Bull. Aust. Math. Soc. **86** (2012), 473–480
doi:10.1017/S0004972712000068

ON THE STABILISATION OF ONE-SIDED KUROSH'S CHAINS

R. R. ANDRUSZKIEWICZ and M. SOBOLEWSKA ✉

(Received 27 November 2011)

Abstract

We construct an example showing that Kurosh's construction of the lower strong radical in the class of associative rings may not terminate at any finite ordinal.

2010 *Mathematics subject classification*: primary 16D25; secondary 16S70.
Keywords and phrases: stabilisation of Kurosh's chains, *-rings, construction of classes.

1. Introduction

All rings in this paper are associative but are not required to have a unity and to be commutative. The concept of a radical was introduced as a tool for the study of the structure of rings; see [11]. It plays an important role in the investigation of various ring constructions; see [1, 4, 7, 10, 14, 18].

If R is a ring then $A <_l R$ ($A <_r R$, $A < R$, $A \triangleleft R$) will mean that A is a left (right, one-sided, two-sided) ideal of R. If A is a subring of R we use A_R to denote the ideal of R generated by A. Given a class $\mathcal{M}$ of rings, we denote by $\{\mathcal{M}\}$ the class of all isomorphic copies of rings in $\mathcal{M}$ and by $l(\mathcal{M})$ the lower radical determined by the class $\mathcal{M}$.

Let $\mathcal{M}$ be a nonempty homomorphically closed class of rings. Define

$$\mathcal{M}_1 = \mathcal{M}^1 = \mathcal{M}_l^1 = \mathcal{M}_r^1 = \mathcal{M},$$

and define $\mathcal{M}_\alpha$ ($\mathcal{M}^\alpha$, $\mathcal{M}_l^\alpha$, $\mathcal{M}_r^\alpha$) as the class of all rings R which have the property that every nonzero homomorphic image of R contains a nonzero ideal (one-sided ideal, left ideal, right ideal) in $\mathcal{M}_\beta$ ($\mathcal{M}^\beta$, $\mathcal{M}_l^\beta$, $\mathcal{M}_r^\beta$) for some $\beta < \alpha$. The chain of classes $\{\mathcal{M}_\alpha\}_{\alpha\geq 1}$ ($\{\mathcal{M}^\alpha\}_{\alpha\geq 1}$, $\{\mathcal{M}_l^\alpha\}_{\alpha\geq 1}$, $\{\mathcal{M}_r^\alpha\}_{\alpha\geq 1}$) is said to be Kurosh's chain (one-sided, left, right Kurosh's chain).

Definition 1.1. A radical $\mathcal{S}$ is said to be *left strong (right strong, strong)* if, for every $A <_l R$ ($A <_r R$, $A < R$), if $A \in \mathcal{S}$ then $A_R \in \mathcal{S}$.

It is obvious that radical $\mathcal{S}$ is strong when $\mathcal{S}$ is left and right strong.

Andruszkiewicz and Puczyłowski proved, in particular, that if $\mathcal{M}$ is a homomorphically closed class of rings with zero multiplication, then radical $l(\mathcal{M})$ is strong [2, Theorem 3]. In particular, Baer's radical $\beta = l(\{R \mid R^2 = 0\})$ is strong.

Given a nonempty homomorphically closed class $\mathcal{M}$ of rings, we denote by $ls(\mathcal{M})$ ($ls_l(\mathcal{M})$, $ls_r(\mathcal{M})$) the smallest strong (left strong, right strong) radical containing $\mathcal{M}$. We call it the *lower strong (left strong, right strong) radical determined by* $\mathcal{M}$.

All of these radicals can be constructed as the sum of the respective Kurosh's chains:

$$l(\mathcal{M}) = \bigcup_{\alpha\geq 1} \mathcal{M}_\alpha, \quad ls(\mathcal{M}) = \bigcup_{\alpha\geq 1} \mathcal{M}^\alpha, \quad ls_l(\mathcal{M}) = \bigcup_{\alpha\geq 1} \mathcal{M}_l^\alpha, \quad ls_r(\mathcal{M}) = \bigcup_{\alpha\geq 1} \mathcal{M}_r^\alpha.$$

Let γ be an ordinal. We say that Kurosh's chain $\{\mathcal{M}_\alpha\}_{\alpha\geq 1}$ stabilises at step γ if $\mathcal{M}_\beta = \mathcal{M}_\gamma$ for every ordinal $\beta \geq \gamma$, which is equivalent to the condition $l(\mathcal{M}) = \mathcal{M}_\gamma$. We say that Kurosh's chain $\{\mathcal{M}_\alpha\}_{\alpha\geq 1}$ stabilises at exactly step γ, if $l(\mathcal{M}) = \mathcal{M}_\gamma \neq \mathcal{M}_\beta$ for every ordinal $\beta < \gamma$. Similarly, we can define a stabilisation of other types of Kurosh's chains.

In 1966 Sulinski *et al.* [17] proved that, for every homomorphically closed class $\mathcal{M}$ of rings, $l(\mathcal{M}) = \mathcal{M}_{\omega_0}$. They asked whether for every ordinal $\gamma \leq \omega_0$ there exists a homomorphically closed class $\mathcal{M}$ of rings such that $\mathcal{M}_\gamma = \mathcal{M}_{\gamma+1}$, but $\mathcal{M}_\gamma \neq \mathcal{M}_\alpha$ for every $\alpha < \gamma$. The first significant result was obtained by Heinicke in 1968 [12]. He answered the question positively for $\gamma = \omega_0$. In the same year Armendariz and Leavitt [5] showed that if $\mathcal{M}$ is a hereditary (which means that if $R \in \mathcal{M}$ and $A \lhd R$, then $A \in \mathcal{M}$) and homomorphically closed class of rings, then $l(\mathcal{M}) = \mathcal{M}_3$. Moreover, Hoffman and Leavitt [13] showed that, if $\mathcal{M}$ is a homomorphically closed and hereditary class of rings, then $l(\mathcal{M})$ is a hereditary class. The general problem was finally solved positively by Beidar in 1982 [6].

Problems of stabilisation for left and right Kurosh's chains are dual, because we can use the ring with inverted multiplication. Therefore we only need to deal with the stability of, for example, left Kurosh's chains. However, since 1971 there remains an open question: are all left (one-sided) Kurosh's chains stabilised at ω_0?

The most valuable results concerning the stabilisation of one-sided Kurosh's chains were obtained by Divinsky *et al.* [8]. However, it was only later that the focus was put on the study for stabilisation of left Kurosh's chains. In [8] it was shown, among other things, that for every homomorphically closed class of rings $\mathcal{M}$ and for every limit ordinal α the class $\mathcal{M}^\alpha$ (also $\mathcal{M}_l^\alpha$) is a radical class; it was also shown that if $\mathcal{M}$ is homomorphically closed and hereditary and contains all zero rings, then the one-sided Kurosh's chain $\{\mathcal{M}^\alpha\}_{\alpha\geq 1}$ stabilises at ω_0 (Theorem 6). In 1987 Puczyłowski [16] reinforced some of these results, showing that if $\mathcal{M}$ is homorphically closed class of rings, then $\mathcal{M}_l^\alpha$ is a radical class for every $\alpha \geq \omega_0$. Puczyłowski showed that if $\mathcal{M}$ is a radical class, then $\mathcal{M}_l^\alpha$ is a radical class for every α, and if $\mathcal{M}$ is also hereditary, then $ls_l(\mathcal{M}) = \mathcal{M}_l^2$. In 1990, by modifying Beidar's example, Andruszkiewicz and Puczyłowski [2] found for each ordinal $\gamma \le \omega_0$ a homomorphically closed class $\mathcal{M}$ of rings, for which the left Kurosh's chain stabilises at exactly step γ. A deeper analysis of their construction shows that it is a good example of one-sided Kurosh's chains.

Reviewing the existing literature on the stabilisation of Kurosh's chains, the authors have not discovered a single example of a radical class $\mathcal{M}$ for which the one-sided (or left) Kurosh's chain stabilises exactly at a given step $\gamma < \omega_0$. Moreover, nobody has published even one example of a radical class $\mathcal{M}$ such that $\mathcal{M}^2 \neq \mathcal{M}^3$ (or $\mathcal{M}_l^2 \neq \mathcal{M}_l^3$). This prompted the first author to bring to the 'Radicals of Rings and Related Topics' workshop [19] questions that can be formulated as follows.

Question 1.2 [16, Question 5]. For any radical class $\mathcal{M}$, does the left Kurosh's chain $\{\mathcal{M}_l^\alpha\}_{\alpha\ge1}$ stabilise at ω_0?

Question 1.3. For every ordinal $\gamma \le \omega_0$, does a radical class $\mathcal{M}$ exist such that the Kurosh's chain $\{\mathcal{M}_l^\alpha\}_{\alpha\ge1}$ stabilises at exactly step γ?

Question 1.4. Does a radical class $\mathcal{M}$ exist such that $ls_l(\mathcal{M}) \neq \mathcal{M}_l^\alpha$ for any ordinal α?

This is a rather difficult subject. We give a partial answer to the above questions, which can be compared to the results of Heinicke [12].

2. Preliminaries

Definition 2.1. Let n be a positive integer. A subring A of a ring R is said to be *n-accessible (left n-accessible, right n-accessible, one-sided n-accessible)* in R if there are subrings $R = A_0, A_1, \ldots, A_{n-1}, A_n = A$ of R such that $A_i \lhd A_{i-1}$ ($A_i <_l A_{i-1}$, $A_i <_r A_{i-1}$, $A_i < A_{i-1}$) for $i = 1, 2, \ldots, n$. A subring A is said to be *accessible (left accessible, right accessible, one-sided accessible)* in R if there exists a positive integer n such that A is n-accessible (left n-accessible, right n-accessible, one-sided n-accessible) in R.

The following proposition is well known (see [3, 15]).

Proposition 2.2. *Let A be a subring of a ring R.*

(i) *A is accessible in R if and only if $(A_R)^m \subseteq A$ for some positive integer m.*
(ii) *If $A = A^2$, then A is accessible in R if and only if $A \lhd R$.*
(iii) *A is left n-accessible (right n-accessible) in R if and only if $RA^n \subseteq A$ ($A^nR \subseteq A$).*
(iv) *A is one-sided n-accessible in R if and only if there exist $k, l \in \mathbb{N} \cup \{0\}$ such that $k + l = n$ and $A^kRA^l \subseteq A$.*

A prime ring A is called a **-ring* if $A/I \in \beta$ for every $0 \neq I \lhd A$ (see [9]). Our next theorem shows in particular that every one-sided accessible subring of a simple domain with a unity is a *-ring.

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Theorem 2.3. *Let R be a simple domain with a unity. If A is a nonzero subring of R which is one-sided n-accessible in R then for, any nonzero ideal I of A, $A^n = I^n = I^{n+1}$. In particular, $A^n = A^{n+1}$ and $(A/I)^n = 0$.*

Proof. The proof is by induction on n. Let $n = 1$. Then $A <_l R$ or $A <_r R$. Let $A <_l R$. Since R has a unity, $A = RA$. But R is a simple ring, so $RAR = R$. Hence $A^2 = (RA)(RA) = (RAR)A = RA = A$. Let I be a nonzero ideal of A. Then $0 \neq AIR \lhd R$. Therefore, $AIR = R$ and hence $AIRA = RA = A$. Thus $AIA = A$. As $AIA \subseteq I$ we have $A \subseteq I$ and $A = I$. The proof for $A <_r R$ is similar.

Let $n > 1$ be a positive integer such that the assertion holds for $n - 1$. Let $A \neq 0$ be a one-sided n-accessible subring of R. There exists a one-sided $(n - 1)$-accessible subring $B \subseteq R$ such that $A < B$. Let $0 \neq I \lhd A$. Assume that $A <_l B$. Then $0 \neq AIB \lhd B$. By assumption, $B^{n-1} = (AIB)^{n-1}$. Also $B^{n-1}A = (AIB)^{n-1}A = (AI)(BAI)^{n-2}BA \subseteq (AI)^{n-1}A \subseteq I$. In particular, $A^n \subseteq I$. Taking $I = A^{n+1}$, we get $A^n = A^{n+1}$. Replacing I^n by I, we obtain $A^n \subseteq I^n$ and thus $A^n = I^n$. Replacing I^{n+1} by I, we have $A^n \subseteq I^{n+1} \subseteq I^n \subseteq A^n$, so $I^n = I^{n+1}$. The proof for $A <_r R$ is similar. □

Corollary 2.4. *Let R be a simple domain with a unity and let A be a nonzero subring of R which is one-sided n-accessible in R. If $C \neq 0$ is an accessible subring of A then $C^n = C^{n+1} = A^n$.*

Proof. By Proposition 2.2, there exists a positive integer m such that $(C_A)^m \subseteq C$. But $0 \neq (C_A)^{nm} \lhd A$ and thus, by Theorem 2.3, $A^n \subseteq (C_A)^{nm} \subseteq C^n \subseteq A^n$; hence $C^n = A^n$. Therefore, $C^{2n} = A^{2n} = A^n = C^n$ and $C^{n+1} = C^n$. □

Proposition 2.5. *Let R be a simple domain with a unity and let $a \in R$. For any positive integers n, m:*

(i) $(Ra^n + [a])^m = Ra^n + [a]^m$;
(ii) $(Ra^n + [a])^m = Ra^n$ *for* $m \geq n$;
(iii) $Ra^n + [a]$ *is the smallest left n-accessible subring in R containing subring* $[a]$.

Proof. The proof is straightforward. □

Corollary 2.6. *Let R be a simple domain with a unity and let $0 \neq a \in R$ be such that $Ra \neq R$. Then $(Ra^n + [a])^{n-1} \neq (Ra^n + [a])^n$ for $n = 2, 3, \ldots$.*

Proof. By Proposition 2.5, $(Ra^n + [a])^{n-1} = Ra^n + [a]^{n-1}$ and $(Ra^n + [a])^n = Ra^n$. If, for some $n = 2, 3, \ldots$, $(Ra^n + [a])^{n-1} = (Ra^n + [a])^n$ then $a^{n-1} \in Ra^n$. But R is a domain and thus $1 \in Ra$ and $R = Ra$, contrary to assumption. □

Proposition 2.7. *Let R be a simple ring with a unity and let $0 \neq e = e^2 \neq 1$, $e \in R$, be such that $eRe \not\approx R$. Then $eRe <_l eR <_r R$ and there are no one-sided ideals of R which are isomorphic to eRe. Moreover, if $eRe \subseteq I \lhd eR$ then $I = eR$.*

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Proof. It is obvious that $eRe <_l eR <_r R$. Assume that there exists $A < R$ such that $A \approx eRe$. Since e is unity in eRe, it follows that A has a unity f. Assume that $A <_l R$. Then $A = Af$, so $A = Rf$ and $A = fA$ and hence $Rf = fRf$. But $Rf = (1 - f)Rf \oplus fRf$, therefore, $(1 - f)Rf = 0$. Since R is simple, $1 - f = 0$ or $f = 0$. But $e \neq 0$; thus $f = 1$. Hence $R \approx eRe$, a contradiction. The proof for $A <_r R$ is similar.

Let $eRe \subseteq I \lhd eR$. Then $(eRe)(eR) \subseteq I$. But $Re^2R = ReR = R$ since $e \neq 0$ and R is simple and thus $eR \subseteq I$. Therefore, $I = eR$. □

Example 2.8. Let K be a field, $R = M_2(K)$ and $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $eRe = \left[\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right] \approx K$ and $eRe \not\approx R$. Moreover, R is simple and $eR = \left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$. Therefore, by Proposition 2.7, in R there are no one-sided ideals of R which are isomorphic to K.

Example 2.9. Let K be a field and let $\mathcal{S} = l(\{K\} \cup \{0\})$. We prove that a radical $\mathcal{S}$ is neither left nor right strong. Note that

$$K \approx \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} <_l \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K \approx \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} <_r \begin{bmatrix} K & 0 \\ K & 0 \end{bmatrix}.$$

Moreover, 0, $\left[\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$ are all the ideals of a ring $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$ and so $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{S}_l^2$; thus $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right] \in ls_l(\mathcal{S})$. If $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{S}$, then there exists a nonzero accessible subring $A \approx K$ of the ring $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$. Then $A^2 = A$ and so $A \lhd \left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$ by Proposition 2.2(ii). Hence $K \approx \left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$, giving a contradiction. Similarly, $\left[\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right] \in \mathcal{S}_r^2 \setminus \mathcal{S}$.

3. Main results

Theorem 3.1. *For every positive integer n there exists a radical class $\mathcal{S}$ such that $\mathcal{S}^n \neq \mathcal{S}^{n+1}$.*

Proof. Let R be a simple domain with a unity which is not a division ring. Then there exists $0 \neq a \in R$ such that $Ra \neq R$. Let n be a fixed positive integer and let $A = Ra^n + [a]$. By Proposition 2.5 and Theorem 2.3, the class $\{A\} \cup \beta$ is homomorphically closed. Let $\mathcal{S} = l(\{A\} \cup \beta)$. Obviously $A \in \mathcal{S}$. Let us observe that

$$A = Ra^n + [a] <_l Ra^{n-1} + [a] <_l Ra^{n-2} + [a] <_l \cdots <_l Ra + [a] <_l R.$$

Thus by Theorem 2.3, proper homomorphic images of rings $Ra^i + [a] \in \beta \subseteq \mathcal{S}$ for every $i = 1, 2, \ldots, n$. Since $A \in \mathcal{S}$, we have by induction $Ra^{n-i} + [a] \in \mathcal{S}^{i+1}$ for $i = 0, 1, \ldots, n-1$. A ring R is simple and thus $R \in \mathcal{S}^{n+1}$. Assume that $R \in \mathcal{S}^n$. If $n \geq 2$, then in R there exists a one-sided $(n-1)$-accessible nonzero subring $B \in \mathcal{S}$. Therefore, there exists a nonzero one-sided subring $C \in \{A\} \cup \beta$ accessible in B. Since R is a domain, $C \approx A$. By Corollary 2.6, $C^{n-1} \neq C^n$. But by Corollary 2.4, $C^{n-1} = C^n$, a contradiction.

If $n = 1$, then $R \in \mathcal{S}$ and in R there exists a nonzero one-sided accessible subring $D \in \{Ra\} \cup \beta$. Since R is a domain, $R \approx Ra$. Therefore, the ring Ra has a unity e and $Ra = Re$. But $e^2 = e$, so $e(1-e) = 0$. Since R is a domain and $a \neq 0$, $e = 1$ and $Ra = R$, a contradiction.

Thus, $\mathcal{S}$ is a radical class such that $\mathcal{S}^n \neq \mathcal{S}^{n+1}$. □

Remark 3.2. The reasoning used in the proof of Theorem 3.1 extends to left Kurosh's chains. So for every natural n there exists a radical class $\mathcal{S}$ such that $\mathcal{S}_l^n \neq \mathcal{S}_l^{n+1}$.

Lemma 3.3. *Let A be a ring such that $A = A^2 \neq 0$. If A is one-sided accessible in a ring R, then $ARA = A$ and in every nonzero homomorphic image of $AR + A$ there exists a nonzero left ideal which is a homomorphic image of A.*

Proof. By Proposition 2.2(iv), there exist $n \in \mathbb{N}$ and $k, l \in \mathbb{N} \cup \{0\}$ such that $k + l = n$ and $A^kRA^l \subseteq A$. Then $A^lA^kRA^lA^k \subseteq A^{k+l+1}$. But $A = A^m$ for all positive integers m; thus $ARA \subseteq A$. Moreover, $A = A^3 \subseteq ARA \subseteq A$ and so $A = ARA$. Hence $(AR + A)A = A$; thus $A <_l AR + A$. Let I be some proper ideal of $AR + A$. If $A \subseteq I$, then $A(AR) \subseteq I$. But $A(AR) = AR$; thus $AR + A \subseteq I$. Hence $I = AR + A$, a contradiction. Therefore, $A \nsubseteq I$. Then $(A + I)/I \neq 0$ and $(A + I)/I <_l R/I$. Moreover, $(A + I)/I \approx A/(A \cap I)$. □

Proposition 3.4. *If $\mathcal{M}$ is a homomorphically closed class of idempotent rings, then $ls(\mathcal{M}) = \mathcal{M}^3$.*

Proof. It suffices to prove that any nonzero ring $R \in ls(\mathcal{M})$ has a nonzero one-sided ideal in class $\mathcal{M}^2$. But R has a nonzero one-sided accessible subring $A \in \mathcal{M}$; thus $A = A^2$ and, by Lemma 3.3, $AR + A \in \mathcal{M}^2$. Since $0 \neq AR + A <_r R$, the proof is complete. □

Example 3.5. Let K be a field and let $\mathcal{S} = l(\{K\} \cup \{0\})$. We prove that in a one-sided Kurosh's chain $\{\mathcal{S}^\alpha\}$, $\mathcal{S}^3 = \mathcal{S}^4$ but $\mathcal{S}^3 \neq \mathcal{S}^2$. Note that $ls(\mathcal{S}) = ls(\{K\} \cup \{0\})$ and thus by Lemma 3.3, $ls(\mathcal{S}) \subseteq \mathcal{S}^3$; hence $ls(\mathcal{S}) = \mathcal{S}^3$. By Example 2.9, $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{S}^2$. But the matrix ring $M_2(K)$ is simple and $\left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right] <_r M_2(K)$, so $M_2(K) \in \mathcal{S}^3$. Assume that $M_2(K) \in \mathcal{S}^2$. Then there exists $0 \neq A < M_2(K)$ such that $A \in \mathcal{S}$. Therefore, there exists a nonzero accessible subring $B \in \{K\} \cup \{0\}$ of the ring A. Hence $B \approx K$. Thus $B = B^2$ and so $B \lhd A$. But $A < M_2(K)$ and thus $K \approx B < M_2(K)$, contrary to Example 2.8. Therefore, $M_2(K) \notin \mathcal{S}^2$. Consequently, $\mathcal{S}^3 \neq \mathcal{S}^2$ and $ls(\mathcal{S}) \neq \mathcal{S}^2$.

Let us observe that the class $\{K\} \cup \{0\}$ is hereditary; so is $\mathcal{S}$. Thus we see that for one-sided Kurosh's chains, we cannot apply [16, Proposition 3.3], which says that if $\mathcal{M}$ is a hereditary radical class, then $ls_l(\mathcal{M}) = \mathcal{M}_l^2$.

Theorem 3.6. *There exists a radical class $\mathcal{S}$ such that $\mathcal{S}^n \neq \mathcal{S}^{n+1}$ for every positive integer n.*

Proof. For every prime number p there exists a simple domain with a unity R_p which is not a division ring and such that $pR_p = 0$. Then for every prime p there is $0 \neq a_p \in R_p$ such that $R_pa_p \neq R_p$. By the proof of Theorem 3.1 for the one-sided Kurosh's chain

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determined by radical class $\mathcal{S}(p) = l(\{R_p a_p^p + [a_p]\} \cup \beta)$, we have $\mathcal{S}(p)^p \neq \mathcal{S}(p)^{p+1}$. We will show that for the one-sided Kurosh's chain determined by a radical class $S = l(\bigcup_{p\in\mathbb{P}} \mathcal{S}(p))$, $S^n \neq S^{n+1}$ for every finite n. Otherwise, there exists a prime p such that $\mathcal{S}^q = ls(\mathcal{S})$. By the proof of Theorem 3.1, $R_q \in ls(\mathcal{S})$ and so $R_q \in \mathcal{S}^q$. Hence there exists a nonzero one-sided subring $B \in \mathcal{S}$, $(q-1)$-accessible in R_q. Therefore, there exists a nonzero accessible subring $C \in \bigcup_{p\in\mathbb{P}} \mathcal{S}(p)$ of B. Thus, $C \in \mathcal{S}(p)$ for some prime p. Since R_q is a domain, there exists a subring $D \approx R_p a_p^p + [a_p]$ accessible in C. But $tR_t = 0$ for every prime t, so $p = q$ and $D \approx R_q a_q^q + [a_q]$ is an accessible subring of B. By Corollary 2.4, $D^{q-1} = D^q$, contrary to Corollary 2.6. □

Remark 3.7. The reasoning used in the proof of Theorem 3.6 extends to left Kurosh's chains. So there exists a radical class $\mathcal{S}$ such that $\mathcal{S}_l^n \neq ls_l(\mathcal{S})$ for every natural n.

We have not been able to answer the following questions.

Question 3.8. Do we have for each hereditary radical class $\mathcal{S}$ that $ls(\mathcal{S}) = \mathcal{S}^3$?

Question 3.9. Does Kurosh's chain constructed in the proof of Theorem 3.1 stabilise at some finite step?

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R. R. ANDRUSZKIEWICZ, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2, Poland
e-mail: randrusz@math.uwb.edu.pl

M. SOBOLEWSKA, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2, Poland
e-mail: magdas@math.uwb.edu.pl

https://doi.org/10.1017/S0004972712000068 Published online by Cambridge University Press