J. Austral. Math. Soc. (*Series A*) **65** (1998), 224–237

GROUPS WHOSE PROPER QUOTIENTS ARE HYPERCENTRAL

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(Received 19 December 1997; revised 27 June 1998)

Communicated by J. R. J. Groves

Abstract

Groups, all proper factor-groups of which are hypercentral of finite torsion-free rank, are studied in this article.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 20F19, 20F14.

0. Introduction

Let G be a group and let N be a normal subgroup of G. The factor-group G/N is said to be a proper factor-group if $N \neq \langle 1 \rangle$. The influence of properties of proper factor-groups on properties of groups was the subject of investigation of many authors. The classic example in this area is the following theorem of Robinson [14, Theorem 10.51]: *a finitely generated soluble group is nilpotent if all its finite factor-groups are nilpotent.* The set of all finite factor-groups also plays an important role in the study of finitely presented groups and in algorithmic problems. The influence of the structure of torsion factor-groups on the structure of some soluble groups has been studied in [21]. But if we consider all proper factor-groups, the influence of their structure on the structure of a group will increase powerfully.

Let $\mathfrak{I}$ be a class of groups. A group G is called a *just-non-$\mathfrak{I}$-group* if $G \notin \mathfrak{I}$, but every proper factor-group of G belongs to $\mathfrak{I}$. The structure of just-non-$\mathfrak{I}$-groups has already been studied for several choices of the class $\mathfrak{I}$. The first research on this topic was done by Newman [11, 12], who considered just-non-abelian groups. Later, the class of just-non-$\mathfrak{I}$-groups was investigated in the cases where $\mathfrak{I}$ is chosen to be the class of finite groups [9, 10, 18], of polycyclic or supersoluble groups [5, 16], of

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https://doi.org/10.1017/S1446788700034972 Published online by Cambridge University Press

Chernikov groups [1], of groups with transitive normality [15], of finite-by-abelian or central-by-finite groups [17]. Franciosi and de Giovanni considered groups, all proper factor-groups of which are nilpotent of class $\leq c$ [2]. Some generalization of this situation can be found in [22]. If G is a non-monolithic group in which all proper factor-groups are nilpotent of class $\leq c$, then G is nilpotent also. But if we reject the bounding of class of nilpotency, the non-monolithic case will be much more complicated. Groups, all proper factor-groups of which are hypercentral of finite 0-rank (torsion-free rank), are studied in this paper. Note that every simple group $G \notin \mathfrak{Z}$ is a just-non-$\mathfrak{Z}$-group. Therefore, in an investigation concerning just-non-$\mathfrak{Z}$-groups, it is natural to consider groups which include a non-identity abelian normal subgroup, that are groups with a non-identity Fitting subgroup.

The main results of our paper are the following theorems.

THEOREM 1. *Let G be a non-monolithic group, all proper factor-groups of which are hypercentral groups of finite 0-rank. If* $\operatorname{Fitt} G \neq \langle 1 \rangle$, *then G is hypercentral. In particular, if every proper factor-group of a non-monolithic group G is periodic and hypercentral, then G is hypercentral.*

THEOREM 2. *Let G be a monolithic group with* $\operatorname{Fitt} G \neq \langle 1 \rangle$. *Then G is a just-non-hypercentral group if and only if G satisfies the following conditions:*

(1) $\operatorname{Fitt} G = M$ *is the monolith of G (in particular, M is abelian);*

(2) *M is a maximal abelian normal subgroup of G;*

(3) *$G = M \lambda H$ where $H = N_G(H)$ is a hypercentral group (we use λ to denote the semidirect product with the normal subgroup M);*

(4) *all complements to M are conjugate in G;*

(5) *the periodic part T of the center $\zeta(H)$ is locally cyclic;*

(6) *if M is an elementary abelian p-group for some prime p then T is a p'-group.*

Moreover, if every proper factor-group of G has a finite 0-rank, then M is an elementary abelian p-subgroup and $T = \zeta(H)$. In particular, if every proper factor-group of G is periodic, then G is also periodic.

1. Some preliminary results

LEMMA 1.1. *Let G be a just-non-hypercentral group. Then*

(1) *G does not include normal non-identity subgroups R_1 and R_2 such that $R_1 \cap R_2 = \langle 1 \rangle$;*

(2) $\zeta(G) = \langle 1 \rangle$;

(3) *if G includes a finite non-identity normal subgroup F, then G is finite.*

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PROOF. (1) From $R_1 \cap R_2 = \langle 1 \rangle$ we obtain the imbedding $G \leq G/R_1 \times G/R_2$, which shows that G is hypercentral.

(2) is obvious.

(3) Suppose that G is infinite. We can assume that F is a finite minimal normal subgroup of G. From (1) we obtain that $F \leq C_G(F)$, in particular F is abelian. The factor-group $G/C_G(F)$ is finite, so $C_G(F) = C$ is infinite. Since G/F is hypercentral, $C/F \cap \zeta(G/F) \neq \langle 1 \rangle$. Let $F \neq aF \in C/F \cap \zeta(G/F)$, $A = \langle F, a \rangle$. If a is an element of an infinite order then $A^k = \langle a^k \rangle \neq \langle 1 \rangle$ for $k = |F|$. Then $A^k \cap F = \langle 1 \rangle$ and we have a contradiction of (1). If $|a|$ is finite then A is finite, in particular, A satisfies Min-G. Since G/A is hypercentral, by [19, Theorem 1′] A has the decomposition $A = A_1 \times A_2$ where A_1 and A_2 are G-invariant subgroups such that every G-chief factor of A_1 is G-central and every G-chief factor of A_2 is not G-central. Since $A \neq F$ and $A/F \leq \zeta(G/F)$, $A_1 \neq \langle 1 \rangle$. This means that $\zeta(G) \neq \langle 1 \rangle$, and we have a contradiction of (2). □

LEMMA 1.2. *Let G be an infinite just-non-hypercentral group and let A be a maximal normal abelian subgroup of G. Assume that $A \neq \langle 1 \rangle$. Then*

(1) *either A is an infinite elementary abelian p-subgroup for some prime p, or A is a torsion-free subgroup;*

(2) $A = C_G(A)$;

(3) *if $A \neq zA \in \zeta(G/A)$, then $C_G(z) = \langle 1 \rangle$.*

PROOF. (1) Let T be the periodic part of A. Assume that $T \neq \langle 1 \rangle$. Lemma 1.1 yields that T is a p-subgroup for some prime p. Put $T_1 = \Omega_1(T) = \{x \in T \mid x^p = 1\}$. Lemma 1.1 implies that T_1 is an infinite elementary abelian p-subgroup. Suppose that $T \neq T_1$. Since G/T_1 is hypercentral, $T/T_1 \cap \zeta(G/T_1) \neq \langle 1 \rangle$. Let $T_1 \neq cT_1 \in T/T_1 \cap \zeta(G/T_1)$, then $[c, g] \in T_1$ for each $g \in G$. It follows that $1 = [c, g]^p = [c^p, g]$. Since $c \notin T_1$, $c^p \neq 1$. This means that $\zeta(G) \neq \langle 1 \rangle$. However, this contradicts Lemma 1.1. Hence $T = T_1$.

If $A \neq T$, then $A = T \times B$ for some subgroup B (see, for example, [3, Theorem 27.5]). In particular, A^p is a non-identity G-invariant torsion-free subgroup. But this contradicts Lemma 1.1. Consequently, if $T \neq \langle 1 \rangle$ then A is elementary abelian.

(2) is almost obvious.

(3) Consider the mapping $\varphi : A \to A$ defined by the rule $a\varphi = [a, z]$, $a \in A$. Since $zA \in \zeta(G/A)$, φ is a G-endomorphism of A. In particular, $\operatorname{Im}\varphi = [A, z]$ and $\operatorname{Ker}\varphi = C_A(z)$ are G-invariant subgroups of A. By (2) $z \notin C_G(A)$ so that $C_A(z) \neq A$. Suppose that $C_A(z) \neq \langle 1 \rangle$. Then $G/C_A(z)$ is hypercentral and $\zeta(G/C_A(z)) \cap A/C_A(z) \neq \langle 1 \rangle$. Let $C_A(z) \neq aC_A(z) \in \zeta(G/C_A(z)) \cap A/C_A(z)$. Since $a \notin C_A(z)$, $a_1 = [a, z] \neq 1$. Let g be an arbitrary element of G. Rewrite the Hall-Witt identity in the form

$$[[a, z], g]^x[[z^{-1}, g^{-1}], a]^g[[g, a^{-1}], z^{-1}]^a = 1, \quad x = z^{-1}.$$

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Since $gA \in \zeta(G/A)$, $[z^{-1}, g^{-1}] \in A$, so that $[[z^{-1}, g^{-1}], a] = 1$. Since $aC_A(z) \in \zeta(G/C_A(z))$, $[g, a^{-1}] \in C_A(z)$, and $[[g, a^{-1}], z^{-1}] = 1$. It follows that $[[a, z]g] = 1$, that is $1 = [a_1, g]$. This means that $\zeta(G) \neq \langle 1 \rangle$. This contradicts Lemma 1.2, so (3) is proved. □

Recall the definition of finite 0-rank.

DEFINITION. We say that a group G has finite 0-rank (or finite torsion-free rank) which is equal to r, if G has a finite subnormal series $\langle 1 \rangle = G_0 < G_1 < \cdots < G_n = G$, r factors of which are infinite cyclic groups, and all remaining factors are torsion groups.

We will denote the 0-rank of group G by $r_0(G)$.

LEMMA 1.3. *Let G be a torsion-free nilpotent group of finite 0-rank and let p be a prime number. Then G has a finite subnormal series $\langle 1 \rangle = H_0 < H_1 < H_2 < \cdots < H_n = G$, in which every factor H_{i+1}/H_i is torsion and p-divisible, $1 \leq i \leq n-1$, and the subgroup H_1 is finitely generated.*

PROOF. Since the factor-group of a torsion-free nilpotent group over its center is torsion-free also (see, for example, [13, Theorem 2.25]), we can use induction on the class of nilpotence c of the group G. If $c = 1$ then G is an abelian torsion-free group of finite 0-rank. Let $\{a_i \mid 1 \leq i \leq r\}$ be a maximal Z-independent subset of G, $B = \langle a_i \mid 1 \leq i \leq r \rangle$. Then G/B is a torsion abelian group of finite Prufer rank, and therefore its Sylow p-subgroup P/B is a Chernikov group. Then P/B includes the finite subgroup H/B such that P/H is a divisible Chernikov p-group. In this case G/H is a p-divisible group.

Let $c > 1$ and $C = \zeta(G)$. Then G/C is a torsion-free nilpotent group of class $c - 1$, and by the induction hypothesis G/C has a finite subnormal series $C = H_2 < H_3 < \cdots < H_n = G$ such that H_2/C is finitely generated and all remaining factors H_{i+1}/H_i are torsion and p-divisible, $2 \leq i \leq n-1$. Since H_2/C is finitely generated, $H_2 = F \cdot C$ for some finitely generated subgroup F. Since $C = \zeta(G)$, F is normal in H_2. By the induction hypothesis C includes the finitely generated subgroup D such that C/D is torsion and p-divisible. Put $H_1 = D \cdot F$. Then H_1 is finitely generated and normal in H_2, and $H_2/H_1 = CF/DF = CDF/DF \cong C/C \cap DF = C/D(C \cap F)$, so that H_2/H_1 is torsion and p-divisible. □

LEMMA 1.4. *Let F be a field, G a hypercentral group, and let A be an FG-module. Suppose that A includes an FG-submodule B satisfying the following conditions:*

(1) *$A(x - 1) \leq B$ for every $x \in G$;*

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(2) *B is a simple FG-submodule;*
(3) $C_G(B) \neq G$.

Then A includes an FG-submodule C such that $A = B \oplus C$.

PROOF. We can assume that $C_G(A) = \langle 1 \rangle$. Let $1 \neq z \in \zeta(G)$. Then the mapping $\varphi : a \to a(z-1), a \in A$, is an FG-endomorphism and $\operatorname{Ker}\varphi = \operatorname{Ann}_A(z-1) = C_A(z)$, $\operatorname{Im}\varphi = A(z-1)$. It follows from (1) that $A(z-1) \leq B$. Since B is a simple FG-submodule, $A(z-1) = B$. If we assume that $B(z-1) = \langle 0 \rangle$, then we have $B \leq \operatorname{Ker}\varphi$, therefore $B = A(z-1) \cong {}_{FG} A/\operatorname{Ker}\varphi$. But in this case, $B(x-1) = \langle 0 \rangle$ for any $x \in G$. This is a contradiction of condition (3). Hence $A(z-1) = B(z-1)$. It follows that $A = \operatorname{Ann}_A(z-1) + B$. Since $B = B(z-1)$, $\operatorname{Ann}_A(z-1) \cap B = \langle 0 \rangle$ so that $A = B \oplus C$ where $C = \operatorname{Ann}_A(z-1)$. □

DEFINITION. Let G be a just-non-hypercentral group, A a non-identity normal abelian subgroup of G, $\mathscr{R}_G(A) = \{B \mid B$ is a non-identity G-invariant subgroup of $A\}$. Let $M = \cap\mathscr{R}_G(A)$. Then either $M = \langle 1 \rangle$ (non-monolithic case) or $M \neq \langle 1 \rangle$. In the second case M is called the monolith of group G.

Lemma 1.4 implies that either A is an elementary abelian p-group or A is torsion-free. Consequently, we must consider the following situations: non-monolithic case of characteristic p, non-monolithic case of characteristic 0, and the monolithic case.

2. Non-monolithic case of characteristic p

Everywhere in this section (except Proposition 2.4), G is a just-non-hypercentral non-monolithic group and A is a maximal normal abelian subgroup of G. We also assume that A is an elementary abelian p-group for some prime p. Lemma 1.1 implies that A is infinite.

LEMMA 2.1. *The factor-group G/A is torsion-free. In particular, if every proper factor-group of G has finite* 0*-rank, then G/A is a nilpotent torsion-free group of finite* 0*-rank.*

PROOF. Let P/A be a Sylow p-subgroup of G/A. Suppose that P/A is a non-identity. Then $P/A \cap \zeta(G/A) \neq \langle 1 \rangle$. Let $gA \neq A$, $gA \in P/A \cap \zeta(G/A)$. Then g is a p-element and the subgroup $\langle g, A \rangle$ is nilpotent (see, for example, [14, Lemma 6.34]). It follows that $C_A(g) \neq \langle 1 \rangle$. However, this is a contradiction of Lemma 1.2.

Let Q/A be a Sylow p'-subgroup of G/A. Suppose that $Q/A \neq \langle 1 \rangle$. Then $\langle 1 \rangle \neq R/A = Q/A \cap \zeta(G/A)$. Let $B \in \mathscr{R}_G(A)$. Since G/B is hypercentral and A/B is the Sylow p-subgroup of R/B, $R/B = A/B \times S/B$ where S/B is a Sylow p'-subgroup

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of R/B. Since $S/B \cong R/A$, R/B is abelian so that $[R, R] \leq \cap\mathscr{R}_G(A) = \langle 1 \rangle$. Thus R is an abelian normal subgroup of G. But $A \leq R$ and $A \neq R$, so we obtain a contradiction with the choice of A. This contradiction shows that G/A is torsion-free.

If G/A has a finite 0-rank, then it is nilpotent (see, for example, [14, Theorem 6.36]). □

Now we need some module-theoretical concepts.

DEFINITION. Let J be a principal ideal domain, A a J-module, $a \in A$, and let $\mathrm{Ann}_J(a) = \{x \in J \mid ax = 0\}$. An element a is called J-torsion if $\mathrm{Ann}_J(a) \neq \langle 0 \rangle$. The set $t_J(A)$ of all J-torsion elements of A is a J-submodule of A. The submodule $t_J(A)$ is called the J-torsion part of A. If $A = t_J(A)$ then A is called the J-torsion module. If $t_J(A) = \langle 0 \rangle$, then we say that A is J-torsion-free.

Let I be an ideal of J. Put $A_I = \{a \in A \mid aI^n = \langle 0 \rangle$ for some $n \in N\}$. It is easy to see that A_I is a J-submodule of A. This J-submodule is called the I-component of module A. Let $\mathrm{Spec}(J)$ be the set of all maximal ideals of J. If $a \in t_J(A)$, then $\mathrm{Ann}_J(a) = P_1^{k_1} \cdots P_l^{k_l}$ for some $P_1, \ldots, P_l \in \mathrm{Spec}(J)$, $k_1, \ldots, k_l \in N$. Put $\Pi_J(a) = \{P_1, \ldots, P_k\}$, $\Pi_J(A) = \cup_{a \in t(A)} \Pi_J(a)$.

As in the case when $J = Z$, we can prove that $t_J(A) = \oplus_{P \in \Theta} A_P$, $\Theta = \Pi_J(A)$.

We can consider A as ZH-module where $H = G/A$ is a hypercentral group.

LEMMA 2.2. (1) *Let $A \neq xA \in \zeta(G/A)$, then A (as $F_p\langle x \rangle$-module) is $F_p\langle x \rangle$-torsion-free.*

(2) *If B is a non-identity G-invariant subgroup of A, then $C_G(B) = A$.*

PROOF. (1) Since $|xA|$ is infinite, by Lemma 2.1, $|x|$ is infinite too and $F_p\langle x \rangle$ is a principal ideal domain. We consider A as F_pH-module where $H = G/A$ and use additive notation for A. Let T be the $F_p\langle x \rangle$-torsion part of A and suppose that $T \neq \langle 0 \rangle$. Since $xA \in \zeta(G/A)$, the I-component of A is a F_pH-submodule for every ideal I of $F_p\langle x \rangle$. Lemma 1.1 yields that $\Pi_J(A) = \{P\}$ for some $P \in \mathrm{Spec}(F_p\langle x \rangle)$. Put $T_1 = \{a \in T \mid aP = \langle 0 \rangle\}$ and assume that $T \neq T_1$. The factor-group G/T_1 is hypercentral, therefore, $\zeta(G/T_1) \cap T/T_1 \neq \langle 1 \rangle$. Let $aT_1 \neq T_1, aT_1 \in T/T_1 \cap \zeta(G/T_1)$. Then $\mathrm{Ann}_{F_p\langle x \rangle}(aT_1) = (x-1)F_p\langle x \rangle = P_1$. Since $\Pi_{F_p\langle x \rangle}(T) = \Pi_{F_p\langle x \rangle}(T/T_1)$, $P = P_1$. However, in this case, $\mathrm{Ann}_{F_p\langle x \rangle}(T_1) = P_1$. In other words, $T_1 \leq \mathrm{Ann}_A(x-1) = C_A(x)$, which is a contradiction of Lemma 1.2. Hence $T = T_1$.

Suppose that $T \neq A$. As in the abelian groups case, we can prove that $A = T \oplus C$ for some $F_p\langle x \rangle$-submodule C. Let $B = AP$. Then $B = CP \leq C$, in particular, $B \cap C = \langle 0 \rangle$. Since $xA \in \zeta(G/A)$, B is a G-invariant subgroup of A and we obtain a contradiction of Lemma 1.1. Hence $A = T$. Since $F_p\langle x \rangle$ is a principal ideal domain, there is an element y such that $P = yF_p\langle x \rangle$. Since P is a maximal ideal of $F_p\langle x \rangle$, y

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is an irreducible polynomial in x. Let $a \in A$. From $aF_p\langle x\rangle \cong F_p\langle x\rangle/P$ we obtain $|aF_p\langle x\rangle| = |\langle a\rangle^{\langle x\rangle}| = p^t$ where $t = \deg y$. It follows that $x^l \in C_G(a)$ where $l = (p^t)!$. Since it is true for each $a \in A$, $x^l \in C_G(A)$. By Lemma 1.2 $C_G(A) = A$. This means that $|xA|$ has finite order. However, Lemma 2.1 implies that G/A is torsion-free. Hence $T = \langle 0\rangle$.

(2) It follows from the choice of B that G/B is hypercentral. If $a \in A$, then the subgroup $\langle aB, xB\rangle$ is nilpotent. It follows that

$$[a, \underbrace{x, \dots, x}_{n}] \in B \quad \text{for some} \quad n \in N.$$

We can rewrite it using the additive notation: $a(x-1)^n \in B$. This means that the factor-module A/B is $F_p\langle x\rangle$-torsion. Let $g \in C_G(B)$, $ag = a_1$, then

$$a_1(x-1)^n = ag(x-1)^n = a(x-1)^n \cdot g = a(x-1)^n, \text{ or } (a_1 - a)(x-1)^n = 0.$$

Since A is $F_p\langle x\rangle$-torsion-free, this means that $a - a_1 = 0$, that is $ag = a$. In other words, $g \in C_G(A) = A$. □

DEFINITION. Let H be a group and let A be a ZH-module. Then A is called the just infinite ZH-module, if A satisfies the following conditions:

(JI 1) if B is a non-zero ZH-submodule of A, then A/B is finite;

(JI 2) $\cap\{B \mid B \text{ is a non-zero } ZH\text{-submodule of } A\} = \langle 0\rangle$.

LEMMA 2.3. *Suppose that all proper factor-groups of G have finite* 0*-rank. Let* $1 \neq a \in A$, *and* $B = \langle a\rangle^G$. *Then B is a just infinite ZH-module where* $H = G/A$.

PROOF. Since G is a non-monolithic group, B satisfies the condition (JI 2). Let C be a G-invariant subgroup of B (that is C is a ZH-submodule of B), $C \neq \langle 1\rangle$. Then G/C is hypercentral. Lemma 2.1 implies that G/A is a torsion-free nilpotent group of finite 0-rank. By Lemma 1.3 the group G possesses a finite subnormal series $A = H_0 < H_1 < H_2 < \dots < H_n = G$ such that H_1/H_0 is finitely generated and H_{i+1}/H_i are torsion and p-divisible, $1 \le i \le n-1$.

Put $B_1/C = \langle a\rangle^{H_1}C/C$, $B_2/C = \langle a\rangle^{H_2}C/C$. Since H_1/A is finitely generated, $H_1 = F_1 \cdot A$ for some finitely generated subgroup F_1. Since G/C is a hypercentral group, its finitely generated subgroup $\langle aC, F_1C/C\rangle$ is nilpotent. Since the torsion part of a finitely generated nilpotent group is finite, B_1/C is finite. [7, Lemma 5] implies that every H_1-invariant subgroup of B_2/C is H_2 - invariant. This means that $B_2/C = B_1/C$, in particular, B_2/C is finite. By [7, Lemma 5] after finitely many steps, we obtain the equation $B/C = B_1/C$. So, B/C is finite. Hence B satisfies the condition (JI 1), and so B is a just infinite ZH-submodule. □

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PROPOSITION 2.4. *Let G be a non-monolithic group, all proper factor-groups of which are hypercentral groups of finite 0-rank, and let A be a non-identity maximal normal abelian subgroup of G. If A is not torsion-free, then G is hypercentral.*

PROOF. Assume the contrary. Let G be non-hypercentral, that is G be just-non-hypercentral. Lemma 1.2 implies that A is an elementary abelian p-subgroup for some prime p. By Lemma 2.1 G/A is a nilpotent torsion-free group of finite 0-rank.

Let $xA \neq A, aA \in \zeta(G/A), 1 \neq a \in A, B = \langle a \rangle$. It follows from Lemma 2.2 that $C_G(B) = A$. By Lemma 2.3 B is a just infinite F_pH-module where $H = G/A$. We can consider B as a JH-module where $J = F_p\langle x \rangle$. By Lemma 2.2 B is J-torsion-free. From [6, Theorem 2′] we obtain that H is finitely generated and abelian-by-finite, and B is a J-minimax module, that is B includes a finitely generated submodule C such that B/C is a J-torsion module with the finite set $\Pi_J(B/C)$. Since Spec J is infinite, there exists a maximal ideal P such that $P \notin \Pi_J(B/C)$. Again $P = yJ$ where y is an irreducible polynomial. We can choose P such that deg $y \geq 2$. In particular C/CP is the P-component of B/CP, hence $B/CP = C/CP \oplus E/CP$, where $E/CP \cong B/C$. It follows that $BP \leq E$, in particular, $B_1 = BP \neq B$. Since $xA \in \zeta(G/A)$, B_1 is a G-invariant subgroup of B. This means that B/B_1 is finite, B/BP is a vector space over the field $J/P = F_1$, so that $B/B_1 = M_1/B_1 \times \cdots \times M_k/B_1$ where M_i/B_1 is a minimal J- submodule $1 \leq i \leq k$. From the choice of P, it follows that $|M_i/B_1| \geq p^2$ for any i. Since $B_1 \neq \langle 1 \rangle$, G/B_1 is hypercentral. Then $\zeta(G/B_1) \cap B/B_1 \neq \langle 1 \rangle$. Since M_i/B_1 is a minimal $\langle x \rangle$-invariant subgroup and $xA \in \zeta(G/A)$, either $M_i/B_1 \leq \zeta(G/B_1)$ or $M_i/B_1 \cap \zeta(G/B_1) = \langle 1 \rangle$. It follows that there is an index t such that $M_t/B_1 \leq \zeta(G/B_1)$. Since $cB_1 \in \zeta(G/B_1)$, $|\langle c \rangle^G B_1/B_1| = p$. On the other hand $|\langle c \rangle^{\langle x \rangle} B_1/B_1| = |M_t/B_1| \geq p^2$. This contradiction shows that G is hypercentral. □

3. Non-monolithic case of characteristic 0

Everywhere in this section (except Proposition 3.5) G is a just-non-hypercentral non-monolithic group, and A is a maximal normal abelian subgroup of G. We assume that A is torsion-free.

Put $\mathscr{P}_G(A) = \{B \mid B$ is a non-identity G-invariant pure subgroup of $A\}$. We have the following two possibilities: $\cap\mathscr{P}_G(A) = \langle 1 \rangle$ and $\cap\mathscr{P}_G(A) \neq \langle 1 \rangle$. Consider the first possibility.

LEMMA 3.1. *If $\cap\mathscr{P}_G(A) = \langle 1 \rangle$, then G/A is torsion-free.*

PROOF. Let T/A be the torsion part of G/A. Suppose that $T/A \neq \langle 1 \rangle$. Then $T/A \cap \zeta(G/A) \neq \langle 1 \rangle$. Therefore T contains an element $x \notin A$ such that $x^p (A$ for some prime p and $xA \in \zeta(G/A)$. Let $V = \langle x, A \rangle$, $B \in \mathscr{P}_G(A)$. Since $B \neq \langle 1 \rangle$

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then G/B is a hypercentral group. If V/B is torsion-free, V/B is abelian (see, for example, [8, Chapter 66]). Suppose that V/B contains elements of finite order. Let Y/B be the torsion part of V/B. Since A/B is torsion-free, $|Y/B| = p$. Then Y/B is normal in G/B. Since $Y/B \cap A/B = \langle 1 \rangle$, $[Y, A] \leq B$. So, $V/B = Y/B \times A/B$ is abelian. Hence in each case V/B is abelian. In other words, $[V, V] \leq B$. Since it is true for every subgroup $B \in \mathscr{P}_G(A)$, it follows that $[V, V] \leq \cap \mathscr{P}_G(A) = \langle 1 \rangle$. Consequently, V is abelian. This is a contradiction with the choice of A. Hence G/A is torsion-free. □

DEFINITION. Let R be a ring, H be a group, and let A be an RH-module. We say that A is an RH-hypercentral (or RH-hypertrivial) module if A has an ascending series of RH-submodules

$$\langle 0 \rangle = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots \leq A_\gamma = A$$

such that $A_{\alpha+1}(x-1) \leq A_\alpha$ for every $x \in H, \alpha < \gamma$.

Let G be a just-non-hypercentral group, A be a non-identity normal abelian subgroup of G, and $H = G/C_G(A)$. Suppose that A is torsion-free. We will consider A as ZH-module. Let $D = A \otimes_Z Q$. We can extend the action of H on A to the action of H on D in only one way. Let E be a non-zero QH-submodule of D, then $E_1 = E \cap A \neq \langle 0 \rangle$. From the relations $A/E_1 = A/A \cap E \cong A + E/E \leq D/E$, we obtain that A/E_1 is Z-torsion-free. The factor-group G/E_1 is hypercentral, therefore the ZH-module A/E_1 is ZH-hypercentral. Let $E_1 = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots \leq C_\gamma = A$ be an ascending series of ZH-submodules such that $A_{\alpha+1}(x-1) \leq A_\alpha$ for each $x \in H$, $0 \leq \alpha < \gamma$. Since A/E_1 is Z-torsion-free, we can choose the submodule C_α such that C_α is pure, $\alpha < \gamma$. Put $Z_\alpha = C_\alpha \otimes_Z Q$. Then, obviously, the series $E = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq Z_{\alpha+1} \leq \cdots \leq Z_\gamma = D$ is a QH-hypercentral series of D/E. Consequently, every proper factor-module of QH-module D is QH-hypercentral. Hence we come to the problem of studying the QH-module D, every QH-factor-module of which is QH-hypercentral, where H is a hypercentral group.

Suppose that $\cap \mathscr{P}_G(A) = \langle 0 \rangle$. Let $L = \cap_{B \in T} B \otimes_Z Q$, where $T = \mathscr{R}_G(A)$ If we assume that $L \neq \langle 0 \rangle$ then $L_1 = L \cap A \neq \langle 0 \rangle$. On the other hand,

$$L \cap A = (\cap_{B \in T} B \otimes_Z Q) \cap A = \cap_{B \in T}(B \otimes_Z Q \cap A) = \cap_{B \in T} B = \langle 0 \rangle.$$

This means that $L = \langle 0 \rangle$ and therefore D is a non-monolithic QH-module.

PROPOSITION 3.2. *Let H be a hypercentral torsion-free group, D a non-monolithic QH-module, and $C_H(D) = \langle 1 \rangle$. Suppose that every proper factor-module of D is QH-hypercentral.*

(1) *If $1 \neq x \in \zeta(H)$, then D is $Q\langle x \rangle$-torsion-free.*

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(2) *If H has finite* 0*-rank, then D is QH-hypercentral*.

PROOF. (1) Let T be the $Q\langle x\rangle$-torsion part of D. Suppose that $T \neq \langle 0\rangle$. Since $x \in \zeta(H)$, the I-component of D is a QH-submodule for every ideal I of ring $Q\langle x\rangle$. It follows that $\Pi_{Q\langle x\rangle}(T) = \{P\}$ for some maximal ideal P of $Q\langle x\rangle$. Put $T_1 = \{a \in T \mid aP = \langle 0\rangle\}$. Assume that $T \neq T_1$. Then D/T_1 is a QH-hypercentral module. Thus for every element $d \in D$, there is a number $n \in \mathbf{N}$ such that $d(x-1)^n \in T_1$. Since $\Pi_{Q\langle x\rangle}(T/T_1) = \{P\}$, this means that $P = (x-1)Q\langle x\rangle$. But in this case $T_1 \leq C_D(x)$; that is $C_D(x) \neq \langle 0\rangle$. This is a contradiction of Lemma 1.2. Hence $T = T_1$. Put $C = DP$, then $T \cap C = \langle 0\rangle$. This means that $C = \langle 0\rangle$; that is $D = T_1$. Since D is non-monolithic, D includes a proper non-zero QH-submodule E. Then D/E is a QH-hypercentral torsion module with $\Pi_{Q\langle x\rangle}(D/E) = \{P\}$. It follows that $P = (x-1)Q\langle x\rangle$, which is impossible, and so (1) is proved.

(2) Assume that D is non-QH-hypercentral. Put $J = Q\langle x\rangle$. We can consider D as JH-module. Let $0 \neq d \in D$, $E = dJH$, and $\pi = \{P \mid P \in \operatorname{Spec}(J) \text{ and } E \neq EP\}$.

Since E is not J-torsion, [20, Theorem 2.3] implies that the set π is infinite. Thus π contains an ideal P such that $P \neq J(x-1)$. From the choice of x, we obtain that EP is a QH-submodule. It follows that D/EP is a QH-hypercentral module. In particular, $\zeta_{QH}(D/EP) \cap E/EP = L/EP \neq \langle 0\rangle$. This means that $L(x-1) \leq EP$. On the other hand, $LP \leq EP$. Since P and $J(x-1)$ are distinct maximal ideals of J, $P + J(x-1) = J$. From the inclusions $L(x-1) \leq EP$, $LP \leq EP$, we obtain that $L \leq EP$, in particular, $L/EP = \langle 0\rangle$. This contradiction proves that D is QH-hypercentral. □

Consideration of the case when $\cap\mathscr{P}_G(A) \neq \langle 1\rangle$ is our next step.

LEMMA 3.3. *If $\cap\mathscr{P}_G(A) \neq \langle 1\rangle$ then $\cap\mathscr{P}_G(A) = A$.*

PROOF. Assume the contrary, and let $B = \cap\mathscr{P}_G(A) \neq \{A\}$. Then B is a proper G-invariant pure subgroup of A. Lemma 1.2 yields that $A = C_G(A)$. Put $H = G/A$. We will consider A as a ZH-module. Put $D = A \otimes_Z Q$. We can consider D as a QH-module. Let $E = B \otimes_Z Q$, then E is a proper QH-submodule of D. If C is a proper G-invariant non-identity subgroup of B, then from the choice of B we obtain that B/C is a torsion group. It follows that E is a simple QH-submodule. Since $E \neq \langle 0\rangle$, the factor-module D/E is QH-hypercentral. Let $V/E = \zeta_{QH}(D/E)$. By Lemma 1.4 there exists a QH-submodule W such that $V = E \oplus W$. It follows from the choice of D that $W_1 = V \cap A \neq \langle 0\rangle$. Hence W_1 is a non-identity G-invariant subgroup of A such that $B \cap W_1 = \langle 1\rangle$. This is a contradiction of Lemma 1.1. So, $\mathscr{P}_G(A) = \{A\}$. □

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PROPOSITION 3.4. *Let G be a non-monolithic group, the proper factor-groups of which are hypercentral groups of finite 0-rank, and let A be maximal normal abelian subgroup of A. If A is a non-identity torsion-free subgroup, then G is hypercentral.*

PROOF. If $\cap\mathscr{P}_G(A) = \langle 1\rangle$, then we can use Proposition 3.2. Suppose that $\cap\mathscr{P}_G(A) \neq \langle 1\rangle$. Assume that G is not-hypercentral. Lemma 3.3 implies that for every non-identity G-invariant subgroup B of A, the factor-group A/B is torsion.

Let $1 \neq a \in A$, $B = \langle a\rangle^G$, $\pi = \{p \mid p \text{ is a prime such that } B \neq B^p\}$. [20, Theorem 2.3] proves that the set π is infinite. Let $p \in \pi$. Since $B/B^p = \langle a\rangle^G B^p/B^p$, B/B^p includes a proper G-invariant maximal subgroup M_p/B^p. Since G/B^p is hypercentral, and any chief factor of a locally nilpotent group is central (see, for example, [13, Theorem 5.27, Corollary 1]), $[B, G] \leq M_p$. It follows that $[B, G] \leq \cap_{p\in\pi} M_p$. If $[B, G] \neq \langle 1\rangle$ then the factor-group $B/[B, G]$ is torsion. Since π is infinite, the set $\Pi(B/[B, G])$ is infinite too. On the other hand, $B/[B, G] = \langle a\rangle^G[B, G]/[B, G] = \langle a\rangle[B, G]/[B, G]$. Hence $B/[B, G]$ is finite. This contradiction shows that $[B, G] = \langle 1\rangle$, that is $B \leq \zeta(G)$. But this is a contradiction of Lemma 1.2. Consequently, G is hypercentral. □

PROOF OF THEOREM 1. Let A be a maximal normal abelian subgroup of G. Since $\operatorname{Fitt} G \neq \langle 1\rangle$, $A \neq \langle 1\rangle$. If A is not torsion-free then G is hypercentral by Proposition 2.4. If A is torsion-free, then G is hypercentral by Proposition 3.4. □

4. Monolithic case

LEMMA 4.1. *Let G be a monolithic just-non-hypercentral group and let M be the monolith of G. If M is abelian, then M is a maximal normal abelian subgroup of G; in particular, $M = C_G(M)$. Moreover, $M = \operatorname{Fitt} G$.*

PROOF. Let A be a maximal normal abelian subgroup of G such that $M \leq A$. Suppose that $A \neq M$. Lemma 1.2 implies that either A is an elementary abelian p-subgroup for some prime p, or A is torsion-free. Consider the first case. Since G/M is hypercentral, $\langle 1\rangle \neq A/M \cap \zeta(G/M)$. Let $aM \neq M$, $aM \in \zeta(G/M) \cap A/M$, $B = \langle a, M\rangle$. We can consider B as F_pH-module, where $H = G/A$. Then M is a simple F_pH-submodule of B, and $[B, g] \leq M$ for any $g \in G$. By Lemma 1.4 there exists a G-invariant subgroup C such that $M \cap C = \langle 1\rangle$. This contradicts Lemma 1.1.

Let A be a torsion-free subgroup. We can consider A as ZH-module. Put $D = A \otimes_Z Q$. We can consider D as QH-module. Since M is a simple ZH-module, the additive group of M is divisible, and $M = M \otimes_Z Q$. Since M is divisible, $A = M \times U$ for some subgroup U (see, for example, [3, Theorem 21.2]). This means that A/M is torsion-free. Since G/M is hypercentral , $\zeta(G/M) \cap A/M$ is non-trivial. Let

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$aM \neq M, aM \in \zeta(G/M) \cap A/M$, $E = \langle a, M\rangle \otimes_Z Q$. Then E/M is a QH-central factor of QH-module D. By Lemma 1.4 there exists a QH-submodule C such that $E = M \oplus C$. It follows from the choice of D that $C_1 = C \cap A = \langle 1\rangle$. But in this case $C_1 \cap M = \langle 1\rangle$, and we obtain a contradiction of Lemma 1.1.

Put $F = \operatorname{Fitt} G$, and assume that $M \neq F$. Since G/M is hypercentral, $F/M \cap \zeta(G/M) \neq \langle 1\rangle$. Let $M \neq xM \in \zeta(G/M) \cap F/M$, $1 \neq a \in M$. The subgroup $L = \langle x, a\rangle$ is nilpotent (see, for example, [13, Theorem 2.18]). It follows that $C_{M\cap L}(x) \neq \langle 1\rangle$. However this is in contradiction with Lemma 1.2. Hence $M = F$. □

LEMMA 4.2. *Let H be a hypercentral group, M a simple ZH-module, $C_H(M) = \langle 1\rangle$, $C = \zeta(H)$, and let T be the torsion part of C.*

(1) *If M is Z-torsion-free, then T is a locally cyclic subgroup.*

(2) *If M is an elementary abelian p-subgroup for some prime p, then T is a locally cyclic p'-subgroup.*

(3) *If H has finite 0-rank, then M is an elementary abelian p-subgroup for some prime p, and C is a locally cyclic p'-subgroup.*

PROOF. Put $E = \operatorname{End}_{ZH}(M)$. Then E is a divisible algebra by Schur's theorem. Let Z be the center of E. Then Z is a subfield of E. For every element $c \in C$ the mapping $\tau_c : a \to ac$, $a \in M$, is a ZH-automorphism of M, and the mapping $\nu : c \to \tau_c$, $c \in C$, is an imbedding of C in the multiplicative group of Z because $C_H(M) = \langle 1\rangle$. It follows from [4, Theorem 127.3] that T is a locally cyclic subgroup (moreover, it is a p'-subgroup if M is an elementary abelian p-group). If $r_0(H)$ is finite, then A is an elementary abelian p-group for some prime p by [20, Theorem 2.3]. From [20, Theorem 2.3], we obtain that C is a torsion subgroup. □

PROOF OF THEOREM 2. Lemma 4.1 implies that M is the hypercentral residual of G. It follows from [19, Theorem 2′] that G includes a subgroup H such that G is a split extension of M by H, and $H = N_G(H)$ is hypercentral. By [19, Theorem 2′] all complements to M are conjugate. Condition (1) follows from Lemma 4.1, condition (2) follows from Lemma 4.1. Conditions (5) and (6) follow from Lemma 4.2. □

The last statement of Theorem 2 follows from previous statements and Lemma 4.2.

The question about the existence of groups from Theorems 1 and 2 is natural. The following theorem clarifies this situation.

THEOREM 3. *Let H be a hypercentral group, $C = \zeta(H)$, and let T be the periodic part of C.*

(1) *If $T = C$ is a locally cyclic p'-subgroup, and p is prime, then there exists a simple F_pH-module M such that $C_H(M) = \langle 1\rangle$.*

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(2) *If H has infinite 0-rank, and T is a locally cyclic group, then there exists a simple ZH-module M such that $C_H(M) = \langle 1 \rangle$ and the additive group of M is torsion-free.*
(3) *If H has infinite 0-rank, and T is a locally cyclic p'-subgroup for some prime p, then there exists a simple F_pH-module M such that $C_H(M) = \langle 1 \rangle$.*

PROOF. (1) There exists a simple F_pC-module B such that $C_C(B) = \langle 1 \rangle$ (see, for example, [17, Section 4]). Consider the F_pH-module $B^* = B \otimes_{F_pC} F_pH$ and identify, in the natural way, B with the F_pC-submodule of $B \otimes 1$. Then $B^* = \oplus_{t \in Y} Bt$ where Y is the transversal to C in H. Let M be a F_pH-composition factor of B^*. Then M is a simple F_pH-module. Since B^* is a semisimple F_pC-module, there exists a non-empty subset S of Y such that M is isomorphic to $M_0 = \oplus_{t \in S} Bt$. If $t \in S$ then $C_C(M) \le C_C(Bt) = t^{-1}C_C(B)t = C_C(B)$. This means that $C_C(M) = \langle 1 \rangle$. Hence $C_H(M) = \langle 1 \rangle$.

(2) Since H has an infinite 0-rank, H includes an abelian subgroup V of infinite 0-rank (see, for example, [14, Theorem 6.36]). We can assume that $C \le V$. Let Q be a maximal periodic subgroup of V with the property $T \cap Q = \langle 1 \rangle$, and let T_1/Q be the periodic part of V/Q. Then $\mathrm{Soc}(T_1/Q) = (\mathrm{Soc}\, T)Q/Q \cong \mathrm{Soc}\, T$, in particular, $\mathrm{Soc}(T_1/Q)$ is locally cyclic. It follows that T_1/Q is locally cyclic. Hence there exists a simple ZV-module B such that $C_V(B) = Q$ and the additive group of B is torsion-free (see [17, Proposition 4.13]). It follows from the choice of B that $C_V(B) \cap C = \langle 1 \rangle$. Put $B^* = B \otimes_{ZC} ZH$, then $B^* = \oplus_{t \in S} Bt$, where S is a transversal to V in H. Let M be a composition ZH-factor of B^*, then M is a simple ZH-module and $M \cong \oplus_{t \in R} Bt$ for some subset R of S. For every $t \in R$, we have $C_H(M) \cap C \le C_C(Bt) = t^{-1}C_C(B)t = \langle 1 \rangle$. This means that $C_H(M) \cap C = \langle 1 \rangle$. Since H is hypercentral, $C_H(M) = \langle 1 \rangle$. The proof of (3) is similar. □

REMARK. Lemma 4.2 shows that if M is a simple ZH-module with $C_H(M) = \langle 1 \rangle$, then M is an elementary abelian p-subgroup for some prime p and $C = \zeta(G)$ is a locally cyclic p'-subgroup. Conversely, Theorem 3 (1) implies that for such group H there exists a simple F_pH-module M with identity centralizer.

Acknowledgments

The authors would like to thank the 'Volkswagen-Stifftung' (RIP program in Oberwolfach) for the support of their research.

https://doi.org/10.1017/S1446788700034972 Published online by Cambridge University Press

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