subset \boldsymbol{H}^2_{\mathrm{cris},s}(1)$ is the subspace generated by the crystalline Chern class of $\boldsymbol{\xi}_s$.

Now, this subspace is a direct summand by the primitivity of the polarization $\boldsymbol{\xi}_s$. Using the trivialization equation (1.6.2) and compatibility with Chern classes once again, one sees that the lattice

$$W \otimes_{\mathfrak{S}, u \mapsto 0} \varphi^* \mathfrak{H}_0 \subset \langle \mathrm{ch}_{\mathrm{cris}}(\boldsymbol{\xi}_s) \rangle [1/p]$$

can be identified with $\langle \mathrm{ch}_{\mathrm{cris}}(\boldsymbol{\xi}_s) \rangle$.

To finish, it remains to show that equation (1.7.2) can be identified with the embedding

$$\langle \mathrm{ch}_{\mathrm{cris}}(\boldsymbol{\xi}_s)\rangle \hookrightarrow \boldsymbol{H}^2_{\mathrm{cris},s}(1)$$

of W-modules underlying equation (1.7.3). But this now follows from [**BMS18**, Theorem 14.6(iii)]. □

REMARK 1.8. To see how equation (1.7.1) can fail, it is instructive to consider the following example, which arises by considering a situation where $\boldsymbol{f}_{\mathrm{dR},s}$ does not generate a direct summand: Suppose that $p = 2$, and choose $\alpha \in u\mathfrak{S}$ such that $\{2, \alpha\}$ is a system of parameters with α invertible in $\mathscr{O}_{\mathscr{E}}$. Let $\mathfrak{L}^{\diamond}$ be the nondegenerate quadratic space over $\mathfrak{S}$ given as the orthogonal sum of two hyperbolic planes spanned by pairs (e_1, f_1) and (e_2, f_2), respectively, where, for each i, we have

$$[e_i, e_i] = [f_i, f_i] = 0; \quad [e_i, f_i] = 1.$$

Set $f = \alpha e_1 + 2(e_2 + f_2)$, and $\mathfrak{L} = \langle f\rangle^{\perp} \subset \mathfrak{L}^{\diamond}$. Note that $\mathfrak{L}$ is not a direct summand of $\mathfrak{L}^{\diamond}$: The quotient $\mathfrak{L}^{\diamond}/\mathfrak{L}$ is isomorphic to the ideal $(2, \alpha) \subset \mathfrak{S}$.

One can now check that

$$\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{L}^{\vee}/\mathfrak{L} \simeq \mathscr{O}_{\mathscr{E}}/8\mathscr{O}_{\mathscr{E}},$$

while

$$W \otimes_{\mathfrak{S}, u\mapsto 0} \mathfrak{L}^{\vee}/\mathfrak{L} \simeq (W/2W)^{\oplus 3}.$$

Therefore, no isomorphism as in equation (1.7.1) can exist in this case.

REMARK 1.9. In the proof above, we have made a reference to 'the' p-adic comparison isomorphism. This should be clarified. At this point, it is best to use the comparison isomorphism arising from the work of Bhatt–Morrow–Scholze again [**BMS18**, Theorem 14.3]. It is checked carefully in [**IIK**, Section 11] that this result does give rise to a canonical crystalline comparison isomorphism, which has all the necessary properties for the arguments needed in the proofs of [**Kis10**, **KM16**], and [**Mad15**], and that it is also compatible with Chern classes of line bundles.

LEMMA 1.10. *The image of equation* (1.6.1) *is a direct summand of the target.*

Proof. Consider the submodule

$$\mathfrak{L}' \subset \mathfrak{L}^{\vee} \oplus \mathfrak{L}_0^{\vee}$$

consisting of elements whose image in $\mathfrak{d}(\mathfrak{L}^{\vee}) \oplus \mathfrak{d}(\mathfrak{L}_0^{\vee})$ is of the form $(\mathfrak{M}(\lambda)(y), y)$, where $\mathfrak{M}(\lambda)$ is the isomorphism from Lemma 1.11.

Note that $\mathfrak{L}'$ contains $\mathfrak{L}$, and that the quotient $\mathfrak{L}'/\mathfrak{L}$ is isomorphic to $\mathfrak{L}_0$. Moreover, since $\mathfrak{M}(\lambda)$ is an isomorphism of φ-modules, $\mathfrak{L}'$ inherits the structure of a Breuil–Kisin module from $\mathfrak{L}^\vee \oplus \mathfrak{L}_0^\vee$.

Now, by the exactness of Fontaine's functor, we have a canonical identification

$$\mathscr{O}_{\mathscr{E}} \otimes_{\mathfrak{S}} \mathfrak{L}' \xrightarrow{\simeq} \mathscr{M}(\boldsymbol{L}^\diamond_{p,\bar{s}})$$

in $\mathrm{Mod}^{\varphi}_{/\mathscr{O}_{\mathscr{E}}}$.

It now follows from Lemma 1.3 that we have a canonical identification $\mathfrak{L}^\diamond = \mathfrak{L}'$ as submodules of $\mathfrak{L}^\vee \oplus \mathfrak{L}_0^\vee$. In particular, $\mathfrak{L}$ is a direct summand of $\mathfrak{L}^\vee$, since it is clearly one in $\mathfrak{L}'$. □

LEMMA 1.11. *With the notation as above, the de Rham realization $\boldsymbol{f}_{\mathrm{dR},s}$ generates a direct summand of $\boldsymbol{L}^\diamond_{\mathrm{dR},s}$. Moreover, if*

$$\boldsymbol{L}_{\mathrm{dR},s} = \langle \boldsymbol{f}_{\mathrm{dR},s}\rangle^{\perp} \subset \boldsymbol{L}^\diamond_{\mathrm{dR},s}$$

is the orthogonal complement of this direct summand, then there is a canonical identification of F-crystals

$$\boldsymbol{L}_{\mathrm{dR},s}(-1) \xrightarrow{\simeq} \boldsymbol{P}^2_{\mathrm{dR},s}$$

characterized by the property that it is compatible with the crystalline comparison isomorphism and the isometry $\alpha_{p,\bar{s}}$.

Proof. Since we have canonical identifications of crystalline and de Rham realizations, it is sufficient to prove this with dR replaced by cris everywhere. We now claim that the lattices

$$W \otimes_{\mathfrak{S},u\mapsto 0} \mathfrak{M}(\boldsymbol{P}^2_{p,\bar{s}}) \subset W \otimes_{\mathfrak{S},u\mapsto 0} \mathfrak{M}(\boldsymbol{P}^2_{p,\bar{s}})[1/p] \simeq \boldsymbol{P}^2_{\mathrm{cris},s}[1/p];$$
$$W \otimes_{\mathfrak{S},u\mapsto 0} \mathfrak{M}(\boldsymbol{L}^\diamond_{p,\bar{s}}(-1)) \subset W \otimes_{\mathfrak{S},u\mapsto 0} \mathfrak{M}(\boldsymbol{L}^\diamond_{p,\bar{s}}(-1))[1/p] \simeq \boldsymbol{L}^\diamond_{\mathrm{cris},s}(-1)[1/p]$$

can be identified with $\boldsymbol{P}^2_{\mathrm{cris},s}$ and $\boldsymbol{L}^\diamond_{\mathrm{cris},s}(-1)$, respectively. Given this, everything is then immediate from Lemma 1.10.

The claim for $\boldsymbol{P}^2_{?}$ was shown in the course of the proof of Lemma 1.11 by using the results of [**BMS18**].

The claim for $\boldsymbol{L}^\diamond_{?}(-1)$ is essentially due to Kisin. We have the Kuga–Satake abelian variety $A^{\diamond,\mathrm{KS}}_s$ associated with the point s equipped with an action by the Clifford algebra $C(L^\diamond)$. It has a p-adic realization $\boldsymbol{H}^\diamond_{p,\bar{s}}$ and a crystalline realization $\boldsymbol{H}^\diamond_{\mathrm{cris},s}$. Moreover, we have a canonical identification

$$\boldsymbol{H}^\diamond_{\mathrm{cris},s} = W \otimes_{u\mapsto 0} \varphi^*\mathfrak{M}(\boldsymbol{H}^\diamond_{p,\bar{s}}).$$

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This certainly follows from [BMS18], but in this special case this was known earlier; see [KM16, Theorem 1.12] for references.

Now, there is a Γ_K-equivariant idempotent projector

$$\boldsymbol{\pi}_{p,\bar{s}} : \mathrm{End}_{C(L^\diamond)}(\boldsymbol{H}^\diamond_{p,\bar{s}}) \to \mathrm{End}_{C(L^\diamond)}(\boldsymbol{H}^\diamond_{p,\bar{s}})$$

whose image is $\boldsymbol{L}^\diamond_{p,\bar{s}}$.

To finish, it suffices to know that $\boldsymbol{L}^\diamond_{\mathrm{cris},s}$ is the image of the projector $\boldsymbol{\pi}_{\mathrm{cris},s}$ on $\mathrm{End}_{C(L^\diamond)}(\boldsymbol{H}_{\mathrm{cris},s})$, defined as the reduction-mod-u of the projector $\varphi^*\mathfrak{M}(\boldsymbol{\pi}_{p,\bar{s}})$ on $\mathrm{End}_{C(L^\diamond)}(\varphi^*\mathfrak{M}(\boldsymbol{H}^\diamond_{p,\bar{s}}))$.

This follows from the very construction of $\boldsymbol{L}^\diamond_{\mathrm{cris}}$ as described in [Mad16, (4.14)] and [KM16, Proposition 3.10]. $\square$

Proof of Proposition A.12 from [KM16]. Following the original attempt at a proof, one finds that there are two main points: The first is to show that the induced map (A.12.1)

$$\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to \mathcal{Z}(2d)_{(p)}$$

factors through the primitive locus $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$, where the orthogonal complement

$$\boldsymbol{L}_{\mathrm{dR}} = \langle \boldsymbol{f}_{\mathrm{dR}} \rangle^{\perp} \subset \boldsymbol{L}^\diamond_{\mathrm{dR}}|_{\mathcal{Z}^{\mathrm{pr}}(m)_{(p)}}$$

restricts to a vector sub-bundle of $\boldsymbol{L}^\diamond_{\mathrm{dR}}$. This is now immediate from the first assertion of Lemma 1.11, which shows that the desired statement is true at every W-valued point of the smooth $\mathbb{Z}_{(p)}$-stack $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$.

The second main point is to show that the isometry

$$\alpha_{\mathrm{dR}} : \boldsymbol{L}_{\mathrm{dR}}(-1)|_{\tilde{\mathsf{M}}_{2d,\mathbb{Q}}} \to \boldsymbol{P}^2_{\mathrm{dR}}|_{\tilde{\mathsf{M}}_{2d,\mathbb{Q}}}$$

extends to an isometry of filtered vector bundles

$$\boldsymbol{L}_{\mathrm{dR}}(-1)|_{\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}} \xrightarrow{\simeq} \boldsymbol{P}^2_{\mathrm{dR}}$$

over $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$. This is now immediate from the second assertion of Lemma 1.11.

Given these two points, the rest of the proof proceeds exactly as in [KM16]: If we have a k-point s of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$, then its deformations to $k[\epsilon]$-points of $\tilde{\mathsf{M}}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ (respectively of $\mathcal{Z}^{\mathrm{pr}}(2d)_{(p)}$) are in bijection with lifts to $\boldsymbol{P}^2_{\mathrm{dR},s} \otimes_k k[\epsilon]$ (respectively $\boldsymbol{L}_{\mathrm{dR},s}(-1) \otimes_k k[\epsilon]$) of the degree 2 part of the Hodge filtration. Therefore, the isometry of filtered vector bundles above exhibits a canonical identification between the tangent spaces at s, thus showing that ι^{KS} is étale at s. $\square$

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