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ON THE HYPERPLANE SECTIONS OF BLOW-UPS OF COMPLEX PROJECTIVE PLANE

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Introduction. Let L be a line bundle on a connected, smooth, algebraic, projective surface X. In this paper we have studied the following questions:

1) Under which conditions is L spanned by global sections? I.e., if $\phi_L : X \longrightarrow \mathbf{P}^N$ denotes the map associated to the space $\Gamma(L)$ of the sections of L, when is ϕ_L a morphism?

2) Under which conditions is L very ample? I.e., when does ϕ_L give an embedding?

These problems arise naturally in the study, and in particular in the classification, of algebraic surfaces (see [**8**], [**3**], [**5**]).

In particular we have restricted our attention to the case in which X is gotten by blowing up s distinct points $x_1, \dots, x_s \in \mathbf{P}^2$. If we denote by $P_1, \dots, P_s$ the corresponding exceptional curves then a line bundle L on X is of the form

$$L \equiv \pi^* O_{\mathbf{P}^2}(d) - \sum_{j=1,\dots,s} t_j P_j$$

where $\pi : X \longrightarrow \mathbf{P}^2$ is the blowing up morphism with center $x_1, \dots, x_s$.

It was classically known that if

$$L \equiv \pi^* O_{\mathbf{P}^2}(3) - \sum_{j=1,\dots,s} P_j,$$

with $x_1, \dots, x_s$ in sufficiently general position, then L is very ample if $s \leqq 6$ and L is spanned by global sections if $s = 7$.

Partial answers to questions (1) and (2) are in [**1**] when $t_1 = \dots = t_s = 1$, in [**7**] when $s = 9$, in [**9**], [**10**], [**11**] when $h^0(L) = 5$.

Note that in our paper we obtain again the very ampleness of

$$L \equiv \pi^* O_{\mathbf{P}^2}(4) - \sum_{j=1,\dots,10} P_j$$

which gives the Bordiga surface in $\mathbf{P}^4$, see [**13**], [**6**], [**9**].

Further applications of our results can be found in [**8**].

In Section 0 we explain our notation and collect background material.

In Section 1 we give a modified version of the Beauville-Reider theorem.

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In Sections 2 and 3 we give sufficient conditions under which L is spanned or very ample.

The similar questions in the case of a ruled surface are examined in [**2**].

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0. Background material. (0.0) Let L be a line bundle on a smooth connected projective surface X. Let $M = L - K_X$, where K_X is the canonical line bundle on X.

(0.1) In order to simplify our notations we give the following definitions: Let X and L be as in (0.0).

1. We say that L is "0-very ample" if L is spanned by global sections;
2. We say that L is "1-very ample" if L is very ample.

THEOREM 0.2. *Let X, L and M be as in* (0.0). *Assume that*

1) *M is big and nef*

2) $M^2 \geqq 5 + 4i, i + 0, 1$

3) *L is not i-very ample.*

Then there is an effective divisor E on X such that

$$M \cdot E - 1 - i \leqq E^2 < M \cdot E/2 < 1 + i.$$

Proof. See [**12**, Theorem 1, pg. 310].

1. Some implications of Reider's method. (1.0) Let L be a line bundle on a smooth connected projective surface X. Let $M = L - K_X$.

Definition 1.0.1. For every $m \in \mathbf{N}$, denote by $\mathcal{D}_m$ the set of all divisors $E \subseteq X$, such that $E \not\equiv 0$ and mE is effective. Moreover we set

$$\mathcal{D} = \bigcup_{m \in \mathbf{N}} \mathcal{D}_m \quad \text{and} \quad \mathcal{D}_M = \{E \in \mathcal{D}_1 \mid M - 2E \in \mathcal{D}\}.$$

THEOREM 1.1. (Reider): *Let:*

1) $M \in \mathcal{D}$

2) $M^2 \geqq 5 + 4i$

3) $(M - E) \cdot E \geqq 2 + i$ *for any $E \in \mathcal{D}_M$ and $i = 0, 1$.*

Then L is i-very ample.

Proof. This is essentially the same as in Theorem (0.2).

(1.2) Let $E \in \mathcal{D}_1$. Then $E = E_1 + \cdots + E_k$ where $E_j, j = 1, \ldots, k$ are all the irreducible and reduced components of E. Denote by $\mathcal{E}_i, i = 0, 1$, the set of all $E \in \mathcal{D}_1$ such that either $k = 1$ or if $k \geqq 2$ then the following inequalities must be satisfied

$$(1.2.1) \qquad \sum_{j=1,\ldots,k} E_j \cdot (E - E_j) \geqq (K - 1)(2 + i) + 1$$

and

(1.2.2) $E' \cdot E'' \geqq 2 \qquad$ if $E = E' + E''$ and $E', E'' \in \mathcal{D}_1$.

LEMMA 1.2.3. *If any* $E \in \mathcal{E}_i \cap \mathcal{D}_M, i = 0, 1$, *verify the inequality*

(1.2.4) $(M - E) \cdot E \geqq 2 + i$

then (1.2.4) *holds also for any* $E \in \mathcal{D}_M$.

Proof. Let $E = E_1 + \cdots + E_k \in \mathcal{D}_M$. where $E_j, j = 1, \ldots, k$, are all the irreducible and reduced components of E. Then $E_j \in \mathcal{E}_i \cap \mathcal{D}_M$. Assume that $E \notin \mathcal{E}_i$. Then $k \geqq 2$. If (1.2.1) is not satisfied then

$$\begin{aligned}(M - E) \cdot E &= \sum_{j=1,\ldots,k} (M - E_j) \cdot E_j - \sum_{j=1,\ldots,k} E_j \cdot (E - E_j) \\ &\geqq k(2 + i) - (k + 1)(2 + i) \geqq 2 + i, \quad i = 0, 1,\end{aligned}$$

i.e., (1.2.4) holds. Assume now that (1.2.2) does not hold. We proceed by induction on k. Let $k = 2$. If (1.2.1) is not satisfied then we are in the above case and thus (1.2.4) holds. Suppose that (1.2.1) is satisfied. Since for $k = 2$ (1.2.1) implies (1.2.2) then (1.2.4) is satisfied by assumption. We now assume that for any $k' \leqq k - 1$ the statement is true. Since (1.2.2) is not satisfied there are $E', E'' \in \mathcal{D}_1$ such that $E' + E'' = E = E_1 + \cdots + E_k$ and $E' \cdot E'' \leqq 1$. Then E' and E'' satisfy (1.2.4) and we have

$$(M - E) \cdot E = (M - E') \cdot E' + (M - E'') \cdot E'' - 2E' \cdot E'' \geqq 2 + 2i.$$

Thus E satisfies (1.2.4).

LEMMA 1.3. *Let* $E \in \mathcal{E}_i, i = 0, 1$. *Then* $g(E) \geqq 0$, *where*

$$g(E) = 1 + (E + K_X) \cdot E/2.$$

Proof. Let $E = E_1 + \cdots + E_k \in \mathcal{D}_1$ where $E_j, j = 1, \ldots, k$ are all the irreducible and reduced components of E. Assume that $g(E) < 0$. Then $k \geqq 2$. Moreover, since

$$g(E) = \sum_{j=1,\ldots,k} g(E_j) - (k - 1) + 1/2 \sum_{j=1,\ldots,k} E_j \cdot (E - E_j)$$

where $g(E_j) \geqq 0$ we have

$$\sum_{j=1,\ldots,k} E_j \cdot (E - E_j) < 2(k - 1) \leqq (k - 1)(2 + i)$$

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which implies $E \notin \mathcal{E}_i$. Thus we have a contradiction.

Remark 1.3.1. Let $E \in \mathcal{D}_1$. Then
1) $(M-E)\cdot E = L\cdot E - 2g(E)+2$
2) If $g(E)=0$ then $E \in \mathcal{E}_i$ if and only if E is smooth. Moreover if L is i-very ample then $L\cdot E \geqq i$.

LEMMA 1.3.2. *Let $E \in \mathcal{D}_M, g(E)=1$ and L be very ample. Then $L\cdot E \geqq 3$.*

Proof. Since L is very ample then $L\cdot E \geqq 1$. If $L\cdot E = 1$ then E is a line relative to L while if $L\cdot E=2$ then E is a conic relative to L. In both cases we have a contradiction since $g(E)=1$.

(1.4) Let $E \in \mathcal{D}_M$. Since

$$M^2 = 4E\cdot(M-E)+(M-2E)^2 \tag{1.4.1}$$

then $E\cdot(M-E) \geqq 2+i$ if and only if $M^2 \geqq 5+4i+(M-2E)^2$. Moreover from (1.4.1) assuming

$$\begin{cases} M^2 \geqq 5+4i \\ (M-E)\cdot E \leqq 1+i \end{cases} \tag{1.4.2}$$

then

$$(M-2E)^2 \geqq 1. \tag{1.4.3}$$

LEMMA 1.4.4. *Let $E \in \mathcal{D}_M, i=0,1$. Assume that*

$$E^2 \geqq 0 \text{ and } (M-2E)\cdot E \geqq 0. \tag{1.4.5}$$

and that (1.4.2) *holds. Then one of the following is satisfied*
1) $i=0, E^2=0, M\cdot E=1$
2) $i=1, E^2=0, M\cdot E=1,2$
3) $i=1, E^2=1, M\equiv 3E$.

Proof. From (1.4.2) and (1.4.5) it follows that

$$0 \leqq E\cdot(M-2E) \leqq 1+i-E^2$$

which combined with Hodge Index Theorem, (1.4.5) and (1.4.3) gives

$$E^2 \leqq E^2\cdot(M-2E)^2 \leqq (E\cdot(M-2E))^2 \leqq (1+i-E^2)^2. \tag{1.4.6}$$

Moreover

$$M\cdot E > 2E^2. \tag{1.4.7}$$

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In fact if $M \cdot E = 2E^2$ then, by Hodge Index Theorem, $M - 2E \equiv \lambda E$ for some $\lambda \in \mathbf{Q}$. Thus $E^2 = 0$ and again, by Hodge Index Theorem, we get $M \equiv \mu E$ for some $\mu \in \mathbf{Q}$. Thus $M^2 = 0$ which contradicts (1.4.2). Applying now (1.4.6) and (1.4.7) we get the statement.

LEMMA 1.4.8. *Let $M^2 \geqq 5 + 4i$ and let $E^2 \geqq -1$ for any $E \in \mathcal{E}_i \cap \mathcal{D}_M$ such that $g(E) = 0$. If there is $E \in \mathcal{E}_i \cap \mathcal{D}_M$ such that $g(E) = 1, E^2 = 0$ and $1 \leqq M \cdot E \leqq 1 + i$, then L is not i-very ample.*

Proof. We have

$$M \cdot E = (M - E) \cdot E = L \cdot E - 2g(E) + 2 = L \cdot E.$$

If $i = 1$ the statement follows from (1.3.2). If $i = 0$ then $M \cdot E = L \cdot E = 1$. Let

$$E = E_1 + \cdots + E_k \in \mathcal{E}_i \cap \mathcal{D}_M,$$

where $E_j, j = 1, \ldots, k$ are all the irreducible and reduced components of E. We study the two cases $k = 1$ and $k \geqq 2$. Let $k = 1$. If E is smooth it follows immediately that L is not spanned. If E is not smooth then there is a singular point $P \in E$. Since if P is a base point we are done, we can suppose that P is not a base point. We have

$$\dim|L - P| = \dim|L| - 1.$$

Furthermore $D'.E \geqq 2$ for any $D' \in |L - P|$. Hence $|L - P| = |L - E|$. If $D \in |L| - |L - P|$ then $Q \in D \cap E$ is a base point. Thus also in this case L is not spanned. Let $k \geqq 2$. Since

$$\begin{aligned} 1 = g(E) &= \sum_{j=1,\ldots,k} g(E_k) - (k-1) + 1/2 \sum_{t=1,\ldots,k} E_t \cdot (E - E_t) \\ &\geqq \sum_{t=1,\ldots,k} g(E_t) + 1 \end{aligned}$$

then $g(E_t) = 0$ for $t = 1, \ldots, k$. Moreover

$$0 = E^2 = \sum_{t=1,\ldots,k} E_t + \sum_{t=1,\ldots,k} E_t \cdot (E - E_t) \geqq -k + 2k = k > 1$$

which gives a contradiction.

2. Rational surfaces. (2.0) Let $x_1, \ldots, x_s$ be distinct points on $\mathbf{P}^2$. Let $\pi : X \longrightarrow \mathbf{P}^2$ expresses X as $\mathbf{P}^2$ with $x_1, \ldots, x_s$ blown up. Denote by $P_j = \pi^{-1}(x_j), j = 1, \ldots, s$ the corresponding exceptional curves. We set

$$L = \pi^*(O_{\mathbf{P}^2}(d)) \otimes [P_1]^{-t_1} \otimes \cdots \otimes [P_S]^{-t_s} \quad \text{and} \quad M = L \otimes K_X$$

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where $t_1, \ldots, t_S \in \mathbf{N}$. Without loss of generality we can assume that $t_1 \geqq \cdots \geqq t_S$. If

$$r \in |\pi^*(O_{\mathbf{P}^2}(1))|$$

then

$$L \equiv dr - \sum_{j=1,\ldots,s} t_j P_j \quad \text{and} \quad M \equiv (d+3)r - \sum_{j=1,\ldots,s} (t_j+1)P_j.$$

Throughout the rest of the paper we will suppose X, L and M being as in (2.0).

LEMMA 2.0.1. *Let $M^2 > 0$ and $d \geqq 0$. Then $M \in \mathcal{D}$.*

Proof. From the Riemann-Roch Theorem it follows that

$$h^0(\alpha M) \geqq \chi(O_X) + (1/2)(\alpha^2 M^2 - \alpha M \cdot K_X) > 0$$

for $\alpha \gg 0$, since

$$h^2(\alpha M) = h^0(K_X - \alpha M) = 0.$$

(2.1) Denote by $\mathcal{D}^*$ the set of all divisors

$$E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j$$

on X such that $y \geqq 0$ and $\alpha_j \leqq y$. Then $\mathcal{D}^* \supseteq \mathcal{D}$. Moreover if we write

$$\mathcal{D}'_M = \{E \in \mathcal{D}_1 | M - 2E \in \mathcal{D}^*\}$$

then $\mathcal{D}'_M \supseteq \mathcal{D}_M$. Let now

$$E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j \in \mathcal{E}_i \cap \mathcal{D}_M, i = 0, 1$$

and let

$$M - 2E \equiv xr - \sum_{j=1,\ldots,s} \lambda_j P_j,$$

i.e., $x = d + 3 - 2y, \lambda_j = t_j + 1 - 2\alpha_j$. Since $E, M - 2E \in \mathcal{D}^*$ then

$$0 \leqq y \leqq (d+3)/2 \quad \text{and} \quad (t_j + 1 - x)/2 \leqq \alpha_j \leqq y.$$

Remark 2.1.1. In view of (1.4.3), if $M^2 \geqq 5 + 4i$ and if $(M - E) \cdot E \leqq 1 + i$, then $x \geqq 1$.

LEMMA 2.1.2. *Let* $M^2 \geqq 5+4i$ *and let* $E \in \mathcal{E}_i \cap \mathcal{D}_M$ *such that* $E^2 \geqq 0$. *If* $E \cdot (M-E) \leqq 1+i$ *then one of the following is verified:*
1) $i = 0, E^2 = 0, M \cdot E = 1$
2) $i = 1, E^2 = 0, M \cdot E = 1, 2$
3) $i = 1, E^2 = 1, M \equiv 3E$.

Proof. By (1.4.4) we have to prove only that $E \cdot (M-2E) \geqq 0$. If

$$E \cdot (M-2E) = xy - \sum_{j=1,\dots,s} \alpha_j \lambda_j < 0$$

then from (1.4.3) it follows that

$$x^2y^2 < \left(\sum_{j=1,\dots,s} \alpha_j\lambda_j\right)^2 \leqq \left(\sum_{j=1,\dots,s} \alpha_j^2\right)\left(\sum_{j=1,\dots,s} \lambda_j^2\right) \leqq (y^2 - E^2)(x^2-1)$$

i.e.,

$$0 < E^2 - E^2x^2 - y^2 = E^2x^2 - \sum_{j=i,\dots,s} \alpha_j^2 \leqq 0.$$

Hence we get a contradiction.

LEMMA 2.1.3. *Let* $M^2 \geqq 5+4i$ *and let*

(2.1.4) $E_M \equiv [(d+3)/2]r - \sum_{j=1,\dots,s} [(t_j+1)/2]P_j.$

If E_M *is effective then*

(2.1.5) $E_M \cdot (M - E_M) \geqq 2 + i$

if and only if one of the following holds:
1) $M^2 \geqq 6+4i$
2) $d+3$ *is even*
3) *if* η *is the number of* $j \in \{1,\dots,s\}$ *such that* t_j *is even then* $\eta \geqq 1$.

Proof. From (1.4.1) it follows that

(2.1.6) $E_M \cdot (M - E_M) = (1/4)(M^2 - (M-2E_M)^2).$

Let $h = d+3-2[(d+3)/2]$ then

(2.1.7) $(M-2E_M)^2 = h - \eta.$

Thus, using (2.1.6) and (2.1.7), it follows that (2.1.5) is satisfied if and only if at least one among 1), 2), and 3) holds.

LEMMA 2.1.8. *Let* $M^2 \geqq 5+4i$ *and let* $x \geqq 1$. *Consider*

$$E \equiv yr - \sum_{j=1,\dots,s} \alpha_j P_j \in \mathcal{E}_i, i = 0, 1.$$

Then:

1) *If* $y = 0$ *then* $\sum_{j=1,\dots,s} \alpha_j = -1$ *and* $\alpha_j \leqq 0, j = 1, \dots, s$
2) *If* $y \geqq 1$ *then* $\alpha_j \geqq 0, j = 1, \dots, s$
3) *If* $y \geqq 2$ *then* $\alpha_j \leqq y - 1, j = 1, \dots, s$
4) *If* $E\wedge \equiv yr - \sum_{j=1,\dots,s} \beta_j P_j$ *where* $\beta_j = \operatorname{Min}\{\alpha_j, (t_j+1)/2\}$ *then* $E^\wedge \in \mathcal{E}_i$ *and*

$$(2.1.9) \quad E^\wedge \cdot (M - E^\wedge) \leqq E \cdot (M - E).$$

Moreover if $(M - 2E) \in \mathcal{D}^*$ *then also* $(M - 2E^\wedge) \in \mathcal{D}^*$.

Proof. 1) Since E is effective and $E \not\equiv 0$ then $\alpha_j \leqq 0$. Moreover if

$$\sum_{j=1,\dots,s} \alpha_j \leqq -2$$

then $g(E) < 0$ and from (1.3) it follows that $E \notin \mathcal{E}_i$. 2) If $\alpha_j < 0$ for some $j \in \{1, \dots, s\}$ then $E_1 = P_j$ and $E_2 = E - E_1$ are effective divisors such that $E_1 \cdot E_2 \leqq O$ and again $E \notin \mathcal{E}_i$. 3) If $\alpha_j = y$ for some $j \in \{1, \dots, s\}$ then $g(E) < 0$and therefore by (1.3) we have $E \notin \mathcal{E}_i$. 4) It is easy to see that (2.1.9) is verified. It remains to prove that $E^\wedge \in \mathcal{E}_i$. If $\alpha_j = 1$ for $j = \{1, \dots, s\}$ then $\beta_j = \alpha_j$ and $E = E^\wedge$. Assume that $\alpha_t \geqq 2$ for some $t \in \{1, \dots, s\}$ then:

$$(2.1.10) \quad E + P_t \in \mathcal{E}_i.$$

To prove (2.1.10) we have to prove that $E + P_t$ satisfies (1.2.1) and (1.2.2). Let $E_{k+1} = P_t$ and $E = E_1 + \cdots + E_k$. Then

$$\sum_{j=1,\dots,k+1} E_j \cdot (E + P_t - E_j) = \sum_{j=1,\dots,k} E_j \cdot (E - E_j) + 2P_t \cdot E$$
$$\geqq (k-1)(2+i) + 1 + 2\alpha_t \geqq k(2+i) + 1.$$

Thus (1.2.1) is satisfied. Let E' and E'' be effective divisors on X such that $E = E' + E''$. To show that $E + P_t$ verifies (1.2.2) it is enough to prove that

$$(2.1.11) \quad (E' + P_t) \cdot E'' \geqq 2.$$

If $E'' \cdot P_t \geqq 0$ then (2.1.11) is verified since $E' \cdot E \geqq 2$. Assume that $E'' \cdot P_t < 0$. Let $F' = E' + P_t$ and $F'' = E'' - P_t$. Then $F'$and F'' are effective divisors such that $F' + F'' = E$ and therefore $F' \cdot F'' \geqq 2$ since $E \in \mathcal{E}_i$. We have

$$(E' + P_t) \cdot E'' = F' \cdot (F'' + P_t) = F' \cdot F'' + F' \cdot P_t \quad \text{and}$$
$$E' \cdot P_t = \alpha_t - E'' \cdot P_t - 1 \geqq 2.$$

Thus (2.1.11) is again verified and consequentelly (2.1.10) is satisfied too. By (2.1.10) and by induction on

$$n = \sum_{j=1,\dots,s} (\alpha_j - \beta_j),$$

we obtain that $E^{\wedge} \in \mathcal{E}_i$. Moreover since

$$(M - 2E^{\wedge}) \cdot P_j = \rho_j = t_j + 1 - 2\beta_j$$

then

$$\rho_j = \begin{cases} \lambda_j & \text{if } (t_j + 1)/2 \geqq \alpha_j \\ 1 & \text{if } (t_j + 1)/2 < \alpha_j \text{ and } t_j \text{ is even} \\ 0 & \text{if } (t_j + 1)/2 < \alpha_j \text{ and } t_j \text{ is odd.} \end{cases}$$

It is easy to check that $\rho_j \leqq x$.

Denote by T_i the set of all

$$E \equiv yr - \sum_{j=1,\dots,s} \alpha_j P_j \in \mathcal{E}_i \cap \mathcal{D}_M \quad \text{such that } 1 \leqq y \leqq (d+2)/2$$

and

$$\operatorname{Max}\{0, (t_j + 2y - d - 2)/2\} \leqq \alpha_j \leqq \begin{cases} 1 & \text{if } y = 1 \\ \operatorname{Min}\{y - 1, (t_j + 1)/2\} & \text{if } y \geqq 2. \end{cases}$$

THEOREM 2.2. *Let $i = 0, 1$ and let:*
1) $d \geqq 0$
2) $M^2 \geqq 5 + 4i$
3) $(M - E) \cdot E \geqq 2 + i$ *for any* $E \in T_i$ *such that* $E^2 < 0$.
Then L is i-very ample unless there is $E \in T_i$ such that either $E^2 = 0$ and $1 \leqq M \cdot E \leqq 1 + i$ or $i = 1, E^2 = 1$ and $M \equiv 3E$.

Proof. The theorem is a direct consequence of (1.1) and of (2.1.2). In fact since $d \geqq 0$ and $M^2 \geqq 5 + 4i$, by (2.0.1), we have $E \in \mathcal{D}$. Moreover applying

(1.2.3), (2.1.8) and (2.1.1), it follows that the condition 3) of (1.1) is satisfied if $(M-E)\cdot E \geqq 2+i$ for any $E \in T_i$. The theorem now follows applying (2.1.2).

THEOREM 2.3. *Let*

1) $2 \geqq t_1 \geqq \cdots \geqq t_s$

2) $M^2 \geqq 5+4i, i=0,1$

Then L is i-very ample if for any y such that $1 \leqq y \leqq (d+2)/2$ *and for any* $D \in |O_{\mathbf{P}^2}(y)|$, *the following in equality holds:*

$$(2.3.1) \quad \sum_{j\in\wedge_\Delta} t_j \leqq y(d+3-y)-2-i$$

where $\wedge_\Delta = \{j \in [1,\ldots,s] | x_j \in D]\}$.

Proof. The statement follows easily from (2.2) and the fact that (2.3.1) is equivalent to

$$(2.3.2) \quad E\cdot(M-E) \geqq 2+i$$

for any $E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j$ such that $1 \leqq y \leqq (d+2)/2$ and $0 \leqq \alpha_j \leqq 1$.

Remark 2.3.3. When $t_1 = \cdots = t_s = 1$, the above theorem improves the result in [**1**]. In particular if $d=4$ we get that $L \equiv 4r - \sum_{j=1,\ldots,s} P_j$ is i-very ample if $s \leqq 11-i, i=0,1$. This bound is sharp (see [**1**]). Hence when $s=10, \phi_L$ embeds X in $\mathbf{P}^4$ provided that at most 3, 7 and 9 of the x_j lie respectively on a line, a conic and a cubic. In this case (X,L) is called "Bordiga Surface" (see [**9**], [**10**], [**11**], [**6**], [**13**]).

THEOREM 2.4. *If*

$$(2.4.1) \quad d \geqq i + \sum_{j=1,\ldots,s} t_j, i=0,1$$

then L is i-very ample.

Proof. We have to proof that:

1) $M^2 \geqq 5+4i$

2) $(M-E)\cdot E \geqq 2+i$ for any $E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j \in \mathcal{E}_i \cap \mathcal{D}_M$.

If $s=0$ then (2.4.1) is trivially true. Assume that

$$\sum_{j=1,\ldots,s} t_j \geqq s \geqq 1.$$

Since

$$\begin{aligned} M^2 = (d+3)^2 - \sum_{j=1,\ldots,s} (t_j+1)^2 &\geqq (4+2i) \sum_{j=1,\ldots,s} t_j + (3+i)^2 - s \\ &\geqq (3+2i) \sum_{j=1,\ldots,s} t_j + (3+i)^2 \\ &\geqq 12+9i > 5+4i, \end{aligned}$$

1) is proved. We want now to prove 2). We have

$$(M-E)\cdot E = y(d+3-y) - \sum_{j=1,\dots,s} \alpha_j(t_j+1-\alpha_j).$$

If $y=0$ then $(M-E)\cdot E \geqq 3$. If $y=1,2$ then $0 \leqq \alpha_j \leqq 1$ and

$$(M-E)\cdot E \geqq (y-1)\sum_{j=1,\dots,s} t_j + y(3+i-y) \geqq 2+i.$$

Thus we assume $y \geqq 3$. If $y \geqq 3$ then by (2.1.8) we may assume

$$\alpha_j \leqq \operatorname{Min}\{y-1,(t_j+1)/2\}, j=1,\dots,s \quad \text{and}$$

$$(M-E)\cdot E \geqq -y^2+(3+i)y+\sum_{j=1,\dots,s}(yt_j-\alpha_j(t_j+1\alpha_j)).$$

We need to consider two cases:

a) $(t_j+1)/2 < y-1$ and b) $y-1 \leqq (t_j+1)/2$.

In case a)

$$yt_j-\alpha_j(t_j+1-\alpha_j) \geqq t_j(t_j+4)/2-((t_j+1)/2)^2 > 0.$$

In case b)

$$yt_j-\alpha_j(t_j+1-\alpha_j) \geqq yt_j-(y-1)(t_j+2-y) = y^2-3y+2+t_j > 0.$$

If $(t_1+1)/2 \geqq y-1$, then

$$\begin{aligned}(M-E)\cdot E &\geqq -y^2+(3+i)y+yt_1-\alpha_1(t_1+1-\alpha_1)\\ &\geqq iy+2+t_1 \geqq 2+i.\end{aligned}$$

Assume now $(t_1+1)/2 < y-1$, then

$$(t_j+4)/2 \leqq y, j=1,\dots,s.$$

By (2.1.1) we may assume $y \leqq (d+2)/2$. Thus we have

$$\begin{aligned}(M-E)\cdot E &\geqq (d+4)y/2-\sum_{j=1,\dots,s}((t_j+1)/2)^2\\ &\geqq y(i+4)/2+\left(\sum_{j=1,\dots,s}(2t_jy-(t_j+1)^2)\right)/4\\ &\geqq y(i+4)/2+\left(\sum_{j=1,\dots,s}(2t_j-1)\right)/4 \geqq 2+i.\end{aligned}$$

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Remark 2.4.2. The bound (2.4.1) is sharp. It can be improved only under the condition that not all the points $x_j, j = 1, \ldots, s$, lie on a line.

Remark 2.4.3. We like to point out that the above theorem is very useful in the investigation of the existence of surfaces whose minimal model is $\mathbf{P}^2$, see [**8**]. However if

$$d < i + \sum_{j=1,\ldots,s} t_j,$$

where $i = 0, 1$ in order to be able to answer to the question if L is i- very ample it is necessary a study of the position of the points $x_1, \ldots, x_s$. A contribution to this problem is given in the following section.

3. General position.

Definition 3.0. We say that $x_1, \ldots, x_s$ are *in general position with respect to* L if for any $E \in |O_{\mathbf{P}^2}(y)|$ such that:
1) E is irreducible and reduced
2) $1 \leqq y \leqq (d+2)/2$
3) $\mu_j(E_j) \leqq (t_j+1)/2, i = 1, \ldots, s$.
Then

$$(3.0.1) \quad (1/2) \sum_{j=1,\ldots,s} \mu_j(E)(\mu_j(E)+1) \leqq h^0(E) - 1 = y(y+3)/2$$

where $\mu_j(E)$ denotes the multiplicity of E at x_j.

Remark 3.0.2. If $2 \geqq t_1 \geqq \cdots \geqq t_s$ then $\mu_j(E) \leqq 1$ and (3.0.1) becomes

$$(3.0.3) \quad \sum_{j=1,\ldots,s} \mu_j(E) \leqq y(y+3)/2$$

which means that there are no more than two points on a line, no more than five points on a conic, no more than nine points on a cubic, etc.

LEMMA 3.1. *Let $x_1, \ldots, x_s$ be in general position with respect to L. Let*

$$E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j \in \mathcal{E}_i$$

be such that $y \leqq (d+2)/2$ and $\alpha_j \leqq (t_j+1)/2, j = 1, \ldots, s$. Then

$$(3.1.1) \quad (1/2) \sum_{j=1,\ldots,s} \alpha_j(\alpha_j+1) \leqq y(y+3)/2.$$

Proof. If $y = 0$ then

$$\sum_{j=1,\ldots,s} \alpha_j = -1 \quad \text{and} \quad \alpha_j \leqq 0, j = 1, \ldots, s,$$

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hence (3.1.1) holds. Assume that $E = E_1 + \cdots + E_k$, where $E_t, t = 1, \ldots, k$ are all the irreducible and reduced components of E. Since $E_t, t = 1, \ldots, k$ satisfies (3.1.1) we can assume $k \geqq 2$. We claim that also E verifies (3.1.1). In fact if E does not satisfies (3.1.1) we get a contradiction since

$$0 > y(y+3) - \sum_{j=1,\ldots,s} \alpha_j(\alpha_j + 1) = \sum_{j=1,\ldots,k} E_t \cdot (E_t - K_X)$$
$$+ \sum_{j=1,\ldots,k} E_t \cdot (E - E_t) \geqq (k-1)(2+i) \geqq 2+i.$$

Note. (3.1.1) is equivalent to

(3.1.2) $E \cdot (E - K_X) \geqq 0.$

PROPOSITION 3.2. *Let $M^2 \geqq 5+4i$ and let that $x_1, \ldots, x_s$ be in general position with respect to L. Consider*

$$E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j \in \mathcal{E}_i \cap \mathcal{D}_M$$

such that $g(E) \geqq 1$. If $E \cdot (M - E) \leqq 1 + i$ then either $g(E) = 1, E^2 = 0$ and $1 \leqq M \cdot E \leqq 1 + i$ or $i = 1, g(E) \leqq 2, E^2 = 1$ and $M \equiv 3E$.

Proof. Since (3.1.2) and $g(E) \geqq 1$ imply that

3.2.1. $E^2 \geqq g(E) - 1 \geqq 0$

the statement follows easily from (2.1.2).

LEMMA (3.2.2) *Let $M^2 \geqq 5+4i$ and let $x_1, \ldots, x_s$ be in general position with respect to L. If there is an $E \in \mathcal{E}_i \cap \mathcal{D}_M$ such that $g(E) = 1, E^2 = 0$ and $1 \leqq M \cdot E \leqq 1 + i$, then L is not i-very ample.*

Proof. We have

$$M \cdot E = (M - E) \cdot E = L \cdot E - 2g(E) + 2 = L \cdot E.$$

Thus when $i = 1$ the statement follows from (1.3.2). Assume that $i = 0$. Then $M \cdot E - L \cdot E = 1$. Moreover if there is $F \in \mathcal{E}_i \cap \mathcal{D}_M$ with $g(F) = 0$ then, since $x_1, \ldots, x_s$ are in general position with respect to L, $F^2 \geqq -1$. So the statement follows from (1.4.8).

THEOREM 3.3. *Let:*
1) $M^2 \geqq 5 + 4i$
2) $x_1, \ldots, x_s$ *are in general position with respect to L*

3) *for any* $E \in \mathcal{E}_i \cap \mathcal{D}_M$ *such that* $g(E) = 2$ *then either* $E^2 \neq 1$ *or* $M \not\equiv 3E$. *Then L is i-very ample if and only if for any* $E \in \mathcal{E}_i \cap \mathcal{D}_M$ *such that* $0 \leqq g(E) \leqq 1$ *we have* $L \cdot E \geqq 2g(E) + i$.

Proof. The statement follows from (1.3.2), (3.2) and (3.2.2).

THEOREM 3.4. *Assume that:*

1) $x_1, \ldots, x_s$ *are in general position with respect to L*

2) $M^2 \geqq 5 + 4i, i = 0, 1$

3) *for any* $E \in \mathcal{E}_i \cap \mathcal{D}_M$ *such that* $g(E) = 2$ *either* $E^2 \neq 1$ *or* $M \not\equiv 3E$.

Then L is i-very ample if $d \geqq 3t_1 + 1$.

Proof. Assume that $d \geqq 3t_1 + 1$ and that there is

$$E \equiv yr - \sum_{j=1,\ldots,s} \alpha_j P_j \in \mathcal{E}_i \cap \mathcal{D}_M$$

such that $g(E) = 0, 1$ and

(3.4.1) $\quad L \cdot E \leqq 2g(E) - 1 + i.$

Then $y \geqq 1$. Moreover by the general position hypothesis on $x_1, \ldots, x_s$ it follows that

(3.4.2) $\quad E \cdot K_X \leqq g(E) - 1 \leqq E^2.$

Therefore

$$(L \cdot E + t_1 E \cdot K_X) = y(d - 3t_1) + \sum_{j=1,\ldots,s} \alpha_j(t_1 - t_j) \geqq y(d - 3t_1).$$

Combining (3.4.1) and (3.4.2) we get that

$$(L \cdot E + t_1 E \cdot K_X) \leqq (2 + t_1)(g(E) - 1) + 1 + i.$$

Hence

$$d \leqq 3t_1 + (A/y)$$

where

$$A = (2 + t_1)(g(E) - 1) + 1 + i.$$

If $g(E) = 0$ then $A < 0$. If $g(E) = 1$ then $y \geqq 3$ and $A = 1 + i$. In both cases we get $d \leqq 3t_1$ which gives a contradiction.

Remark 3.4.1. Let X, L and M be as in Theorem (3.4). Assume that $t_1 \leqq 2$. Then L is i-very ample if $d \geqq 7$. If $1 \leqq d \leqq 6$ a direct computation shows that L is i-very ample if it satisfies the conditions in the following table I:

i	d	L is i-very ample if
0, 1	1	$p = 0$ and $q \leqq 1 - i$
0, 1	2	$p \leqq 1 - i, 1 \leqq 2 - i$ if $p = 1 - i$
0, 1	3	$p \leqq 1, q \leqq i - 1$ if $p = 2 - i$
1	4	$p \leqq 1$
0	4	$p \leqq 4, q = 0$ if $p = 4$
1	5	$p \leqq 4$
0, 1	6	$p \leqq 8 - i, q \leqq i$ if $p = 8 - 1$

where $p, q \in \mathbf{Z}_+$ are such that $p + q = s$ and $t_1 = \cdots = t_p = 2, t_{p+1} = \cdots = t_s = 1$. Conversely if L is not as in table I, L is not i-very ample. (Remember that we are supposing $M^2 \geqq 5 + 4i$). For example, consider

$$L_i = 6r - 2 \sum_{j=1,\ldots,7} P_j - (2 - i)P_8 - P_9, i = 0, 1$$

and let

$$E \equiv 3r - \sum_{j=1,\ldots,9} P_j \in \mathcal{D}_1.$$

Then $g(E) - 1, L \cdot E = 1 + i, E^2 = 0$. Therefore, from (3.3) it follows that L_i is not i-very ample.

Note. After this paper was written, R. Weinfurtner, a student of K. Hulek, has generalized our results to the case of infinitesimally near points.

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