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A GEOMETRIC APPROACH TO THE HEINE-BOREL THEOREM

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1. Introduction. In a topological plane with strong enough topological properties one can use [**6**] open triangular regions to define a base for the topology. Similarly, one can use these regions to define boundedness of a set. In this setting we show that in the absolute plane geometry, the holding of the Heine-Borel theorem is equivalent to every four points being contained in some such region and that this second condition is equivalent to the parallel postulate. Thus we give two new conditions equivalent to the parallel postulate.

2. Preliminaries

Definition. If β is a base for a topological space (X, τ), then we say a set S is β-bounded if and only if S is contained in some member of β.

THEOREM (Heine-Borel). *If (X, τ) is a topological space with base β such that:*
(1) *the space is Hausdorff;*
(2) *the closure of each member of β is compact;*
(3) *every finite union of members of β is contained in an element of β;*
then a set S is compact if and only if S is closed and β-bounded.

The proof follows readily from standard properties (see [**5**, pp. 140–141]). (This observation was made by Dr. Bray.)

There are two approaches which give us what we want for our geometry, but some readers may prefer one to the other. Needless to say we leave out much of the detail of these approaches.

A flat plane [**6**] is a set of points and lines such that the following conditions hold.

I Each two distinct points are joined by a line.
II No two distinct lines join the same pair of distinct points.
III The set of points and set of lines each has a Hausdorff topology.
IV The operations of join and intersection are continuous.
V The set of points with the topology is a two-dimensional manifold.

Furthermore, it is known [**7**] that the basic open triangular regions may not be well-defined, so as an additional condition we add the requirement that the flat plane does have uniquely defined open triangular regions for each triple of non-collinear points.

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Another approach is from [**4**] or the equivalent systems in [**3**] or in [**1**]. In only the last reference is topology mentioned; so much has to be done to obtain the conditions above. It was this latter approach, essentially using the axioms of connection and order of Hilbert with Pasch as the closure axiom to two dimensions and a Dedekind cut axiom for each line which the first two authors originally used.

From either approach we obtain that the open triangular regions are convex and form a base for the point topology, which is Hausdorff, and that the closure of such regions, the closed triangular regions, add only the closed line segments joining their vertices. Furthermore, these closed regions are compact (see [**6**, p. 15] for the former approach; the details for the latter approach are contained in Frand's Master's Thesis.) Thus we see that only one of the conditions for the Heine-Borel theorem remains to be considered.

3. The four point condition. In any topological space any set of four points is compact. Therefore, a necessary condition for our Heine-Borel theorem to hold is for any set of four distinct points to lie in some open triangular region. On the other hand, suppose that any four points lie in some open triangular region; then, by induction and convexity of these regions, it is clear that any n points lie in some open triangular region. But by convexity, a triangular region contains n such regions if and only if it contains their $3n$ vertices. Therefore, we have:

THEOREM 1. *A necessary and sufficient condition for the Heine-Borel theorem to hold for open triangular regions bounding sets in a flat plane is that every four points are contained in some open triangular region.*

THEOREM 2 (Giles). *If S is any non-empty open convex subset of a flat affine plane such that every four points are contained in some open triangular region, then there are at most three vertices and the boundary, if non-empty, consists of line segments and rays.*

Proof. Let T_1 be a closed triangular region contained in the region and let Σ, a countable dense subset of the region, be ordered. Let T_2 contain T_1 and the first (in the ordering) element of Σ be not in T_1, etc. Thus we have a sequence T_n of closed triangular regions, so $T_n \subset T_{n+1}$ for all $n \geqq 1$ and $S = \bigcup_{n=1}^{\infty} T_n$.

Let P be an interior point of T_1 and let k be any ray with P as endpoint. If $k \cap S$ is bounded, then let Q_k be the unique point not in $k \cap S$ but in its closure. Let h be a line of support for S at Q_k. Construct a triangular neighbourhood $A_1B_1C_1$ as follows: A_1 is between P and Q_k while B_1C_1 is parallel to h and contains a point D_1 where Q_k is between D_1 and A_1. We can form a nest of triangular neighbourhoods $A_iB_iC_i$ of Q_k such that for each point of S there is at least one such neighbourhood not containing it and such that $A_iB_i || A_jB_j$, $B_iC_i || B_jC_j$, $A_iC_i || A_jC_j$ for all i, j.

Now for a particular ray k, one of two things happen: either there is a neighbourhood $A_mB_mC_m$ which contains no vertex of any of the T_n; or there is a neighbourhood $A_mB_mC_m$ containing an infinite number of vertices. In either event, the lines joining the vertices of T_n intersects A_mB_m in points L_n and A_mC_m in points R_n. Let L and R be the limits of L_n and R_n, respectively. It is clear that no vertex contained in the neighbourhood forces L and R to lie on h. Moreover, if L and R lie on h (angles of vertices converging to a straight angle is also a possibility), then it is clear that in this neighbourhood, the boundary of S is a straight line.

On the other hand, L, R, and Q_k may not be collinear. In this case there is some vertex V_s of T_s in open triangular region L, R and Q_k. Then it follows that for all $n \geqq s$ there is a vertex V_n of T_n in $A_mB_mC_m$, since any line joining L_n and R_n with $n > s$ would intersect L_sV_s as well as R_sV_s. By the pigeon-hole principle, there are at most three such. This completes the proof of the theorem.

COROLLARY. *The geometries obtained as open convex subsets of E_2 for which the Heine-Borel theorem holds in our sense are: E_2, half-plane, strip, solid angle, triangular region, and half-strip (not necessarily rectangular).*

Proof. This follows from inspection of the possible regions given in the theorem.

THEOREM 3. *Four points in an absolute plane geometry lie in some triangular region if and only if there is a unique parallel.*

Proof. Using Klein's model for a hyperbolic plane it follows from the corollary above that the four point condition fails in the hyperbolic case. Also it is clear that the four point condition holds in E_2. From [**2**, p. 186], we note that there are just these two absolute plane geometries.

THEOREM 4. *In any affine flat plane any four points always lie in some triangular region.*

Proof. The technique for surrounding three non-collinear points is found from several iterations of the following. From triangle ABC on BC find D so that C is between B and D; then closed triangular region ABD contains closed triangular region ABC. We now consider the case when three of the four points lie in some triangular region ABC and the fourth lies in one of the six exterior open regions bounded by the sides of the triangle, since all other cases can be resolved by the above technique. These six open regions can be placed in two classes: if the vertex V (one of A,B,C) is between a point of X the region and the point Y obtained as the intersection of VX and the side opposite vertex V then we say it is a "near-vertex" region; it is a "far-vertex" region otherwise. The near vertex case is resolved thus: region XBC contains region ABC. The far vertex case uses the parallel postulate. The line through X parallel to BC intersects AB in D and AC in E, so region ADE contains ABC. The rest follows as before.

COROLLARY. *In any affine flat plane our Heine-Borel theorem holds.*

Note that the corollary to Theorem 2 shows that having a unique parallel is not a necessary condition. It is the presence of congruence in the absolute geometry which forces the equivalence.

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