Canad. Math. Bull. Vol. 39 (1), 1996 pp. 95–105

MEAN CURVATURE OF RIEMANNIAN FOLIATIONS

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ABSTRACT. It is shown that a suitable conformal change of the metric in the leaf direction of a transversally oriented Riemannian foliation on a closed manifold will make the basic component of the mean curvature harmonic. As a corollary, we deduce vanishing and finiteness theorems for Riemannian foliations without assuming the harmonicity of the basic mean curvature.

1. **Introduction.** Let $\mathcal{F}$ denote a transversally oriented Riemannian foliation on a closed manifold. Reinhart [R] introduced basic differential forms to provide a generalized notion of forms on the quotient space $M/\mathcal{F}$, which, in general, is not a manifold. In particular the deRham cohomology $H_B^*(\mathcal{F})$ of the complex of basic differential forms is of great interest and has been studied extensively. In contrast to the special case of Riemannian manifolds, the operators d and δ defined as usual on the local quotients, are not in general adjoint operators. The defect is related to the mean curvature of the leaves.

In [KT1,2,3] Kamber and Tondeur studied this basic cohomology, $H_B^*(\mathcal{F})$, under the additional assumption that the mean curvature form κ of the leaves of $\mathcal{F}$ is a basic 1-form. Alvarez López showed in [A] that the work of Kamber and Tondeur can essentially be done without the restrictive assumption of basic mean curvature, by replacing κ by its basic component.

Our aim in this paper is to present an independent and intrinsically Riemannian approach to many familiar problems in the theory of foliations. As the leaves of a foliation correspond to points on the quotient manifold and points do not have any particular quality in Riemannian geometry, we want to ignore the particularities of the leaves and concentrate on transversal differential forms and their operators. Our motivation for this point of view is that we want to treat the transversal geometry of a foliation as an example of a somewhat singular space. The problems treated here are: (i) vanishing theorems; (ii) a generalization of the theorem of Hebda [H] to differential forms of degree ≥ 1 (compare also [MRT]); (iii) the Hodge decomposition (see also [EH] and [A]); (iv) the relation of tautness with the non-vanishing of the top degree basic cohomology (compare [Ma] and [A]); and (v) Poincaré duality (compare [EH]). This paper is an updated version of a 1991 Ohio State University preprint [MMR].

The proof of these results is based on the solution of a certain parabolic equation, associated to a diffusion process with a drift term. Some of the ideas in [MRT] are used, but the present paper is independent in a technical sense.

This work was partially supported by a US-NSF grant DMS-89-3474 (first author), a Canadian NSERC grant OGP-7873 (second author) and a Swiss NSF grant #2000-042054.94/1 (third author).

Received by the editors May 11, 1994.

AMS subject classification: 53C12, 57R30.

2. **Definitions and Results.** Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact manifold M^n without boundary, defined by an integrable distribution $L \subset TM$ with normal bundle $Q = TM/L$ of dimension q. We assume throughout this paper that the holonomy invariant transversal Riemannian metric g_Q for Q is induced by a bundle-like metric $g_M = g_L + g_Q$ on TM. This means that the local projections on distinguished charts $\pi\colon U \longrightarrow B_U$ are Riemannian submersions such that on overlaps the transition functions are isometries. We will also identify Q with $L^\perp$. The complex $\Omega_B^*(\mathcal{F})$ of basic differential forms is, by definition, the space of forms which in a distinguished coordinate chart U are pullbacks of forms defined on the local quotient B_U. We follow Reinhart and define the operators d, δ and the Laplacian $\Delta = d\delta + \delta d$ on $\Omega_B^*(\mathcal{F})$ to be the pullbacks of the corresponding operators defined on the local quotient Riemannian manifold (B_U, g_Q). The local formulas are consistent because the transition functions of distinguished charts are isometries. To be able to apply global analysis we need extensions of these basic operators to operators which act on all differential forms of the compact manifold M. The symbols of the extended operators will coincide with the corresponding standard operators of the Riemannian manifold (M, g). Although these extensions are not unique, the geometry of a foliation suggests some appropriate extensions. The extension of d is simply the exterior derivative d defined on M, and is independent of any metric. To define the appropriate extension $\tilde{\delta}$ of δ that we want to use in this paper, we need some preparations.

The orthogonal splitting $TM = L \oplus L^\perp$ defines a reduction of the principal bundle of orthonormal frames on M to the subgroup $O(p) \times O(q)$ imbedded diagonally in $O(n)$ ($p = n - q = \dim(L)$). The projection of the Levi-Civita connection ∇^M on to this reduction is the connection $\tilde{\nabla}$ which was introduced in [MRT]. This connection $\tilde{\nabla}$ is a metric connection respecting the splitting $TM = L \oplus L^\perp$, but with torsion. The difference tensor $\gamma = \tilde{\nabla} - \nabla^M$ between the two connections is a 1-form with values in $O(TM)$, the bundle of skew-symmetric endomorphisms of TM. γ can be described in terms of the second fundamental form α of the leaves of L and the O'Neill tensor β of the distribution $L^\perp \approx Q$ as a direct sum $\gamma = j_1\alpha + j_2\beta$, where j_1 and j_2 are the following natural imbeddings:

$$j_1\colon \mathrm{Hom}(L, Q) \subset o(TM) = o(L \oplus Q); \quad j_2\colon \mathrm{Hom}(Q, L) \subset o(TM) = o(L \oplus Q)$$

$$\phi \longmapsto \begin{pmatrix} 0 & -\phi^t \\ \phi & 0 \end{pmatrix}; \qquad \psi \longmapsto \begin{pmatrix} 0 & \psi \\ -\psi^t & 0 \end{pmatrix}.$$

In terms of covariant derivatives, $\tilde{\nabla}$ is given by the formulas:

$$\text{(2.1)} \qquad \begin{aligned} &\text{(i) } \tilde{\nabla}_V W = \nabla_V W - \alpha(V, W) = \mathrm{pr}_L(\nabla_V W); \\ &\text{(ii) } \tilde{\nabla}_X V = \mathrm{pr}_L(\nabla_X V); \\ &\text{(iii) } \tilde{\nabla}_X Y = \nabla_X Y - \beta(X, Y) = \mathrm{pr}_Q(\nabla_X Y); \\ &\text{(iv) } \tilde{\nabla}_V X = \mathrm{pr}_Q(\nabla_V X); \end{aligned}$$

where $V, W \in \Gamma(L)$ are vertical and $X, Y, \in \Gamma(Q)$ are horizontal vector fields. The torsion of $\tilde{\nabla}$ is:

(2.2) $$\tilde{T}(X, Y) = -\operatorname{pr}_L[X, Y] \text{ for } X, Y \in \Gamma(Q), \text{ and is zero otherwise.}$$

The trace of γ is equal to the trace of α, which is, by definition, the mean curvature vector field of the leaves, and will be denoted by B. (B is horizontal, but is not necessarily a basic vector field.) The 1-form dual to B, with respect to the metric g_M, is denoted by κ.

As proved in [MRT], $\tilde{\nabla}$ restricted to basic vector fields coincides with the pullback of the Levi-Civita connection of the local Riemannian quotient B_U with metric g_Q and hence it is natural to define:

(2.3) $$\tilde{\delta}\omega(A_2, \ldots, A_r) = -\sum_{k=1}^{n}\Big\{e_k\big(\omega(e_k, A_2, \ldots, A_r)\big) - \omega(\tilde{\nabla}_{e_k}e_k, A_2, \ldots, A_r)\Big\} + \sum_{k=1}^{n}\sum_{i=2}^{r}\omega(e_k, \ldots, \tilde{\nabla}_{e_k}A_i, \ldots, A_r)$$

where ω is an r-form on $M, A_2, \ldots, A_r \in \Gamma(TM)$ and $\{e_k\}$ is a local orthonormal frame for TM. The definition is independent of the choice of the frame $\{e_k\}$.

This operator $\tilde{\delta}$, although not the exact adjoint of d still has the same symbol as δ_M, the adjoint of d on the compact Riemannian manifold M. The main property of $\tilde{\delta}$ is that it leaves the complex of basic form $\Omega_B^*(M)$ invariant and when restricted to basic forms it coincides locally with the usual operator δ on the local Riemannian quotient. Compare [MRT, Prop. 2.16]. We will simply denote $\tilde{\delta}|_{\Omega_B^*}$ by δ.

We also note that in case α is a 1-form the above formula can be written as $(\tilde{\delta} - \delta_M)\alpha = \kappa \vee \alpha$, where $\vee$, interior multiplication, is the adjoint of the exterior product $\wedge$.

The Laplacian defined by

(2.4) $$\tilde{\Delta} = d\tilde{\delta} + \tilde{\delta}d$$

although not self-adjoint, is still a second order elliptic operator with self-adjoint principal symbol, and differs from the Hodge-deRham Laplacian Δ of the metric g^M only by a lower order term determined by the tensor $\gamma = \tilde{\nabla} - \nabla$.

All the results in this paper are based on the following technical theorem.

MAIN THEOREM 2.5. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact manifold M^n with a bundle-like metric $\mathring{g}_M = g_L + g_Q$, where $L \subset TM$ is the distribution tangent to the leaves and $Q = L^{\perp}$. Then, there exists a smooth positive function ϕ on M such that, with respect to the measure μ_M defined by the Riemannian metric $g_M = \phi g_L + g_Q$, $\int_M \tilde{\Delta} f \mu_M = 0$ for any basic function f.*

For the rest of this section all computations will be done exclusively w.r.t. the conformally modified bundle-like metric g_M provided by Theorem 2.5. Note that g_M and $\mathring{g}_M$ define the same transversal Riemannian metric g_Q. This metric g_M induces an L^2-inner product on $\Omega^*(M)$, the space of all forms on M and hence by restriction also on the basic forms $\Omega_B = \Omega_B^*(\mathcal{F}) \subset \Omega^*(M)$. Let $\Omega^*(M) = \Omega_B \oplus \Omega_B^\perp$ be the L^2-orthogonal splitting. Let d^* and δ^* be the formal adjoints of d and $\tilde{\delta}$, *restricted to* Ω_B, with respect to the L^2-inner product. Since on basic forms, in terms of local distinguished charts U, the operator $\tilde{\delta}$ agrees with the usual δ of the local quotient B_U, we write δ^* instead of $\tilde{\delta}^*$. With this notation we also wish to emphasize that d^* and δ^* are defined on Ω_B only. Later, for technical reasons, it will be convenient to extend d^* and δ^* to $\Omega^*(M)$, the whole complex of all differential forms on M. In general, however, the extension of δ^* to $\Omega^*(M)$ will not coincide with the adjoint of $\tilde{\delta}$, considered as operators on $\Omega^*(M)$.

Next, we define the basic mean curvature $\tilde{\kappa}$. If the mean curvature of the leaves is a basic form as in [KT1], $\tilde{\kappa}$ agrees with the usual mean curvature form κ. In the general case, $\tilde{\kappa}$ coincides with the basic component of the mean curvature as defined, *e.g.*, in [A].

DEFINITION 2.6. $\tilde{\kappa} = -\delta^* 1$ (Basic mean curvature)

The calculations of [T, Chap. 12] apply in our more general situation and yield:

$$\delta^* = d - \tilde{\kappa}\wedge \tag{2.7}$$

$$d^* = \delta + \tilde{\kappa}\vee \tag{2.8}$$

on basic forms, where $\vee$ is the adjoint of $\wedge$.

The next proposition states that $\tilde{\kappa}$ is a closed form. If the mean curvature is basic, the corresponding assertion, $d\kappa = 0$, is usually proved by using Rummler's formula [Ru], [T, p. 66]. In our set-up, it will turn out to be a formal consequence of the definitions.

PROPOSITION 2.9. The basic mean curvature is a closed form, *i.e.*, satisfies $d\tilde{\kappa} = 0$.

PROOF. $d\tilde{\kappa} = (d - \tilde{\kappa}\wedge)\tilde{\kappa} = \delta^*\tilde{\kappa} = -\delta^*\delta^* 1 = -(\delta\delta)^* 1 = 0$, since $\delta\delta = 0$ on basic forms.

THEOREM 2.10. *The basic mean curvature satisfies* $d^*\tilde{\kappa} = 0$.

PROOF. $\langle d^*\tilde{\kappa}, f\rangle = \langle d^*\delta^* 1, f\rangle = \langle 1, \delta d f\rangle = \int_M \tilde{\delta}\, df\, \mu_M = 0$, for any basic function f.

REMARK. The identities $d\tilde{\kappa} = 0, d^*\tilde{\kappa} = 0$ assert that the mean curvature $\tilde{\kappa}$ (which, in the terminology of [A], is the basic component of the mean curvature) is harmonic with respect to the Laplacian $\Delta^* = dd^* + d^*d$. The reason for this, of course, is the conformal change of the metric in the leaf direction provided by the Main Theorem. We note, however, that $\delta\tilde{\kappa} = d^*\tilde{\kappa} - \tilde{\kappa} \vee \tilde{\kappa} = -|\tilde{\kappa}|^2$, and $\tilde{\kappa}$ is not harmonic with respect to $d\delta + \delta d$, where δ is given by the usual formula on a distinguished local chart B_U. In keeping with our point of view of ignoring the geometry of the leaves, we propose the following:

DEFINITION 2.11. $\mathcal{F}$ is taut iff $\tilde{\kappa} \equiv 0$.

A closer inspection of the geometry of the leaves will verify that this definition agrees with the usual definition of tautness for Riemannian foliations on compact manifolds. The condition $\tilde{\kappa} \equiv 0$, however, is simpler to state and verify.

Another application of the Theorem is a Bochner-Weitzenböck formula for basic differential forms. Our point of view is that we can apply the usual pointwise formulas of

Riemannian geometry on the local quotient and for integration, we use the modified metric provided by the Main Theorem, which is adapted to the extended Laplacian $\tilde{\Delta}$ that we use. The following Weitzenböck formula is well known:

$$\langle \omega, \Delta\omega \rangle = \frac{1}{2}\Delta(|\omega|^2) + |\nabla\omega|^2 + \langle \omega, \mathcal{R}(\omega)\rangle \tag{2.12}$$

where ω is a r-form, Δ is the Hodge-deRham-Laplacian, ∇ is the Levi-Civita connection, and the algebraic operator $\mathcal{R}$ is given by:

$$\mathcal{R} = -\sum e_a.\hat{R}(e_a). \tag{2.13}$$

where $\{e_a\}$ is an orthornormal base for 2-forms, $\hat{R}$ is the Riemannian curvature tensor viewed as a symmetric endomorphism on 2-forms and the dots indicate that the two forms e_a and $\hat{R}(e_a)$ are acting as skew-symmetric endomorphisms on r-forms.

In the case of a foliation with a bundle like metric, the local Weitzenböck formulas on distinguished charts are consistent, since the transition functions are local isometries. Using the extension $\tilde{\Delta}$ and the adapted metric g_M of Theorem 2.5 to integrate, we have $\int \tilde{\Delta}(|\omega|^2) = 0$. Since $\tilde{\Delta}$ agrees with the usual Laplacian $\Delta = d\delta + \delta d$ on the local Riemannian quotient B_U, we obtain:

PROPOSITION 2.14. If $\omega \in \Omega_B^*(\mathcal{F})$ is a harmonic form then

$$\int_M \big(|\nabla\omega|^2 + \langle \omega, \mathcal{R}(\omega)\rangle\big)\mu_M = 0$$

where μ_M is the volume form of the metric g_M, provided by Theorem 2.5.

As an application of this proposition, we obtain extensions of well known vanishing theorems of Riemannian geometry to the case of Riemannian foliations with bundle-like metrics. See [MRT] for such results. The Main Theorem also implies the following well known criterion for Poincaré duality to hold for the basic cohomology $H_B^*(\mathcal{F})$.

THEOREM 2.15. *If $H_B^q(\mathcal{F}) \neq 0$, where q is the codimension of the foliation, then Poincaré duality holds for the basic cohomology $H_B^*(\mathcal{F})$.*

PROOF. To prove Proposition 2.14, we show first that every cohomology class in $H_B^q(\mathcal{F})$ is represented by a $\tilde{\Delta}$-harmonic q-form. Let ω_0 denote an arbitrary basic q-form and let $\omega(t)$ be the solution of the heat equation

$$\Big(\frac{\partial}{\partial t} + \tilde{\Delta}\Big)\omega = 0 \qquad \text{with initial condition } \omega(0) = \omega_0.$$

If ν_Q denotes the transversal volume form defined by the metric g_Q and if we set $\omega(t) = u(t)\nu_Q$ then u satisfies

$$\Big(\frac{\partial}{\partial t} + \tilde{\delta d}\Big)u = 0 \qquad \text{with initial condition } u(0) = u_0.$$

The solution is given by $u(t) = P_t u_0 = e^{-t\tilde{\Delta}}u_0$, or in other words by

$$u(t,x) = \int P_t(x,y)u_0(y)\mu_M(dy)$$

where $P_t(x,y) = P_t\delta_x(y)$ is the heat kernal of $\tilde{\Delta}$ (δ_x denotes the Dirac delta function at x). By the results of the next section (Lemma 3.8), the solution $u(t)$ remains basic for all t, and in the limit as $t \longrightarrow \infty$, $u(t)$ converges to the constant function $\bar{u} = \int u_0\mu_M$.

This shows that any cohomology class in $H^q_B(\mathcal{F})$ is represented by $c\nu_Q$ for some constant c. Since $H^q_B(\mathcal{F}) \neq 0$ by assumption, $\dim\big(H^q_B(\mathcal{F})\big) = 1$. After making a choice of a generator $[\![\nu_Q]\!]$ for $H^q_B(\mathcal{F}) \approx \mathbb{R}$ we define the cohomological inner product $(\alpha,\beta)_{\text{cohom}}$ of two basic forms α and $\beta \in \Omega^k_B(\mathcal{F})$ by:

$$[\![\alpha \wedge *\beta]\!] = (\alpha,\beta)_{\text{cohom}}[\![\tilde{\nu}_Q,]\!] \tag{2.16}$$

where $[\![\ ]\!]$ denotes the cohomology class and $*$ is the transversal Hodge duality operator on basic forms defined locally in distinguished charts by means of the transversal volume form ν_Q. The local definitions are consistent on overlaps, since the transition functions are isometries of the local Riemannian quotient. (In [T], $*$ is denoted by $\bar{*}$). If $\alpha\wedge *\beta = f\nu_Q$, then the L^2-inner product induced on Ω_B by g_M is given by $\langle\alpha,\beta\rangle = \int f\mu_M$ and since the measure μ_M is invariant under the flow $P_t = e^{-t\tilde{\Delta}}$, (see the next section (3.4)), the cohomological product $(\ ,\ )_{\text{cohom}}$ coincides (up to a normalization constant) with the L^2-inner product. In particular, it is a *non-degenerate* pairing and hence Poincaré duality holds for the basic cohomology. Finally, we also recover the well known Hodge decomposition of basic forms for the pair of adjoint operators d, d^* and analogously also for the other pair δ, δ^*.

THEOREM 2.17. *Let d be the exterior derivative on basic forms, and d^* be its adjoint. Then*

$$\Omega^*_B(\mathcal{F}) = \mathcal{H}^*_B \oplus d\big(\Omega^*_B(\mathcal{F})\big) \oplus d^*\big(\Omega^*_B(\mathcal{F})\big),$$

where $\mathcal{H}^*_B$ denotes harmonic forms.

PROOF. The proof rests on the fact that the pair of adjoint operators d^* and δ^* which are defined on the basic forms Ω_B can be extended to a pair of adjoint elliptic operators defined on the space of all forms $\Omega(M)$ as follows:

First we write $\tilde{\delta} - \delta_M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with respect to the L^2-orthogonal splitting $\Omega(M) = \Omega_B \oplus \Omega_B^{\perp}$ and define:

$$\tilde{d}^* = \delta_M + \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \tilde{\delta} - \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \tag{2.18}$$

where δ_M is the adjoint of d on $\Omega(M)$. $\tilde{d}^*$ restricted to basic forms coincides with d^*. Similarly writing $\tilde{\delta}^* - d = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix}$ and setting:

$$\delta^* = d + \begin{bmatrix} A^* & 0 \\ 0 & D^* \end{bmatrix} = \tilde{\delta}^* - \begin{bmatrix} 0 & B^* \\ C^* & 0 \end{bmatrix} \tag{2.19}$$

defines an extension of δ^*. Denoting these extended operators by the same symbols, we define the Laplacian:

$$\Delta^* = d^*\delta^* + \delta^* d^* \tag{2.20}$$

https://doi.org/10.4153/CMB-1996-012-4 Published online by Cambridge University Press

which is self-adjoint and leaves Ω_B^* invariant. Moreover by Lemma 3.10 of the next section, the operators $A, A^*, B, \ldots$, are all bounded operators and therefore Δ^* is a bounded perturbation of the standard Laplacian Δ_M and hence elliptic.

Let ω_0 be a closed basic form. The heat flow

$$\left(\frac{\partial}{\partial t} + \Delta^*\right)\omega = 0 \qquad \text{with initial condition } \omega(0) = \omega_0 \in \Omega_B^*(\mathcal{F}) \tag{2.21}$$

has a solution $\omega(t)$, which is also closed and basic for $0 \le t < \infty$. Furthermore, the cohomology class $[\![\omega(t)]\!]$ stays constant and since Δ^* is a self-adjoint and non-negative elliptic operator, the limit $\omega_\infty = \lim_{t\to\infty} \omega(t) \in [\![\omega_0]\!]$ exists, and is harmonic with respect to Δ^*. This can be checked in local distinguished coordinates, since in these coordinates the operator Δ^* coincides with the usual Hodge-deRham Laplacian of the local Riemannian quotient. The Hodge decomposition now follows by standard arguments.

As a corollary of Theorem 2.17, we obtain the following characterization of the vanishing of the basic mean curvature $\tilde{\kappa}$. This corresponds to a minimality theorem of Masa [Ma].

THEOREM 2.22. *If q is the codimension of the foliation, then the basic mean curvature $\tilde{\kappa}$ vanishes if and only if $H_B^q(\mathcal{F}) \neq 0$.*

PROOF. If $\tilde{\kappa} \equiv 0$, then by (2.8) $d^* = \delta$, and the transversal volume form ν is harmonic and represents a non-zero element in $H_B^q(\mathcal{F})$ by Theorem 2.17. On the other hand, if $H_B^q(\mathcal{F}) \neq 0$, then there is a basic function, positive at some point of M, such that $f\nu$ is harmonic. $d^*(f\nu) = 0 \Rightarrow df = \tilde{\kappa} f \vee \nu \Rightarrow |df| > \text{const}\,|f| \Rightarrow f$ is strictly positive on the compact manifold M. Since $\tilde{\kappa} = d(\log f)$ and $\tilde{\kappa}$ is harmonic, 2.17 implies that $\tilde{\kappa} \equiv 0$.

REMARK. Theorem 2.22 is also proved in [A]. Note that in contrast to [A], we have arranged $\tilde{\kappa}$ to be harmonic. As is pointed by Alvarez Lopez [A], there are basically 2 methods to deal with problems on Riemannian foliations. The first relies heavily on the structure theorem of Molino and the second uses the mean curvature form. Masa's paper [Ma] is an example of the first method and [A] is an example of the second, although Masa's minimality theorem is used to prove the equivalent of Theorem 2.22 in [A, Cor. 6.2]. As mentioned in the introduction, the purpose of this paper is to show that an independent and intrinsically Riemannian approach is strong enough on its own. We would like to thank the referee for pointing out that we should mention and compare these different methods.

3. **Proof of the Main Theorem.** To find the function ϕ required in Theorem 2.5, we study the diffusion semi-group

$$P_t = e^{-t\tilde{\Delta}} \tag{3.1}$$

acting on functions. The operator $\tilde{\Delta} = \tilde{\delta} d$ acting on a function f is given by

$$\tilde{\Delta} f = \Delta_M f - B(f) \tag{3.2}$$

where $\Delta_M f = -\text{div}\,(\text{grad} f)$ is the standard self-adjoint Laplacian of the metric g_M and the drift term $B(f)$ is the directional derivative of f in the direction of the mean curvature vector field B, which is dual to κ. The adjoint operator $\tilde{\Delta}^*$ of $\tilde{\Delta}$ is given by:

$$\tilde{\Delta}^* f = \Delta_M f + B(f) - \text{div}\,(B).f \tag{3.3}$$

where the divergence is with respect to the volume form $\mathring{\mu}_M$ of the initial metric $\mathring{g}_M$. We prove below that the diffusion semi-group P_t has an invariant measure μ_M; that is a probability measure on M such that for all $t > 0$ and $f \in L^1(\mathring{\mu}_M)$,

$$\int P_t f \mu_M = \int f \mu_M. \tag{3.4}$$

We prove as well that μ_M has a smooth strictly positive density ψ which satisfies on account of (3.4), the equation:

$$\int (\tilde{\Delta} f)\psi \mathring{\mu}_M = 0, \text{ for all } f \in L^1(\mathring{\mu}_M). \tag{3.5}$$

which is a weak form of the adjoint equation

$$\tilde{\Delta}^* \psi = 0 \tag{3.6}$$

Indeed, we find μ by first solving (3.6) for ψ, using the Krein-Rutman theorem [KR] as applied to the resolvent operators $A_\lambda = (\tilde{\Delta} + \lambda)^{-1}$ acting on the scale of Sobolev spaces of M. Finally, the proof of the theorem is a consequence of the dual role of the μ_M as the invariant measure for P_t and as the volume form of the conformally changed metric $g_M = \phi g_L + g_Q$, where $\phi = \psi^{2/p}$.

LEMMA 3.7. *There is a unique smooth solution $\psi \in C^\infty(M)$ of the elliptic equation*

$$\tilde{\Delta}^* \psi = \Delta_M \psi + B(\psi) - \text{div}\,(B).\,\psi = 0$$

satisfying

$$\textit{(i)}\quad \psi > 0 \qquad \textit{and} \qquad \textit{(ii)}\quad \int \psi \mathring{\mu}_M = 1.$$

PROOF. For $k = 0, 1, \ldots$, let H^k denote the Sobolev space of functions on M obtained by completing $C^\infty(M)$ with respect to the norm:

$$\|f\|_k^2 = \sum_{i=0}^{k} \int_M |\nabla^{(i)} f|^2 \mu_M,$$

where ∇ is the Levi-Civita connection of g_M.

By standard elliptic theory, $\tilde{\Delta} + \lambda$ is a bounded operator from H^{k+2} to H^k for any $\lambda \geq 0$ and is invertible if λ is sufficiently large. Choose $k > n$ so that $H^k \subset C^2(M)$ and also λ large enough so that, by Rellich's lemma, the resolvent $A_\lambda = (\tilde{\Delta} + \lambda)^{-1}$ is a compact operator on H^k.

We can now apply the Krein-Rutman theorem [KR] to A_λ relative to the positive cone:

$$K = \{f \in H^k \mid f > 0 \text{ on } M\}.$$

If $f \geq 0$, then $u = A_\lambda f$ satisfies $(\tilde{\Delta} + \lambda)u = f \geq 0$, and hence by the strong maximum principle for the operator $(\tilde{\Delta} + \lambda)$, u is everywhere ≥ 0 with u vanishing at a point iff f vanishes identically. This shows that A_λ is a strongly positive operator on H^k with respect to the cone K. The theorem of Krein-Rutman [KR, Thm. 6.3] asserts that:

(a) the top of the spectrum of A_λ and the top of the spectrum of its adjoint A_λ^*, acting on $H^{-k} = (H^k)^*$, are attained at the same simple eigenvalue $\rho_\lambda > 0$;

(b) the cone K contains, up to scalar multiples, exactly one eigenvector f_λ for A_λ and f_λ has eigenvalue ρ_λ;

(c) the dual cone K^* contains, up to scalar multiples, exactly one eigenvector ψ_λ for A_λ^* and ψ_λ has eigenvalue ρ_λ. Furthermore, ψ_λ is strictly positive in the sense that $\langle \psi_\lambda, f \rangle \geq 0$ for all $f \in \bar{K}$, with equality only if $f \equiv 0$.

Since the constant function 1 satisfies $(\tilde{\Delta} + \lambda)1 = \lambda 1$, we have $1 = \lambda A_\lambda 1$; and since $1 \in K$, it follows from (b) above that $\rho_\lambda = \frac{1}{\lambda}$. This fact and the resolvent identity

$$A_{\lambda_2} - A_{\lambda_1} = (\lambda_1 - \lambda_2) A_{\lambda_2} A_{\lambda_1}$$

now yield

$$A_{\lambda_1}^* \psi_{\lambda_2} = \lambda_2 A_{\lambda_1}^* A_{\lambda_2}^* \psi_{\lambda_2} = \frac{\lambda_2}{\lambda_1 - \lambda_2}(A_{\lambda_2}^* - A_{\lambda_1}^*)\psi_{\lambda_2}.$$

Hence,

$$(1 + \frac{\lambda_2}{\lambda_1 - \lambda_2}) A_{\lambda_1}^* \psi_{\lambda_2} = \frac{\lambda_2}{\lambda_1 - \lambda_2} A_{\lambda_2}^* \psi_{\lambda_2} = \frac{1}{\lambda_1 - \lambda_2} \psi_{\lambda_2}$$

and so

$$A_{\lambda_1}^* \psi_{\lambda_2} = \frac{1}{\lambda_1} \psi_{\lambda_2}.$$

Therefore the eigenfunction $\psi = \psi_\lambda$ is actually independent of λ. The identity $A_\lambda^* \psi = \frac{1}{\lambda}\psi$ implies $\psi = \frac{1}{\lambda}(\tilde{\Delta}^* + \lambda)\psi$ and hence $\tilde{\Delta}^* \psi = 0$. Now, ψ is a smooth function, since it is a solution of an elliptic equation and we can normalize it to satisfy $\int \psi \mathring{\mu}_M = 1$. It is then uniquely determined. This also follows from the fact that $\dim\big(\mathrm{Ker}(\tilde{\Delta}^*)\big) = \dim\big(\mathrm{Ker}(\tilde{\Delta})\big)$ by index theory and $\dim\big(\mathrm{Ker}(\tilde{\Delta})\big) = 1$ by the maximum principle. This proves the lemma.

We realize this invariant measure as the volume form of a Riemannian metric on M by making the following conformal change of the original metric along the leaves L of the foliation. Let $\phi = \psi^{2/p}$, and define $g_M = \phi g_L + g_Q$. Then the volume form of g_M is the invariant measure μ_M with the property:

$$\int_M \tilde{\Delta} f \mu_M = \int_M (\tilde{\Delta} f)\psi \mathring{\mu}_M = \int_M f(\tilde{\Delta}^* \psi)\mathring{\mu}_M = 0$$

for any function $f \in L^1(\mathring{\mu}_M)$ and the main Theorem 2.5 is now proved. Notice that L is still perpendicular to Q and the transversal structure g_Q is left unchanged.

We also need the following:

LEMMA 3.8. *For any given smooth initial basic function u_0 on $M, u(t) = P_t u_0 = e^{-t\tilde{\Delta}}u_0$ exists for all $t > 0$. The solution $u(t)$ remains basic for all t and converges smoothly to a constant function $\bar{u} = \int u_0 \mu_M$ as $t \longrightarrow \infty$.*

PROOF. The solution $u(t)$ exists for all t, since $\tilde{\Delta}$ is a linear elliptic operator with a self-adjoint symbol and it remains basic because $\tilde{\Delta}$ preserves Ω_B. Since $\tilde{\Delta}$ has the form (3.2), it satisfies the maximum principle on the compact manifold M. This implies, in particular, that the solution $u(t)$ stays bounded in the C^0-norm for all time and hence by parabolic regularity also in every higher order Sobolev-norm for say $t \geq 1$. This implies convergence as $t \longrightarrow \infty$ to a $\tilde{\Delta}$-harmonic form, which has to be a constant $\bar{u}$, again by the maximum principle. Thus, $\bar{u} = \int u_0 \mu_M$ since by (3.4) the L^1-norm is preserved by the flow.

We now turn to some properties of the projection operator $\Omega^*(M) \longrightarrow \Omega^*_B(\mathcal{F})$. We first deal with the case of functions and recall a few well known facts about conditional expectation. Let X be a topological space, $\mathcal{X}$ the σ-algebra of Borel subsets of X and let μ be a finite measure on $\mathcal{X}$. Let $A_2 \subset A_1$ be algebras of real valued continuous functions on X and let $\mathcal{A}_2 \subset \mathcal{A}_1$ be the σ-algebras of subsets which they generate; *i.e.*, $\mathcal{A}_i$ is the smallest σ-algebra for which every function of A_i is measurable, for $i = 1, 2$. Then the spaces $L^p(X, \mathcal{A}_i, \mu)$ are exactly the norm closure of the algebra A_i in $L^p(X, \mathcal{X}, \mu)$. In this situation the following fact is well known (Compare [N]).

LEMMA 3.9.

(i) *For every $f \in L^1(X, \mathcal{A}_1, \mu)$ there exists a unique element $\hat{f} \in L^1(X, \mathcal{A}_2, \mu)$ such that*
$$\int_X g\hat{f}\,d\mu = \int_X gf\,d\mu \qquad \text{for every } g \in L^\infty(X, \mathcal{A}_2, \mu).$$

(ii) *The function $f \longmapsto \hat{f}$, when restricted to $L^p(X, \mathcal{A}_1, \mu)$ is a projection operator of norm 1 for all $p \geq 1$.*

(iii) *For $p = 2$, $f \longmapsto \hat{f}$ is the orthogonal projection of $L^2(X, \mathcal{A}_1, \mu)$ onto $L^2(X, \mathcal{A}_2, \mu)$.*

PROOF.

(i) If $f \in L^1(X, \mathcal{A}_1, \mu)$, then the set function $\nu(E) = \int_E f\,d\mu$ is absolutely continuous with respect to μ, when both are considered as measures on the smaller σ-algebra $\mathcal{A}_2$. By the Radon-Nikodym theorem there is a unique (up to sets of μ-measure zero) function $\hat{f}$, measurable in $\mathcal{A}_2$, such that $\nu(E) = \int_E \hat{f}\,d\mu$ for all $E \in \mathcal{A}_2$. But this means that $\int_X \chi_E \hat{f}\,d\mu = \int_X \chi_E f\,d\mu$, where $\chi_E \in L^\infty(X, \mathcal{A}_2, \mu)$ is the characteristic function of E. By linearity and dominated convergence this equality holds for all $g \in L^\infty(X, \mathcal{A}_2, \mu)$.

(ii) Since μ is a finite measure $L^p(X, \mathcal{A}_i, \mu) \subset L^1(X, \mathcal{A}_i, \mu)$ for all $p \geq 1$. For $f \in L^p(X, \mathcal{A}_2, \mu)$, $\hat{f} = f$ by uniqueness and therefore $f \longmapsto \hat{f}$ is an idempotent linear operator. We compute its norm:
$$\begin{aligned}\|\hat{f}\|_{L^p(X,\mathcal{A}_1,\mu)} &= \|\hat{f}\|_{L^p(X,\mathcal{A}_2,\mu)}\\ &= \sup\Big\{\int_X g\hat{f}\,d\mu \mid g \in L^\infty(X, \mathcal{A}_2, \mu),\ \|g\|_{p^*} \leq 1\Big\}\\ &= \sup\Big\{\int_X gf\,d\mu \mid g \in L^\infty(X, \mathcal{A}_2, \mu),\ \|g\|_{p^*} \leq 1\Big\} \leq \|f\|_{L^p(X,\mathcal{A}_1,\mu)}\end{aligned}$$

and since $\hat{f} = f$ for $f \in L^\infty(X, \mathcal{A}_2, \mu)$, the norm of the operator $f \longmapsto \hat{f}$ is 1.

(iii) For $f, h \in L^2(X, \mathcal{A}_1, \mu)$, $\int_X h\hat{f}\, d\mu = \int_X \hat{h}\hat{f}\, d\mu = \int_X \hat{h} f\, d\mu$.

We can now apply the above Lemma to the compact Riemannian manifold (M, g_M) with $A_1 = \Omega^0(M)$ and $A_2 = \Omega^0_B(M)$ to get the L^p-estimates that we needed in the previous section in the case of functions. In order to get the same estimates for all forms, we simply regard forms on M as $O(n)$-equivariant functions on the total space P of the frame bundle of (M, g_M), with values in the exterior algebra $\Lambda^*(\mathbb{R}^n) = \mathbb{R}^{2^n}$. They form an (graded commutative) algebra of functions A_1 on P. We let A_2 be the subalgebra of all forms that are basic and apply the above Lemma.

LEMMA 3.10. *The orthogonal projection map $L^2\big(\Omega^*(M)\big) \longrightarrow L^2\big(\Omega^*_B(M)\big)$ is continuous with respect to any L_p-norm, for $p \geq 2$. In fact, the L_p-norm is equal to* 1.

This Lemma was used in the proof of Theorem 2.16 and it also shows that the basic mean curvature $\tilde{\kappa}$ is bounded in any L_p-norm $p \geq 1$. Therefore by elliptic regularity $\tilde{\kappa}$ is smooth. This fact is also proved in [A].

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https://doi.org/10.4153/CMB-1996-012-4 Published online by Cambridge University Press