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L^p BOUNDS FOR NONISOTROPIC MARCINKIEWICZ INTEGRALS ASSOCIATED TO SURFACES

FENG LIU and SUZHEN MAO✉

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Abstract

In an extrapolation argument, we prove certain L^p $(1 < p < \infty)$ estimates for nonisotropic Marcinkiewicz operators associated to surfaces under the integral kernels given by the elliptic sphere functions $\Omega \in L(\log^+ L)^\alpha(\Sigma)$ and the radial function $h \in \mathcal{N}_\beta(\mathbb{R}^+)$. As applications, the corresponding results for parametric Marcinkiewicz integral operators related to area integrals and Littlewood–Paley g_λ^*-functions are given.

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1. Introduction

As is well known, Marcinkiewicz integral operators belong to a broad class of Littlewood–Paley g-functions and L^p bounds regarding them are useful in the study of smoothness properties of functions and behavior of integral transformations, such as Poisson integrals, singular integrals and, more generally, singular Radon transforms. In this paper we focus on the L^p mapping properties for a class of nonisotropic Marcinkiewicz integral operators associated to surfaces.

Before establishing our main results, let us recall and introduce some notation. Let $n \geq 2$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space with a nonisotropic dilation. Precisely, let P be an $n \times n$ real matrix whose eigenvalues have positive real parts and let $\alpha = \mathrm{trac}P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^P = \exp((\log t)P)$. There is a nonnegative function r on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$. The function r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \backslash \{0\}$; furthermore, it satisfies:

(i) $r(A_t x) = t r(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;

(ii) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;

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(iii) if $\Sigma = \{x \in \mathbb{R}^n | r(x) = 1\}$, then $\Sigma = \{\theta \in \mathbb{R}^n | \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$: then, the Lebesgue measure can be written as $dx = t^{\alpha-1}\, d\sigma\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\alpha-1}\, d\sigma(\theta)\, dt$$

for appropriate functions f, where $d\sigma$ is a C^∞ measure on Σ;

(iv) there are positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$c_1|x|^{\alpha_1} \le r(x) \le c_2|x|^{\alpha_2} \quad \text{if } r(x) \ge 1,$$
$$c_3|x|^{\beta_1} \le r(x) \le c_4|x|^{\beta_2} \quad \text{if } r(x) \le 1.$$

See [5, 16, 19] for more details.

Let Ω be a locally integrable function and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$. We assume that

$$\int_\Sigma \Omega(\theta)\, d\sigma(\theta) = 0. \tag{1.1}$$

For a suitable mapping $\Phi : (0, \infty) \to (0, \infty)$, we define the parametric Marcinkiewicz integral operator along the surfaces $\{A_{\Phi(r(y))}y'; y \in \mathbb{R}^n\}$ by

$$\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)(x) := \bigg(\int_0^\infty \bigg|\frac{1}{t^\varrho}\int_{r(y)\le t} \frac{h(r(y))\Omega(y)}{r(y)^{\alpha-\varrho}} f(x - A_{\Phi(r(y))}y')\, dy\bigg|^2 \frac{dt}{t}\bigg)^{1/2}, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\varrho = \sigma + i\tau$ ($\sigma, \tau \in \mathbb{R}$ with $\sigma > 0$), $y' = A_{r(y)^{-1}}y$, $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class) and $h \in \Delta_1(\mathbb{R}^+)$. Here $\Delta_\gamma(\mathbb{R}^+) (\gamma \ge 1)$ denotes the collection of measurable functions h on $\mathbb{R}^+ := (0, \infty)$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{j\in\mathbb{Z}} \bigg(\int_{2^j}^{2^{j+1}} |h(t)|^\gamma \frac{dt}{t}\bigg)^{1/\gamma} < \infty.$$

It is easy to check that $L^\infty(\mathbb{R}^+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}^+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}^+)$ for any $1 \le \gamma_2 < \gamma_1 < \infty$. Let $\mathcal{N}_\delta(\mathbb{R}^+)$ $(\delta > 0)$ be the set of all measurable functions h on $\mathbb{R}^+$ satisfying

$$N_\delta(h) = \sum_{m=1} m^\delta 2^m d_m(h) < \infty \quad \text{with } d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k, m)|,$$

where $E(k, 1) = \{t \in (2^k, 2^{k+1}] : |h(t)| \le 2\}$ and

$$E(k, m) = \{t \in (2^k, 2^{k+1}] : 2^{m-1} < |h(t)| \le 2^m\} \quad \text{for } m \ge 2.$$

It follows from [18] that

$$\Delta_\gamma(\mathbb{R}^+) \subsetneq \mathcal{N}_{\delta_1}(\mathbb{R}^+) \subsetneq \mathcal{N}_{\delta_2}(\mathbb{R}^+), \quad \forall \delta_1 > \delta_2 > 0 \text{ and } 1 < \gamma < \infty. \tag{1.3}$$

We denote by $L(\log^+ L)^\beta(\Sigma)$ $(\beta > 0)$ the space of all those functions Ω on Σ which satisfy

$$\int_\Sigma |\Omega(\theta)| \log^\beta(2 + |\Omega(\theta)|)\, d\sigma(\theta) < \infty.$$

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Also, we consider the $L^q(\Sigma)$ spaces and write $\|\Omega\|_q = (\int_\Sigma |\Omega(\theta)|^q \, d\sigma(\theta))^{1/q}$ for $\Omega \in L^q(\Sigma)$. Note that

$$L^q(\Sigma) \subsetneq L(\log^+ L)^{\beta_1}(\Sigma) \subsetneq L(\log^+ L)^{\beta_2}(\Sigma), \quad q > 1 \text{ and } \beta_2 < \beta_1. \tag{1.4}$$

When $\Phi(t) = t$, we denote $\mathscr{M}_{h,\Omega,\Phi,\varrho}$ by $\mathscr{M}_{h,\Omega,\varrho}$. When $A_t = tE$ with E being the identity matrix and $r(x) = |x|$ (the Euclidean norm), Σ recovers the unit sphere in $\mathbb{R}^n$ denoted by S^{n-1}, and the operator $\mathscr{M}_{h,\Omega,\varrho}$ reduces to the classical parametric Marcinkiewicz integral operator, which has been studied by many authors. For example, see [4, 20] for the case $h(t) = \varrho = 1$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$, [8, 9] for the case $\varrho \equiv 1$, $h(t) \in \Delta_\infty(\mathbb{R}^+)$ and $\Omega \in H^1(S^{n-1})$, [1] for the case $h(t) \in \Delta_\gamma(\mathbb{R}^+)$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and [1, 13] for the case $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$. When $A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \ldots, t^{\alpha_n} x_n)$ with $\alpha_1, \ldots, \alpha_n$ being integers greater than one and $r(x) = \rho(x)$ with $\rho(x)$ being the solution to the equation $\sum_{j=1}^n x_j^2 \rho(x)^{-2\alpha_j} = 1$, the operator $\mathscr{M}_{h,\Omega,\varrho}$ recovers the parabolic parametric Marcinkiewicz integral operators denoted by $\mu_{h,\Omega,\varrho}$, and then Σ recovers S^{n-1}. The L^p mapping properties of $\mu_{h,\Omega,\varrho}$ have been discussed extensively by many authors. Xue *et al.* [21] proved that $\mu_{h,\Omega,\varrho}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $h(t) = \varrho = 1$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Chen and Ding [6] (respectively, [7]) extended the above result to the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ (respectively, $\Omega \in H^1(S^{n-1})$). The investigation of the parabolic parametric Marcinkiewicz integral operators $\mu_{h,\Omega,\varrho}$ with additional roughness in the radial direction has also received a large amount of attention by many authors (see [14, 15] for example).

On the other hand, to study further the singular integral operator with rough kernel both on the unit sphere and in the radial direction, Sato [17] first introduced the radial condition $\mathcal{N}_\beta(\mathbb{R}^+)$ and proved the following result.

THEOREM A. *Let $\Omega \in L \log^+ L(\Sigma)$ satisfy (1.1) and $h \in \mathcal{N}_1(\mathbb{R}^+)$; then the nonisotropic singular integral operator $T_{h,\Omega}$ defined by*

$$T_{h,\Omega}(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(r(y))\Omega(y)}{r(y)^\alpha} f(x-y)\, dy, \quad x \in \mathbb{R}^n,$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Based on the above, a natural question is the following.

QUESTION. Is $\mathscr{M}_{h,\Omega,\varrho}$ bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ under the condition that $\Omega \in L(\log^+ L)^\alpha(\Sigma)$ and $h \in \mathcal{N}_\beta(\mathbb{R}^+)$?

In this paper, we will give an affirmative answer to this question by considering a class of operators broader than $\mathscr{M}_{h,\Omega,\varrho}$. More precisely, we denote by $\mathfrak{F}$ the set of all functions φ satisfying the following conditions (a) or (b):

(a) $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a increasing C^1 function such that $t\varphi'(t) \geq C_\varphi \varphi(t)$ and $\varphi(2t) \leq c_\varphi \varphi(t)$ for all $t > 0$, where C_φ and c_φ are independent of t. Moreover, φ' is monotonic.

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(b) $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing C^1 function such that $t\varphi'(t) \le -C_\varphi \varphi(t)$ and $\varphi(t) \le c_\varphi \varphi(2t)$ for all $t > 0$, where C_φ and c_φ are independent of t. Moreover, φ' is monotonic.

Remark 1.1. There are some model examples on the class $\mathfrak{F}$ satisfying (a), such as t^β $(\beta > 0)$, $t^\beta(\ln(1+t))^\gamma$ $(\beta, \gamma > 0)$, $t \ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. The model example of functions $\phi \in \mathfrak{F}$ which satisfy (b) are t^δ $(\delta < 0)$, $t^{-1}\ln(1+1/t)$. It should be pointed out that there are two important facts, as follows.

(i) If $\varphi(t) \in C^1(\mathbb{R}^+)$ is nonnegative and increasing (respectively, decreasing) on $\mathbb{R}^+$ and $\varphi(t)/(t\varphi'(t))$ is bounded on $\mathbb{R}^+$, then $\lim_{t\to 0}\varphi(t) = 0$ (respectively, $\lim_{t\to 0}\varphi(t) = +\infty$) and $\lim_{t\to+\infty}\varphi(t) = +\infty$ (respectively, $\lim_{t\to+\infty}\varphi(t) = 0$) (see [11]).

(ii) If $\varphi \in \mathfrak{F}$ and satisfies (a), there exists a constant $B_\varphi > 1$ such that $\varphi(2t) \ge B_\varphi \varphi(t)$ (see [2, 3] for example). Similarly, one can easily check that if $\varphi \in \mathfrak{F}$ and satisfies (b), then there exists a constant $B_\varphi > 1$ such that $\varphi(t) \ge B_\varphi \varphi(2t)$.

Our main results can be stated as follows.

Theorem 1.2. *Let $\mathscr{M}_{h,\Omega,\Phi,\varrho}$ be as in (1.2) and $\Phi \in \mathfrak{F}$. Suppose that $\Omega \in L^q(\Sigma)$ for some $q \in (1,2]$ satisfying (1.1) and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma \in (1,2]$. Then:*

(i) *for $2 \le p < \infty$,*

$$\|\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C_p(\gamma-1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *for $1 < p < 2$,*

$$\|\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C_p(\gamma-1)^{-1}(q-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}.$$

The constants $C_p > 0$ are independent of h, Ω, q and γ, but depend on Φ.

Theorem 1.3. *Let $\mathscr{M}_{h,\Omega,\Phi,\varrho}$ be as in (1.2) and $\Phi \in \mathfrak{F}$. Suppose that Ω satisfies (1.1).*

(i) *If $\Omega \in L(\log^+ L)^{1/2}(\Sigma)$ and $h \in N_{1/2}(\mathbb{R}^+)$, then, for $2 \le p < \infty$,*

$$\|\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(\Sigma)})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) *If $\Omega \in L\log^+ L(\Sigma)$ and $h \in N_1(\mathbb{R}^+)$, then, for $1 < p < 2$,*

$$\|\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C(1 + \|\Omega\|_{L\log^+ L(\Sigma)})(1 + N_1(h))\|f\|_{L^p(\mathbb{R}^n)}.$$

The constants $C_p > 0$ depend on Φ.

Remark 1.4. When $A_t = tE$ with E being the identity matrix and $r(x) = |x|$ (the Euclidean norm), Theorem 1.3 was shown by Liu and Wu in more general form (see [13, Theorem 1.6]) (also see [1] for the case $\Phi(t) = t$). When $A_t x = (t^{\alpha_1}x_1, t^{\alpha_2}x_2, \ldots, t^{\alpha_n}x_n)$ with $\alpha_1, \ldots, \alpha_n$ being integers greater than one and $r(x) = \rho(x)$ with $\rho(x)$ being the solution to the equation $\sum_{j=1}^n x_j^2 \rho(x)^{-2\alpha_j} = 1$, Theorem 1.3 was proved by Liu and Zhang in more general form (see [15, Theorem 1]). It should be pointed out that our main results are also new, even in the special case $\Phi(t) = t$ and $\varrho = 1$.

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The rest of this paper is organized as follows. In Section 2 we present some preliminary lemmas. The proofs of main results will be given in Section 3. Finally, we consider the L^p bounds of the corresponding parametric Marcinkiewicz integral operators related to area integrals and Littlewood–Paley g_λ^*-functions in Section 4. We remark that the proof of Theorem 1.2 is based on the method of [1], but we add some new techniques. The main ingredients of our proofs in Theorem 1.2 are to give two sharp estimates for two maximal operators (see Lemma 2.3). As a consequence of Theorem 1.2, we can prove Theorem 1.3 via an extrapolation method which was originally by Yano (see [22]) and developed by Sato (see [17]).

Throughout the paper, we let p' denote the conjugate index of p which satisfies $1/p + 1/p' = 1$. For $x \in \mathbb{R}$, we set $[x] = \max\{k \in \mathbb{Z} : k \le x\}$. The letter C will stand for positive constants that are not necessarily the same at each occurrence but that are independent of the essential variables.

2. Preliminary lemmas

Following the notation in [17], let P^* denote the adjoint of the matrix P. Then $A_t^* = \exp((\log t)P^*)$. We can define a nonnegative function s from $\{A_t^*\}$ in exactly the same way as we define r from $\{A_t\}$.

We will use the following estimates (see [19]):

$$d_1|\xi|^{a_1} < s(\xi) < d_2|\xi|^{a_2} \quad \text{if } s(\xi) \ge 1, \tag{2.1}$$

$$d_3|\xi|^{b_1} < s(\xi) < d_4|\xi|^{b_2} \quad \text{if } 0 < s(\xi) \le 1, \tag{2.2}$$

where $d_j\ (j = 1, 2, 3, 4)$, $a_k, b_k\ (k = 1, 2)$ are positive constants. It follows from (2.1)–(2.2) that

$$|\xi| \le C_1(s(\xi)^{1/a_1} + s(\xi)^{1/b_1}), \tag{2.3}$$

$$|\xi|^{-1} \le C_2(s(\xi)^{-1/a_2} + s(\xi)^{-1/b_2}). \tag{2.4}$$

First we give the following estimate, which follows from [17, Corollary 4.2] via an integration by parts argument.

Lemma 2.1. *Let L be the degree of the minimal polynomial of P and $\Psi \in C^1([a,b])$ with $0 < a < b$. Then, for $\xi, \eta \in \mathbb{R}^n \backslash \{0\}$,*

$$\left|\int_a^b \exp(i\eta \cdot A_t\xi)\Psi(t)\,dt\right| \le C|\eta \cdot P\xi|^{-1/L}\Big(\sup_{t\in[a,b]} |\Psi(t)| + \int_a^b |\Psi'(t)|\,dt\Big)$$

for some positive constant C independent of ξ, η and Ψ. Applying Lemma 2.1, we shall establish the following result.

Lemma 2.2. *Let L be as in Lemma 2.1 and $\Phi \in \mathfrak{F}$. Then, for $\xi, \eta \in \mathbb{R}^n \backslash \{0\}$ and $t > 0$, there exists a constant $C > 0$ such that*

$$\left|\int_{t/2}^t \exp(i\eta \cdot A_{\Phi(u)}\xi)\frac{du}{u}\right| \le C|\eta \cdot PA_{\Phi(t)}\xi|^{-1/L}.$$

The constant C is independent of ξ, η, but depends on Φ.

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PROOF. We only consider the case $\Phi \in \mathfrak{F}$ satisfying the condition (a), since the other case can be proved similarly. By a change of variables,

$$\begin{aligned}\int_{t/2}^{t} \exp(i\eta \cdot A_{\Phi(u)}\xi)\frac{du}{u} &= \int_{\Phi(t/2)}^{\Phi(t)} \exp(i\eta \cdot A_u\xi)\frac{du}{\Phi^{-1}(u)\Phi'(\Phi^{-1}(u))}\\ &= \Phi(t)\int_{\varsigma}^{1} \exp(iA^*_{\Phi(t)}\eta \cdot A_u\xi)\phi(u)g(u)\,dt,\end{aligned}$$

where $\varsigma = \Phi(t/2)/\Phi(t)$, $\phi(u) = 1/\Phi^{-1}(\Phi(t)u)$ and $g(u) = (\Phi'(\Phi^{-1}(\Phi(t)u)))^{-1}$. Let

$$I(u) = \int_{\varsigma}^{u} \exp(iA^*_{\Phi(t)}\eta \cdot A_v\xi)\phi(v)\,dv, \quad \varsigma \le u \le 1.$$

By Lemma 2.1 and the fact that $PA_u = A_uP$ for any $u > 0$, there exists $C > 0$ which is independent of ξ, η such that for $\varsigma \le u \le 1$,

$$\begin{aligned}|I(u)| &\le C|A^*_{\Phi(t)}\eta \cdot P\xi|^{-1/L}\Big(\sup_{s\in[\varsigma,u]} |\phi(s)| + \int_{\varsigma}^{u} |\phi'(v)|\,dv\Big)\\ &\le \frac{C}{t}|\eta \cdot PA_{\Phi(t)}\xi|^{-1/L}.\end{aligned}$$

Thus, by integration by parts and the properties of Φ,

$$\begin{aligned}\Big|\int_{t/2}^{t} \exp(i\eta \cdot A_{\Phi(u)}\xi)\frac{du}{u}\Big| &= \Phi(t)\Big|\int_{\varsigma}^{1} g(u)\,dI(u)\Big|\\ &\le \Phi(t)\Big(|I(1)g(1)| + \int_{\varsigma}^{1} |I(u)|\,|g'(u)|\,du\Big)\\ &\le C\Phi(t)|\eta \cdot PA_{\Phi(t)}\xi|^{-1/L}((t\Phi'(t))^{-1} + (t\Phi(t/2))^{-1})\\ &\le \frac{C(1+2c_\Phi)}{C_\Phi}|\eta \cdot PA_{\Phi(t)}\xi|^{-1/L}\\ &\le C(\Phi)|\eta \cdot PA_{\Phi(t)}\xi|^{-1/L}.\end{aligned}$$

This proves Lemma 2.2. □

For $q, \gamma \in (1, \infty)$ and $t > 0$, we define the family of measures $\{\sigma_{h,t}\}_{t>0}$ and the related maximal operators σ_h^* and $M_{h,q,\gamma}$ on $\mathbb{R}^n$ by

$$\widehat{\sigma_{h,t}}(\xi) = \frac{1}{t^{\varrho}}\int_{t/2<r(y)\le t} \exp(-2\pi i\xi \cdot A_{\Phi(r(y))}y')\frac{h(r(y))\Omega(y')}{r(y)^{\alpha-\varrho}}\,dy,$$

$$\sigma_h^*(f)(x) = \sup_{t\in\mathbb{R}^+} |\,|\sigma_{h,t}| * f(x)|,$$

$$M_{h,q,\gamma}(f)(x) = \sup_{k\in\mathbb{Z}} \int_{2^{q'\gamma'k}}^{2^{q'\gamma'(k+1)}} |\,|\sigma_{h,t}| * f(x)|\frac{dt}{t},$$

where $|\sigma_{h,t}|$ is defined in the same way as $\sigma_{h,t}$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$.

In what follows, we will establish some lemmas, which will play key roles in the proofs of our main results.

Lemma 2.3. *Let $\Omega \in L^q(\Sigma)$ for some $1 < q < \infty$ and satisfy (1.1). Suppose that $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma > 1$ and $\Phi \in \mathfrak{F}$. Then, for any $t > 0$ and $\xi \in \mathbb{R}^n$, there exists $C > 0$ such that*

$$\max\{|\widehat{\sigma_{h,t}}(\xi)|, |\,|\widehat{\sigma_{h,t}}|(\xi)|\} \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)} \max\{1, |A^*_{\Phi(t)}\xi|^{-1/(4q'\gamma'L)}\}, \tag{2.5}$$

$$\max\{|\widehat{\sigma_{h,t}}(\xi)|, |\,|\widehat{\sigma_{h,t}}|(\xi) - |\widehat{\sigma_{h,t}}|(0)|\} \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{1/(4q'\gamma'L)}. \tag{2.6}$$

The constant C is independent of h, Ω, q, γ, but depends on Φ.

Proof. We only consider the case $\Phi \in \mathfrak{F}$ satisfying the condition (a), since the other case can be proved similarly. By a change of variable and Hölder's inequality,

$$\begin{aligned} |\widehat{\sigma_{h,t}}(\xi)| &= \left|\frac{1}{t^\varrho}\int_{t/2}^t \int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\frac{h(u)}{u^{1-\varrho}}\,du\right| \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}. \end{aligned} \tag{2.7}$$

Similarly,

$$|\,|\widehat{\sigma_{h,t}}|(\xi)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}. \tag{2.8}$$

On the other hand, by a change of variable and Hölder's inequality,

$$\begin{aligned} |\widehat{\sigma_{h,t}}(\xi)| &= \left|\frac{1}{t^\varrho}\int_{t/2}^t \int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\frac{h(u)}{u^{1-\varrho}}\,du\right| \\ &\le \int_{t/2}^t \left|\int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\right||h(u)|\,\frac{du}{u} \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\Big(\int_{t/2}^t \left|\int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\right|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\ &\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}^{\max\{0,1-2/\gamma'\}} \\ &\quad\times\Big(\int_{t/2}^t \left|\int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\right|^2\frac{du}{u}\Big)^{1/\max\{2,\gamma'\}}. \end{aligned} \tag{2.9}$$

By Lemma 2.1 and Hölder's inequality, for any $0 < \epsilon < \min\{1/(2q'), 1/L\}$,

$$\begin{aligned} &\int_{t/2}^t \left|\int_\Sigma \exp(-2\pi i\xi \cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\right|^2\frac{du}{u} \\ &\quad= \int_{t/2}^t \iint_{\Sigma\times\Sigma} \exp(-2\pi i A^*_{\Phi(u)}\xi \cdot (\theta - w))\Omega(\theta)\overline{\Omega(w)}\,d\sigma(\theta)\,d\sigma(w)\frac{du}{u} \\ &\quad\le \iint_{\Sigma\times\Sigma}\left|\int_{t/2}^t \exp(-2\pi i\xi \cdot A_{\Phi(u)}(\theta - w))\frac{du}{u}\right||\Omega(\theta)\overline{\Omega(w)}|\,d\sigma(\theta)\,d\sigma(w) \\ &\quad\le C\iint_{\Sigma\times\Sigma}|\xi \cdot (A_{\Phi(t)}P(\theta - w))|^{-\epsilon}|\Omega(\theta)\overline{\Omega(w)}|\,d\sigma(\theta)\,d\sigma(w) \\ &\quad\le C\|\Omega\|_{L^q(\Sigma)}^2\Big(\iint_{\Sigma\times\Sigma}|P^*A^*_{\Phi(t)}\xi \cdot (\theta - w)|^{-\epsilon q'}\,d\sigma(\theta)\,d\sigma(w)\Big)^{1/q'} \\ &\quad\le C\|\Omega\|_{L^q(\Sigma)}^2|A^*_{\Phi(t)}\xi|^{-\epsilon}, \end{aligned} \tag{2.10}$$

where the last inequality follows from [12, page 533] (also see [17, proof of Lemma 1]). If follows from (2.9) and (2.10) that

$$|\widehat{\sigma_{h,t}}(\xi)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{-1/(2q'\max\{2,\gamma'\}L)}, \tag{2.11}$$

where we take $\epsilon = 1/(2q'L)$. Similarly,

$$|\,|\widehat{\sigma_{h,t}}|(\xi)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{-1/(2q'\max\{2,\gamma'\}L)}.$$

This, together with (2.7), (2.8) and (2.11), implies (2.5). On the other hand, by a change of variables, (1.1) and Hölder's inequality,

$$\begin{aligned}
|\widehat{\sigma_{h,t}}(\xi)| &= \bigg|\frac{1}{t^\varrho}\int_{t/2}^{t}\int_\Sigma (\exp(-2\pi i\xi\cdot A_{\Phi(u)}\theta)-1)\Omega(\theta)\,d\sigma(\theta)\frac{h(u)}{u^{1-\varrho}}\,du\bigg| \\
&\le C\int_{t/2}^{t}\int_\Sigma |\Omega(\theta)|\,|\xi\cdot A_{\Phi(u)}\theta|\,d\sigma(\theta)|h(u)|\,\frac{du}{u} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\bigg(\int_{t/2}^{t}\bigg|\int_\Sigma |\Omega(\theta)|\,|\xi\cdot A_{\Phi(u)}\theta|\,d\sigma(\theta)\bigg|^{\gamma'}\frac{du}{u}\bigg)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\bigg(\int_{\Phi(t/2)}^{\Phi(t)}\bigg|\int_\Sigma |\Omega(\theta)|\,|\xi\cdot A_u\theta|\,d\sigma(\theta)\bigg|^{\gamma'}\frac{du}{\Phi'(\Phi^{-1}(u))\Phi^{-1}(u)}\bigg)^{1/\gamma'} \\
&\le C_\Phi^{-1/\gamma'}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\bigg(\int_{\Phi(t/2)}^{\Phi(t)}\bigg|\int_\Sigma |\Omega(\theta)|\,|\xi\cdot A_u\theta|\,d\sigma(\theta)\bigg|^{\gamma'}\frac{du}{u}\bigg)^{1/\gamma'} \\
&\le C_\Phi^{-1/\gamma'}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\bigg(\int_{\varsigma}^{1}\bigg|\int_\Sigma |\Omega(\theta)|\,|A^*_{\Phi(t)}\xi\cdot A_u\theta|\,d\sigma(\theta)\bigg|^{\gamma'}\frac{du}{u}\bigg)^{1/\gamma'},
\end{aligned}$$

where ς is as in Lemma 2.2. Note that $\varsigma \ge c_\Phi^{-1}$ and $|A_u\theta| \le C$ for $u \in [\varsigma, 1]$ and $\theta \in \Sigma$. Thus,

$$|\widehat{\sigma_{h,t}}(\xi)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|. \tag{2.12}$$

It follows from (2.7) and (2.12) that

$$|\widehat{\sigma_{h,t}}(\xi)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{1/(4q'\gamma'L)}. \tag{2.13}$$

Similarly, we can prove that

$$|\,|\widehat{\sigma_{h,t}}|(\xi) - |\widehat{\sigma_{h,t}}|(0)| \le C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{1/(4q'\gamma'L)},$$

which, combined with (2.13), implies (2.6). This proves Lemma 2.3. □

Lemma 2.4. *Let h, Ω, Φ be as in Lemma 2.3. Then, for any $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$\|\sigma_h^*(f)\|_{L^p(\mathbb{R}^n)} \le Cq'\gamma'\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.14}$$

$$\|M_{h,q,\gamma}(f)\|_{L^p(\mathbb{R}^n)} \le Cq'\gamma'\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.15}$$

The constant C is independent of h, Ω, q, γ, but depends on Φ.

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Proof. We only prove the case $\Phi \in \mathfrak{F}$ satisfying the condition (a); the other case can be obtained similarly. By Remark 1.1, there exists $B_\Phi > 1$ such that $\Phi(2t) \geq B_\Phi \Phi(t)$ for any $t > 0$. For convenience, we set $N_{q,\gamma} = q'\gamma' \|h\|_{\Delta_\gamma(\mathbb{R}^+)} \|\Omega\|_{L^q(\Sigma)}$. For $k \in \mathbb{Z}$, we define the family of measures $\{\mu_k\}_{k\in\mathbb{Z}}$ and a maximal operator μ^* on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f(x)\,d\mu_k(x) = \int_{2^{q'\gamma' k} < r(y) \leq 2^{q'\gamma'(k+1)}} \frac{h(r(y))\Omega(y')}{\rho(y)^\alpha} f(A_{\Phi(r(y))} y')\,dy,$$
$$\mu^*(f)(x) = \sup_{k\in\mathbb{Z}} \big| |\mu_k| * f(x) \big|,$$

where $|\mu_k|$ is defined in the same way as μ_k, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$. One can easily check that

$$\sigma_h^*(f) \leq \mu^*(|f|).$$

Therefore, to prove (2.14), it suffices to prove that

$$\|\mu^*(f)\|_{L^p(\mathbb{R}^n)} \leq C_p N_{q,\gamma} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{2.16}$$

Below we estimate $|\widehat{\mu_k}(\xi)|$. By a change of variable, (2.3) and the same argument as in getting (2.12),

$$\begin{aligned}
\big| |\widehat{\mu_k}|(\xi) - |\widehat{\mu_k}|(0) \big| &= \Big| \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \int_\Sigma (\exp(-2\pi i \xi \cdot A_{\Phi(u)}\theta) - 1) |\Omega(\theta)|\,d\sigma(\theta) |h(u)| \frac{du}{u} \Big| \\
&\leq \sum_{i=0}^{[q'\gamma']} \int_{2^{q'\gamma' k+i}}^{2^{q'\gamma' k+i+1}} \int_\Sigma |\Omega(\theta)|\,|\xi \cdot A_{\Phi(u)}\theta|\,d\sigma(\theta) |h(u)| \frac{du}{u} \\
&\leq \sum_{i=0}^{[q'\gamma']} \|h\|_{\Delta_\gamma(\mathbb{R}^+)} \|\Omega\|_{L^q(\Sigma)} |A^*_{\Phi(2^{q'\gamma' k+i+1})}\xi| \\
&\leq ([q'\gamma'] + 1) \|h\|_{\Delta_\gamma(\mathbb{R}^+)} \|\Omega\|_{L^q(\Sigma)} ((\Phi(2^{q'\gamma' k+[q'\gamma']+1}) s(\xi))^{1/a_1} \\
&\qquad + (\Phi(2^{q'\gamma' k+[q'\gamma']+1}) s(\xi))^{1/b_1}) \\
&\leq C N_{q,\gamma} (c_\Phi^{([q'\gamma']+1)/a_1} (\Phi(2^{q'\gamma' k}) s(\xi))^{1/a_1} \\
&\qquad + c_\Phi^{([q'\gamma']+1)/b_1} (\Phi(2^{q'\gamma' k}) s(\xi))^{1/b_1}).
\end{aligned} \tag{2.17}$$

One can easily check that

$$\big| |\widehat{\mu_k}|(\xi) \big| \leq C N_{q,\gamma}, \quad \forall\, \xi \in \mathbb{R}^n. \tag{2.18}$$

Interpolating between (2.17) and (2.18) leads to

$$\big| |\widehat{\mu_k}|(\xi) - |\widehat{\mu_k}|(0) \big| \leq C N_{q,\gamma} ((\Phi(2^{q'\gamma' k}) s(\xi))^{1/(4q'\gamma' a_1 L)} + (\Phi(2^{q'\gamma' k}) s(\xi))^{1/(4q'\gamma' b_1 L)}). \tag{2.19}$$

On the other hand, by a change of variable, Hölder's inequality and (2.10), for any $0 < \epsilon < \min\{1/(2q'), 1/L\}$,

$$\begin{aligned}
||\widehat{\mu_k}|(\xi)| &= \Big|\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \int_\Sigma \exp(-2\pi i\xi\cdot A_{\Phi(u)}\theta)|\Omega(\theta)|\,d\sigma(\theta)|h(u)|\frac{du}{u}\Big| \\
&\le \Big(\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |h(u)|^\gamma \frac{du}{u}\Big)^{1/\gamma} \\
&\quad\times \Big(\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_{\Phi(u)}\theta)|\Omega(\theta)|\,d\sigma(\theta)\Big|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\
&\le C(q'\gamma')^{1/\gamma}\|h\|_{\Delta_\gamma(\mathbb{R}^+)} \\
&\quad\times \Big(\sum_{i=0}^{[q'\gamma']}\int_{2^{q'\gamma' k+i}}^{2^{q'\gamma' k+i+1}} \Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_{\Phi(u)}\theta)|\Omega(\theta)|\,d\sigma(\theta)\Big|^{\gamma'}\frac{du}{u}\Big)^{1/\gamma'} \\
&\le C(q'\gamma')^{1/\gamma}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^1(\Sigma)}^{\max\{0,1-2/\gamma'\}} \\
&\quad\times \Big(\sum_{i=0}^{[q'\gamma']}\Big(\int_{2^{q'\gamma' k+i}}^{2^{q'\gamma' k+i+1}} \Big|\int_\Sigma \exp(-2\pi i\xi\cdot A_{\Phi(u)}\theta)\Omega(\theta)\,d\sigma(\theta)\Big|^{2}\frac{du}{u}\Big)^{\gamma'/\max\{2,\gamma'\}}\Big)^{1/\gamma'} \\
&\le C(q'\gamma')^{1/\gamma}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}^{\max\{0,1-2/\gamma'\}} \\
&\quad\times \Big(\sum_{i=0}^{[q'\gamma']}(\|\Omega\|_{L^q(\Sigma)}^2|A^*_{\Phi(2^{q'\gamma' k+i+1})}\xi|^{-\epsilon})^{\gamma'/\max\{2,\gamma'\}}\Big)^{1/\gamma'} \\
&\le C(q'\gamma')^{1/\gamma}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\Big(\sum_{i=0}^{[q'\gamma']}|A^*_{\Phi(2^{q'\gamma' k+i+1})}\xi|^{-\epsilon\gamma'/\max\{2,\gamma'\}}\Big)^{1/\gamma'}.
\end{aligned}$$

This, together with (2.4) and (2.18), leads to

$$||\widehat{\mu_k}|(\xi)| \le CN_{q,\gamma}((\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' a_2 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' b_2 L)}). \quad (2.20)$$

We can choose a nonnegative $C_0^\infty(\mathbb{R}^n)$ function ψ such that $\hat{\psi}(0) = 1$ and $\mathrm{supp}(\psi) \subset \{x \in \mathbb{R}^n : r(x) \le 1\}$. Define the family of measures $\{\nu_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\nu_k(\xi) = |\mu_k|(\xi) - \psi_k(\xi)|\widehat{\mu_k}|(0), \quad (2.21)$$

where $\psi_k(x) = \Phi(2^{q'\gamma' k})^{-\alpha}\psi(A_{\Phi(2^{q'\gamma' k})^{-1}}x)$. Let $\Psi_k = |\widehat{\mu_k}|(0)\psi_k$. One can easily check that

$$\mu^*(f) \le G(f) + \Psi^*(|f|), \quad (2.22)$$
$$\nu^*(f) \le \mu^*(f) + \Psi^*(|f|), \quad (2.23)$$

where $\nu^*(f) = \sup_{k\in\mathbb{Z}}||\nu_k| * f|$, $\Psi^*(f) = \sup_{k\in\mathbb{Z}}||\Psi_k| * f|$ and $G(f) = (\sum_{k\in\mathbb{Z}}|\nu_k * f|^2)^{1/2}$. By the L^p boundedness of the Hardy–Littlewood maximal function on $\mathbb{R}^n$ with respect to the function $r(\cdot)$,

$$\Big\|\sup_{k\in\mathbb{Z}}|\psi_k * f|\Big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \quad (2.24)$$

where positive C is independent of γ and q. Thus, by (2.18),

$$\|\Psi^*(f)\|_{L^p(\mathbb{R}^n)} \le CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{2.25}$$

where C is independent of γ and q. By (2.22) and (2.25), to prove (2.16), it suffices to prove that

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \le CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{2.26}$$

where C is independent of γ and q. By a well-known property of Rademacher's function, (2.26) follows from

$$\|\tau_\epsilon(f)\|_{L^p(\mathbb{R}^n)} \le CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

where $\tau_\epsilon(f) = \sum_{k\in\mathbb{Z}} \epsilon_k \nu_k * f$ with $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1 (the inequality is uniform in ϵ) and C is independent of γ and q. It follows from (2.3)–(2.4) and (2.18)–(2.20) that

$$|\widehat{\nu_k}(\xi)| \le CN_{q,\gamma} \min\{1, (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' a_2 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' b_2 L)}\}, \tag{2.27}$$

$$|\widehat{\nu_k}(\xi)| \le CN_{q,\gamma}((\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' a_1 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' b_1 L)}). \tag{2.28}$$

Let $\{\Gamma_k\}_{k\in\mathbb{Z}}$ be a sequence of nonnegative functions in $C_0^\infty((0,\infty))$ such that

$$\operatorname{supp}(\Gamma_k) \subset [\Phi(2^{q'\gamma'(k+1)})^{-1}, \Phi(2^{q'\gamma'(k-1)})^{-1}], \quad \sum_{k\in\mathbb{Z}} \Gamma_k^2(t) = 1,$$

$$|(d/dt)^j \Gamma_k(t)| \le C_j/t^j \quad \text{for } j = 1, 2, \ldots,$$

where C_j $(j = 1, 2, \ldots)$ are independent of q and γ. Define the Fourier multiplier operators S_k by

$$\widehat{S_k(f)}(\xi) = \Gamma_k(s(\xi))\hat{f}(\xi). \tag{2.29}$$

By Littlewood–Paley theory, for any $1 < p < \infty$, $\{g_k\} \in L^p(\mathbb{R}^n, \ell^2)$ and $f \in L^p(\mathbb{R}^n)$, there exists $C_p > 0$ which is independent of q and γ such that

$$\Big\|\sum_{k\in\mathbb{Z}} S_k(g_k)\Big\|_{L^p(\mathbb{R}^n)} \le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}, \tag{2.30}$$

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |S_k(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.31}$$

By the definition of S_k, we can write

$$\tau_\epsilon(f) = \sum_{k\in\mathbb{Z}} \epsilon_k \nu_k * S_{j+k} S_{j+k}(f) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \epsilon_k S_{j+k}(\nu_k * S_{j+k}(f)) := \sum_{j\in\mathbb{Z}} \tau_j(f). \tag{2.32}$$

Then, by Plancherel's theorem, (2.27)–(2.28) and (2.30)–(2.31),

$$\begin{aligned}\|\tau_j(f)\|_{L^2(\mathbb{R}^n)}^2 &\le C\sum_{k\in\mathbb{Z}} \int_{\{\Phi(2^{q'\gamma'(k+j+1)})^{-1} \le s(\xi) \le \Phi(2^{q'\gamma'(k+j-1)})^{-1}\}} |\hat{f}(\xi)|^2 |\widehat{\nu_k}(\xi)|^2\, d\xi \\ &\le C(N_{q,\gamma} D_j)^2 \|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

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where $D_j = (B_\Phi^{-(j-1)/(8a_1L)} + B_\Phi^{-(j-1)/(8b_1L)})\chi_{\{j\geq 1\}}(j) + (B_\Phi^{(j+1)/(4a_2L)} + B_\Phi^{(j+1)/(4b_2L)})$ $\chi_{\{j<1\}}(j)$. Then

$$\|\tau_j(f)\|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma}B_\Phi^{-c|j|}\|f\|_{L^2(\mathbb{R}^n)}, \tag{2.33}$$

where C and c are independent of γ and q. This, together with (2.32), implies that

$$\|\tau_\epsilon(f)\|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^2(\mathbb{R}^n)}.$$

We also obtain that

$$\|G(f)\|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^2(\mathbb{R}^n)},$$

which, by combining (2.22) and (2.23) with (2.25), yields

$$\|\nu^*(f)\|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^2(\mathbb{R}^n)}.$$

This, together with the trivial estimate $\sup_{k\in\mathbb{Z}}\|\nu_k\| \leq CN_{q,\gamma}$ and the proof of [12, Lemma, page 544], implies that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\nu_k * g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}$$

holds for arbitrary functions $\{g_k\} \in L^p(\mathbb{R}^n, \ell^2)$ with $p = 4$ or $p = 4/3$. This, combining (2.30) with (2.31), implies that

$$\|\tau_j(f)\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)} \tag{2.34}$$

for $p = 4$ or $p = 4/3$. By (2.32) and the interpolation between (2.33) and (2.34),

$$\|\tau_\epsilon(f)\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 4/3 < p < 4.$$

Consequently,

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 4/3 < p < 4.$$

Reasoning as above, (2.22)–(2.23), (2.25), (2.30)–(2.33), the trivial estimate $\sup_{k\in\mathbb{Z}}\|\nu_k\| \leq CN_{q,\gamma}$, the proof of [12, Lemma, page 544] and an interpolation argument yield

$$\|G(f)\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 8/7 < p < 8.$$

By using this argument repeatedly, we can obtain ultimately (2.26). Equation (2.14) is proved.

It remains to prove (2.15). Let ψ_k be as in (2.21). Define the family of measures $\{\omega_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\omega_k(\xi) = \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\widehat{\sigma_{h,t}}|(\xi)\,\frac{dt}{t} - \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\widehat{\sigma_{h,t}}|(0)\,\frac{dt}{t}\psi_k(\xi). \tag{2.35}$$

By Lemma 2.3, one can easily check that

$$\begin{aligned} &\big|\,|\widehat{\sigma_{h,t}}|(\xi) - |\widehat{\sigma_{h,t}}|(0)\widehat{\psi_k}(\xi)\big| \\ &\quad\leq C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\min\{1, |A^*_{\Phi(t)}\xi|^{1/(4q'\gamma' L)} + |A^*_{\Phi(2^{q'\gamma' k})}\xi|^{1/(4q'\gamma' L)}\}, \end{aligned} \tag{2.36}$$

$$\begin{aligned} &\big|\,|\widehat{\sigma_{h,t}}|(\xi) - |\widehat{\sigma_{h,t}}|(0)\widehat{\psi_k}(\xi)\big| \\ &\quad\leq C\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}(|A^*_{\Phi(t)}\xi|^{-1/(4q'\gamma' L)} + |A^*_{\Phi(2^{q'\gamma' k})}\xi|^{-1/(4q'\gamma' L)}). \end{aligned} \tag{2.37}$$

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It follows from (2.3)–(2.4) and (2.36)–(2.37) that

$$|\widehat{\omega_k}(\xi)| \le CN_{q,\gamma} \min\{1, (\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' a_1 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' b_1 L)}\}, \quad (2.38)$$
$$|\widehat{\omega_k}(\xi)| \le CN_{q,\gamma}((\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' a_2 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' b_2 L)}). \quad (2.39)$$

We get from (2.35) that

$$M_{h,q,\gamma}(f) \le g(f) + \Theta^*(|f|), \quad (2.40)$$
$$\omega^*(f) \le M_{h,q,\gamma}(f) + \Theta^*(|f|), \quad (2.41)$$

where $\omega^*(f) = \sup_{k\in\mathbb{Z}} ||\omega_k| * f|$, $g(f) = (\sum_{k\in\mathbb{Z}} |\omega_k * f|^2)^{1/2}$ and $\Theta^*(f) = \sup_{k\in\mathbb{Z}} ||\Theta_k| * f|$ with $\Theta_k = \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\widehat{\sigma_{h,t}}|(0)(dt/t)\psi_k$. It follows from (2.5) and (2.24) that

$$\|\Theta^*(f)\|_{L^p(\mathbb{R}^n)} \le CN_{q,\gamma}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \quad (2.42)$$

where C is independent of q and γ. By (2.38)–(2.42), the trivial estimate $\sup_{k\in\mathbb{Z}} \|\omega_k\| \le CN_{q,\gamma}$ and the same arguments as in getting (2.14), we obtain (2.15). This completes the proof of Lemma 2.4. □

Applying Lemma 2.4, we obtain the following result.

Lemma 2.5. *Let Ω, Φ be as in Lemma 2.3 and $h \in \Delta_\gamma(\mathbb{R}^+)$ for some $\gamma \in (1, 2]$. Then there exists $C > 0$ such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le C(q'\gamma')^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}, \quad 2 \le p < \infty, \quad (2.43)$$

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le Cq'\gamma'\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2. \quad (2.44)$$

The constant C is independent of h, Ω, q, γ, but depends on Φ.

Proof. The idea of the proof is similar to the one appearing in the proof of [1, Lemma 3.7]. First we prove (2.43). For fixed $2 \le p < \infty$, by duality, there exists a nonnegative function $f \in L^{(p/2)'}(\mathbb{R}^n)$ with $\|f\|_{L^{(p/2)'}(\mathbb{R}^n)} \le 1$ such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t} f(x)\, dx. \quad (2.45)$$

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By a change of variable and Hölder's inequality, we obtain

$$\begin{aligned}
&|\sigma_{h,t} * g_k(x)|^2 \\
&\quad \le \Big(\int_{t/2<r(y)\le t} \frac{|h(r(y))\Omega(y)|}{r(y)^\alpha} |g_k(x - A_{\Phi(r(y))}y')|\,dy\Big)^2 \\
&\quad \le \Big(\int_{t/2}^t \int_\Sigma |g_k(x - A_{\Phi(u)}y'))|\,|\Omega(y')|\,d\sigma(y')|h(u)|\,\frac{du}{u}\Big)^2 \\
&\quad \le C\|h\|^\gamma_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\Big(\int_{t/2}^t \int_\Sigma |g_k(x - A_{\Phi(u)}y')|^2|\Omega(y')|\,d\sigma(y')|h(u)|^{2-\gamma}\,\frac{du}{u}\Big). \quad (2.46)
\end{aligned}$$

Thus, by (2.45), (2.46) and Hölder's inequality, one can check that

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} \\
&\quad \le C\|h\|^\gamma_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |g_k(x)|^2 \tilde{M}_{|h|^{2-\gamma},q,\gamma}(\tilde{f})(-x)\,dx \\
&\quad \le C\|h\|^\gamma_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)} \Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} \|\tilde{M}_{|h|^{2-\gamma},q,\gamma}(\tilde{f})\|_{L^{(p/2)'}(\mathbb{R}^n)}, \quad (2.47)
\end{aligned}$$

where $\tilde{f}(x) = f(-x)$ and $\tilde{M}_{|h|^{2-\gamma},q,\gamma}(f)$ denotes $M_{|h|^{2-\gamma},q,\gamma}$ with $\varrho = 1$. It is easy to check that $|h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)$; thus, by (2.15),

$$\begin{aligned}
\|\tilde{M}_{|h|^{2-\gamma},q,\gamma}(|\tilde{f}|)\|_{L^{(p/2)'}(\mathbb{R}^n)} &\le Cq'\Big(\frac{\gamma}{2-\gamma}\Big)'\|\,|h|^{2-\gamma}\|_{\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^{(p/2)'}(\mathbb{R}^n)} \\
&\le Cq'\gamma'\|h\|^{2-\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)},
\end{aligned}$$

which, combined with (2.47), implies (2.43).

Next, we prove (2.44). Assume that $1 < p < 2$; by duality, there exist functions $\{f_k(x,t)\}$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ with $\|\{f_k(\cdot,\cdot)\}\|_{L^{p'}(\mathbb{R}^n,\ell^2(L^2([2^{q'\gamma' k},2^{q'\gamma'(k+1)}],dt/t)))} \le 1$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \\
&\quad \le \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} \sigma_{h,t} * g_k(x) f_k(x,t) \frac{dt}{t}\,dx \\
&\quad \le C(q'\gamma')^{1/2}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \|H\|^{1/2}_{L^{p'/2}(\mathbb{R}^n)}, \quad (2.48)
\end{aligned}$$

where

$$H(x) = \sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * \tilde{f}_k(x,t)|^2 \frac{dt}{t} \quad \text{and} \quad \tilde{f}_k(x,t) = f(-x,t).$$

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Since $p' > 2$, there exists a nonnegative function $u \in L^{(p'/2)'}(\mathbb{R}^n)$ such that

$$\|H\|_{L^{p'/2}(\mathbb{R}^n)} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * \tilde{f}_k(x,t)|^2 \frac{dt}{t} u(x)\,dx.$$

By (2.14), Hölder's inequality and the fact that $|h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)$,

$$\begin{aligned}\|H\|_{L^{p'/2}(\mathbb{R}^n)} &\le C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)} \int_{\mathbb{R}^n} \tilde{\sigma}^*_{|h|^{2-\gamma}}(\tilde{u})(-x)\Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\tilde{f}_k(x,t)|^2 \frac{dt}{t}\Big)\,dx \\ &\le C\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|\tilde{\sigma}^*_{|h|^{2-\gamma}}(\tilde{u})\|_{L^{(p'/2)'}(\mathbb{R}^n)} \\ &\le Cq'\Big(\frac{\gamma}{2-\gamma}\Big)'\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}^+)}\|\,|h|^{2-\gamma}\|_{\Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+)}\|\Omega\|^2_{L^q(\Sigma)} \\ &\le Cq'\gamma'\|h\|^2_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|^2_{L^q(\Sigma)}, \end{aligned} \tag{2.49}$$

where $\tilde{u}(x) = u(-x)$ and $\tilde{\sigma}^*_{|h|^{2-\gamma}}(\tilde{u})$ denotes $\sigma^*_{|h|^{2-\gamma}}(\tilde{u})$ with $\varrho = 1$. Equation (2.44) follows from (2.48) and (2.49). This proves Lemma 2.5. □

3. Proof of main results

This section is devoted to the proofs of the main results.

Proof of Theorem 1.1. Let h, Ω, Φ be as in Theorem 1.2. By Minkowski's inequality, we can write

$$\begin{aligned}\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)(x) &= \Big(\int_0^\infty \Big|\sum_{k=-\infty}^{0} \frac{1}{t^\varrho}\int_{2^{k-1}t<r(y)\le 2^k t} \frac{\Omega(y)h(r(y))}{r(y)^{\alpha-\varrho}} f(x - A_{\Phi(r(y))}y')\,dy\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\le \sum_{k=-\infty}^{0}\Big(\int_0^\infty \Big|\frac{1}{t^\varrho}\int_{2^{k-1}t<r(y)\le 2^k t} \frac{\Omega(y)h(r(y))}{r(y)^{\alpha-\varrho}} f(x - A_{\Phi(r(y))}y')\,dy\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\le (1-2^{-\sigma})^{-1}\Big(\int_0^\infty |\sigma_{h,t} * f(x)|^2 \frac{dt}{t}\Big)^{1/2}. \end{aligned} \tag{3.1}$$

Let

$$S_{h,\Omega,\varrho}(f)(x) := \Big(\int_0^\infty |\sigma_{h,t} * f(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

By Lemma 2.3 and (2.3)–(2.4), one can verify that

$$\begin{aligned}&\Big(\int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\widehat{\sigma_{h,t}}(\xi)|^2 \frac{dt}{t}\Big)^{1/2} \\ &\quad\le C(\gamma-1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)} \\ &\qquad\times \min\{1, (\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' a_1 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{1/(4q'\gamma' b_1 L)}, \\ &\qquad\qquad (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' a_2 L)} + (\Phi(2^{q'\gamma' k})s(\xi))^{-1/(4q'\gamma' b_2 L)}\}. \end{aligned} \tag{3.2}$$

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Let S_k be as in (2.29). Then, by Minkowski's inequality and the definition of S_k,

$$\mathcal{S}_{h,\Omega,\varrho}(f)(x) \le \sum_{j\in\mathbb{Z}} T_j(f)(x), \tag{3.3}$$

where

$$T_j(f)(x) = \Big(\sum_{k\in\mathbb{Z}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\sigma_{h,t} * S_{j+k} S_{j+k} f(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

Using (2.30)–(2.31) and invoking Lemma 2.5,

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \le C(\gamma-1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \le p < \infty, \tag{3.4}$$

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \le C(\gamma-1)^{-1}(q-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2. \tag{3.5}$$

The constant $C > 0$ is independent of q and γ, but depends on Φ. On the other hand, by Plancherel's theorem and (3.2),

$$\begin{aligned}\|T_j(f)\|_{L^2(\mathbb{R}^n)}^2 &\le \sum_{k\in\mathbb{Z}} \int_{\{\Phi(2^{q'\gamma'(k+j+1)})^{-1} \le s(\xi) \le \Phi(2^{q'\gamma'(k+j-1)})^{-1}\}} \int_{2^{q'\gamma' k}}^{2^{q'\gamma'(k+1)}} |\widehat{\sigma_{h,t}}(\xi)|^2 \frac{dt}{t} |\hat{f}(\xi)|^2\, d\xi \\ &\le C B_\Phi^{-2|j|\delta}(\gamma-1)^{-1}(q-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}^2\|\Omega\|_{L^q(\Sigma)}^2\|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

where $B_\Phi > 1$ is as in Lemma 2.4. The constants C and δ are independent of q and γ. That is,

$$\|T_j(f)\|_{L^2(\mathbb{R}^n)} \le C B_\Phi^{-|j|\delta}(\gamma-1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^2(\mathbb{R}^n)}. \tag{3.6}$$

Interpolating between (3.4)–(3.5) and (3.6),

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \le C B_\Phi^{-|j|\delta_p}(\gamma-1)^{-1/2}(q-1)^{-1/2}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \le p < \infty, \tag{3.7}$$

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \le C B_\Phi^{-|j|\epsilon_p}(\gamma-1)^{-1}(q-1)^{-1}\|h\|_{\Delta_\gamma(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2. \tag{3.8}$$

The constants C, δ_p and ϵ_p are independent of q and γ. Theorem 1.2 follows from (3.1), (3.3) and (3.7)–(3.8). □

Proof of Theorem 1.2. Theorem 1.3 follows directly from Theorem 1.2 and an extrapolation argument as in the proof of [17, Theorem 1.2]) (also see [18, Theorem 1.2]). We omit the details. □

4. Additional results

As applications of our main results, we consider the corresponding parametric Marcinkiewicz integral operators $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}$ and $\mathscr{M}_{h,\Omega,\Phi,S,\varrho}$ related to the Littlewood–Paley g^*_λ-function and the area integral S, respectively, which are interesting in themselves. More precisely, let Φ be as in (1.2); we define the operators $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}$ and $\mathscr{M}_{h,\Omega,\Phi,S,\varrho}$ by

$$\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)(x) := \bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+r(x-y)}\Big)^{\alpha\lambda}\bigg|\frac{1}{t^\varrho}\int_{r(z)\le t}\frac{h(r(z))\Omega(z)}{r(z)^{\alpha-\varrho}}f(y-A_{\Phi(r(z))}z')\,dz\bigg|^2\frac{dy\,dt}{t^{\alpha+1}}\bigg)^{1/2},$$

where $\lambda > 0$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$;

$$\mathscr{M}_{h,\Omega,\Phi,S,\varrho}(f)(x) := \bigg(\iint_{\Gamma(x)}\bigg|\frac{1}{t^\varrho}\int_{r(z)\le t}\frac{h(r(z))\Omega(z)}{r(z)^{\alpha-\varrho}}f(y-A_{\Phi(r(z))}z')\,dz\bigg|^2\frac{dy\,dt}{t^{\alpha+1}}\bigg)^{1/2},$$

where $\Gamma(x) = \{(y,t)\in\mathbb{R}^{n+1}_+ : r(x-y) < t\}$.

Theorem 4.1. *Let $\Omega \in L(\log^+ L)^{1/2}(\Sigma)$ satisfying (1.1) and $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$. Suppose that $\Phi \in \mathfrak{F}$ and $\lambda > 1$. Then, for $2 \le p < \infty$,*

$$\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C(\lambda,\alpha,\varrho,\Phi)(1+\|\Omega\|_{L(\log^+L)^{1/2}(\Sigma)})(1+N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)}, \quad (4.1)$$

$$\|\mathscr{M}_{h,\Omega,\Phi,S,\varrho}(f)\|_{L^p(\mathbb{R}^n)} \le C(\alpha,\varrho,\Phi)(1+\|\Omega\|_{L(\log^+L)^{1/2}(\Sigma)})(1+N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)}. \quad (4.2)$$

Remark 4.2. Because of (1.3) and (1.4), Theorem 4.1 essentially improves and generalizes [10, Theorem 2], even in the special case $r(x) = |x|$ and $\Phi(t) = t$.

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.3. *Let $\lambda > 1$. Then there exists a constant $C(\lambda,\alpha)$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n}(\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)(x))^2 g(x)\,dx \le C(\lambda,n)\int_{\mathbb{R}^n}(\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)(x))^2 M(g)(x)\,dx,$$

where M is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$ with respect to the function $r(\cdot)$.

Proof. By the definition of $\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}(\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)(x))^2 g(x)\,dx \\
&\quad= \int_{\mathbb{R}^n}\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+r(x-y)}\Big)^{\alpha\lambda} \\
&\qquad\times\bigg|\frac{1}{t^\varrho}\int_{r(z)\le t}\frac{h(r(z))\Omega(z)}{r(z)^{\alpha-\varrho}}f(y-A_{\Phi(r(z))}z')\,dz\bigg|^2\frac{dy\,dt}{t^{\alpha+1}}g(x)\,dx \\
&\quad\le \int_{\mathbb{R}^n}\int_0^\infty\bigg|\frac{1}{t^\varrho}\int_{r(z)\le t}\frac{h(r(z))\Omega(z)}{r(z)^{\alpha-\varrho}}f(y-z)\,dz\bigg|^2 \\
&\qquad\times\bigg(\sup_{t>0}\frac{1}{t^\alpha}\int_{\mathbb{R}^n}\Big(\frac{t}{t+r(x-y)}\Big)^{\alpha\lambda}g(x)\,dx\bigg)\frac{dt}{t}\,dy \\
&\quad\le C(\lambda,\alpha)\int_{\mathbb{R}^n}(\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)(y))^2 M(g)(y)\,dy
\end{aligned}$$

for $\lambda > 1$. This proves Lemma 4.1. □

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PROOF OF THEOREM 4.1. First we prove (4.1). For $2 \le p < \infty$, by duality,

$$\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)\|^2_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^q(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} (\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)(x))^2 g(x)\,dx,$$

where $q = (p/2)'$ and the supremum is taken over all g satisfying $\|g\|_{L^q(\mathbb{R}^n)} \le 1$. By the L^p bounds of M, Hölder's inequality, Lemma 4.3 and Theorem 1.3, t

$$\begin{aligned}\|\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)\|^2_{L^p(\mathbb{R}^n)} &\le C(\lambda,\alpha) \sup_{\|g\|_{L^q(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} (\mathscr{M}_{h,\Omega,\varrho,\Phi}(f)(x))^2 M(g)(x)\,dx \\ &\le C(\lambda,\alpha)\|\mathscr{M}_{h,\Omega,\Phi,\varrho}(f)\|^2_{L^p(\mathbb{R}^n)} \\ &\le C(\lambda,\alpha,\varrho,\Phi)(1+\|\Omega\|_{L(\log^+ L)^{1/2}(\Sigma)})^2 \\ &\quad \times (1+N_{1/2}(h))^2\|f\|^2_{L^p(\mathbb{R}^n)}, \quad 2\le p<\infty.\end{aligned}$$

Thus, (4.1) holds. On the other hand, it is easy to check that

$$\mathscr{M}_{h,\Omega,\Phi,S,\varrho}(f)(x) \le 2^{\alpha\lambda/2}\mathscr{M}^*_{h,\Omega,\Phi,\lambda,\varrho}(f)(x),$$

which, combined with (4.1), implies (4.2). Theorem 4.1 is proved. □

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References

[1] H. Al-Qassem and Y. Pan, 'On certain estimates for Marcinkiewicz integrals and extrapolation', *Collect. Math.* **60** (2009), 123–145.

[2] A. Al-Salman, 'A note on parabolic Marcinkiewicz integrals along surfaces', *Proc. A. Razmadze Math. Inst.* **27** (2010), 21–36.

[3] A. Al-Salman, 'Parabolic Marcinkiewicz integrals along surfaces on product domains', *Acta Math. Sin. (Engl. Ser.)* **27** (2011), 1–18.

[4] A. Al-Salman, H. Al-Qassem, L. C. Cheng and Y. Pan, 'L^p bounds for the function of Marcinkiewicz', *Math. Res. Lett.* **9** (2002), 697–700.

[5] A. P. Calderón and A. Torchinsky, 'On singular integrals', *Amer. J. Math.* **78** (1956), 289–309.

[6] Y. Chen and Y. Ding, 'L^p bounds for parabolic Marcinkiewicz integral with rough kernels', *J. Korean Math. Soc.* **44** (2007), 733–745.

[7] Y. Chen and Y. Ding, 'The parabolic Littlewood–Paley operator with Hardy space kernels', *Canad. Math. Bull.* **52** (2009), 521–534.

[8] Y. Ding, D. Fan and Y. Pan, 'L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel', *Acta Math. Sin. (Engl. Ser.)* **16** (2000), 593–600.

[9] Y. Ding, D. Fan and Y. Pan, 'On the L^p boundedness of Marcinkiewicz integrals', *Michigan Math. J.* **50** (2002), 17–26.

[10] Y. Ding, S. Lu and K. Yabuta, 'A problem on rough parametric Marcinkiewicz functions', *J. Aust. Math. Soc.* **72** (2002), 13–21.

[11] Y. Ding, Q. Xue and K. Yabuta, 'Boundedness of the Marcinkiewicz integrals with rough kernel associated to surfaces', *Tohoku Math. J.* (2) **62** (2010), 233–262.

[12] J. Duoandikoetxea and J. L. Rubio de Francia, 'Maximal and singular integral operators via Fourier transform estimates', *Invent. Math.* **84** (1986), 541–561.

[13] F. Liu and H. Wu, 'On Marcinkiewicz integrals associated to compound mappings with rough kernels', *Acta Math. Sin. (Engl. Ser.)* **30** (2014), 1210–1230.
[14] F. Liu, H. Wu and D. Zhang, 'L^p bounds for parametric Marcinkiewicz integrals with mixed homogeneity', *Math. Inequal. Appl.* **18** (2015), 453–469.
[15] F. Liu and D. Zhang, 'Parabolic Marcinkiewicz integrals associated to polynomials, compound curves and extrapolation', *Bull. Korean Math. Soc.* **52** (2015), 771–788.
[16] N. Riviére, 'Singular integrals and multiplier operators', *Ark. Mat.* **9** (1971), 243–278.
[17] S. Sato, 'Estimates for singular integrals along surfaces of revolution', *J. Aust. Math. Soc.* **86** (2009), 413–430.
[18] S. Sato, 'Estimates for singular integrals and extrapolation', *Studia Math.* **192** (2009), 219–233.
[19] E. M. Stein and S. Wainger, 'Problems in harmonic analysis related to curvature', *Bull. Amer. Math. Soc. (N.S.)* **84** (1978), 1239–1295.
[20] T. Walsh, 'On the function of Marcinkiewicz', *Studia Math.* **44** (1972), 203–217.
[21] Q. Xue, Y. Ding and K. Yabuta, 'Parabolic Littlewood–Paley g-function with rough kernel', *Acta Math. Sin. (Engl. Ser.)* **24** (2008), 2049–2060.
[22] S. Yano, 'An extrapolation theorem', *J. Math. Soc. Japan* **3** (1951), 296–305.

FENG LIU, College of Mathematics and Systems Science,
Shandong University of Science and Technology, Qingdao,
Shandong 266590, China
e-mail: liufeng860314@163.com

SUZHEN MAO, School of Mathematical Sciences,
Xiamen University, Xiamen,
Fujian 361005, China
e-mail: suzhen.860606@163.com