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ON H-SETS AND OPEN FILTER ADHERENCES[(1)]

BY

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ABSTRACT. The relationship between H-sets and open filter adherences is considered. The open filter adherences of an H-closed space are shown to be H-sets; and, a necessary and sufficient condition is given for an H-set S, of a Hausdorff space X, to be an open filter adherence. A necessary condition is determined for the existence of a minimal adherent set which contains S; and, in the case that X is H-closed, sufficient conditions are determined. As a related result, an H-closed space X is shown to be seminormal if every H-set of X possesses a neighborhood base consisting of regular open sets.

1. **Introduction.** H-sets were introduced by N. Veličko in [5], and independently by J. Porter and J. Thomas in [4]. Since their introduction, H-sets have played a major role in the development of H-closed spaces. Because of their importance, attempts have been made to obtain characterizations of H-sets using various techniques; and, as a result, new insight has been gained regarding their nature. (Of special note is the recent work of J. Vermeer, which focuses attention upon the Iliadis and Hausdorff absolutes (see [6])). However, it appears that such characterizations are difficult to obtain. Although H-sets cannot be characterized as open filter adherences, the two concepts are closely related (see below). Since the open filter adherences are fundamental to the theory of H-closed spaces, consideration should (and will herein) be given to their relationship with H-sets.

A nonempty subset S of a space X is an *H-set* (of X) if every cover of S by open subsets of X contains a finite subfamily whose closures in X cover S; and, X is *H-closed* if X is an H-set of X. (All spaces considered herein are assumed to be Hausdorff.) It is easily seen that the H-sets of a space X are closed subsets of X.

Regarding notation, cl S (resp. int S) will be used to denote the closure (resp. interior) of a set S in a space X, and η_S will be used to denote the open

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neighborhood filter of S. (For the case $S = \{x\}$, η_x will be used.) Also, for two subsets S_1 and S_2 of X, $S_1 \backslash S_2$ will denote the complement of S_2 in S_1.

In [5], Veličko showed that the θ-closed subsets of an H-closed space are H-sets. (A subset $S \subseteq X$ is *θ-closed* if $S = \text{cl}_\theta S \equiv \{x \in |\text{cl } N \cap S \neq \emptyset$ for all $N \in \eta_x\}$.) J. Joseph (in [3]) improved upon Veličko's result by showing that $\text{cl}_\theta S = \text{ad } \eta_S$ (where $\text{ad } \eta_S \equiv \cap_{N \in \eta_s} \text{cl } N$ denotes the adherent set of η_S). Thus, if X is H-closed and $S \subseteq X$, then $\text{cl}_\theta S$ is both, an H-set, and an open filter adherence. (A subset S of a space X will be called *an open filter adherence* if $S = \text{ad } \mathscr{F}$ for some open filter $\mathscr{F}$ on X).

Two questions naturally arise:

(1) Is every open filter adherence of an H-closed space necessarily an H-set?

(2) Is every H-set of an H-closed space X, the adherent set of an open filter on X?

The first question is answered affirmatively by Proposition (2.2). On the other hand, the second question cannot be answered in its present form. An answer to the question is dependent upon the H-closed space under consideration. Although every H-set of the well-known Urysohn space is an open filter adherence (see Remark (2.6)), the H-closed space constructed in Example (2.8) contains an H-set which is not an open filter adherence (see Remark (2.9)).

The construction of Example (2.8) quite naturally leads to consideration of open filter adherences which are minimal with respect to containing a given H-set. And, for the case that S is an H-set of an H-closed space, a sufficient condition for an open filter to have a minimal adherence with respect to S, is developed herein. (For a subset S of a space X and an open filter base $\mathscr{F}$ on X such that $S \subseteq \text{ad } \mathscr{F}$, it will be said that *$\mathscr{F}$ has a minimal adherence with respect to S* if there does not exist an open filter base $\mathscr{G}$ on X such that $S \subseteq \text{ad } \mathscr{G} \subsetneq \text{ad } \mathscr{F}$).

The remainder of this section is given over to terminology which will be used in what follows below.

If $\mathscr{F}$ is an open filter base on X and $S \subseteq X$, then $\mathscr{F}$ meets S if $F \cap S \neq \emptyset$ for every $F \in \mathscr{F}$, and $\mathscr{F}$ has a cluster point in S if there exists an $x \in \text{ad } \mathscr{F} \cap S$. If $\mathscr{F}$ and $\mathscr{G}$ are open filter bases on X, then $\mathscr{F}$ meets $\mathscr{G}$ if $\mathscr{F}$ meets each $G \in \mathscr{G}$.

A subset $S \subseteq X$ is regular closed (resp. regular open) if $S = \text{cl(int } S)$ (resp. $S = \text{int(cl } S)$). The regular closed subsets of an H-closed space X are H-closed (see [4]) and are therefore, H-sets of X. Moreover, every open filter base on X has a cluster point in X, and this property characterizes the family of H-closed spaces.

2. **Main Results.**

LEMMA (2.1). [2, (1.1)(e)]. *A subset S of a space X is an H-set if and only if every open filter base on X, which meets S, has a cluster point in S.*

PROPOSITION (2.2). *If $\mathscr{F}$ is an open filter base on an H-closed space X, then* ad $\mathscr{F}$ *is an H-set of X.*

PROOF. Let $\mathscr{G}$ be an open filter base on X which meets ad $\mathscr{F}$. Then for $G_1 \in \mathscr{G}$ and $F_1 \in \mathscr{F}$, G_1 contains an element $x \in \text{ad}\,\mathscr{F} \subseteq \text{cl}\,F_1$, so that $G_1 \cap F_1 \neq \emptyset$. Since $\mathscr{F}$ and $\mathscr{G}$ are filter bases, so also is $\omega \equiv \{G \cap F | G \in \mathscr{G}, F \in \mathscr{F}\}$; and since X is H-closed, $\emptyset \neq \text{ad}\,\omega \subseteq \text{ad}\,\mathscr{F} \cap \text{ad}\,\mathscr{G}$. By (2.1), ad $\mathscr{F}$ is an H-set.

PROPOSITION (2.3). *An H-set S of a space X is an open filter adherence if and only if, for $x \in (\text{cl}_\theta\, S \backslash S)$, there exists $N_x \in \eta_x$ such that, for all finite subsets $\{x_1, x_2, \ldots, x_n\} \subseteq (\text{cl}_\theta\, S \backslash S)$, $[\text{int}(\cup_{i=1}^n \text{cl}\, N_{x_i})] \cap S = \emptyset$.*

PROOF OF THE SUFFICIENCY. Suppose that the condition is satisfied for an H-set $S \subseteq X$. If $(\text{cl}_\theta\, S \backslash S) = \emptyset$, then $S = \text{cl}_\theta\, S = \text{ad}\,\eta_S$; so assume that $(\text{cl}_\theta\, S \backslash S) \neq \emptyset$, and let N_x be determined for each $x \in (\text{cl}_\theta\, S \backslash S)$. Let $\mathscr{F} \equiv \{N \backslash \text{cl}\, N_x | N \in \eta_S,\ x \in (\text{cl}_\theta\, S \backslash S)\}$, and let $F \equiv \cap_{i=1}^n (N_i \backslash \text{cl}\, N_{x_i})$ be an arbitrary finite intersection of elements of $\mathscr{F}$. For $s \in S$, $s \in [(\cap_{i=1}^n N_i) \backslash \text{int}(\cup_{i=1}^n \text{cl}\, N_{x_i})] \subseteq \text{cl}\, F$. It follows that $\mathscr{F}$ is an open filter base and that $S \subseteq \text{ad}\,\mathscr{F}$. Moreover, for $x \in (\text{cl}_\theta\, S \backslash S)$, $N_x \in \eta_x$ which fails to meet each $(N \backslash \text{cl}\, N_x) \in \mathscr{F}$ so that $(\text{ad}\,\mathscr{F}) \cap (\text{cl}_\theta\, S \backslash S) = \emptyset$. Hence $S \subseteq \text{ad}\,\mathscr{F} = (\text{ad}\,\mathscr{F}) \backslash (\text{cl}_\theta\, S \backslash S) \subseteq \text{ad}\,\eta_S \backslash (\text{cl}_\theta\, S \backslash S) = \text{cl}_\theta\, S \backslash (\text{cl}_\theta\, S \backslash S) = S$. This proves sufficiency.

PROOF OF THE NECESSITY. Now suppose that $S = \text{ad}\,\mathscr{F}$ for some open filter base $\mathscr{F}$ on X. If $(\text{cl}_\theta\, S \backslash S) = \emptyset$ there is nothing to prove. So assume that $(\text{cl}_\theta\, S \backslash S) \neq \emptyset$. For $x \in (\text{cl}_\theta\, S \backslash S)$, $x \notin \text{ad}\,\mathscr{F}$, so that there exists $N_x \in \eta_x$ which fails to meet some $F_x \in \mathscr{F}$. If $\{x_1, x_2, \ldots, x_n\} \subseteq (\text{cl}_\theta\, S \backslash S)$ and $s \in [S \cap \text{int}(\cup_{i=1}^n \text{cl}\, N_{x_i})]$, then $[\text{int}(\cup_{i=1}^n \text{cl}\, N_{x_i})] \in \eta_s$ which fails to meet $\cap_{i=1}^n F_{x_i}$, so that $s \notin \text{ad}\,\mathscr{F}$, a contradiction. This proves necessity.

EXAMPLE (2.4). [1, Example 3.14]. Let

$$A = \left\{a_{ij} = \left(\frac{1}{i}, \frac{1}{j}\right) | i, j \in \mathbf{N}\right\},$$

$$B = \left\{b_{ij} = \left(\frac{1}{i}, \frac{-1}{j}\right) | i, j \in \mathbf{N}\right\},$$

and

$$C = \left\{c_i = \left(\frac{1}{i}, 0\right) | i \in \mathbf{N}\right\},$$

where $\mathbf{N}$ denotes the set of positive integers. Let $a = (0, 1)$ and $b = (0, -1)$. The topology for $W = A \cup B \cup C$ is the topology inherited from the plane, and a basic open set containing a (resp. b) is of the form

$$U_n(a) = \{a\} \cup \{a_{ij} | i \geqq n, j \in \mathbf{N}\}$$

$$(\text{resp. } V_n(b) = \{b\} \cup \{b_{ij} | i \geqq n, j \in \mathbf{N}\}).$$

The space $hW = W \cup \{a, b\}$ with this topology constitutes a well-known H-closed space due to Urysohn.

LEMMA (2.5). [2, (1.2)(*a*)]. *If S is an H-set of X and E is a θ-closed subset of $X \backslash S$, then there exists $N_E \in \eta_E$ such that* $\text{int}(\text{cl } N_E) \cap S = \emptyset$.

REMARK (2.6). The space hW of (2.4) has the property that every H-set of hW is an open filter adherence. To see this, let S be an H-set of hW and let $x \in (\text{cl}_\theta S \backslash S)$. Note that for $y \in hW \backslash \{a, b\}$ the basic neighborhoods of y are both open and closed, so that $y \in \text{cl}_\theta S$ implies $y \in \text{cl } S = S$. Hence $(\text{cl}_\theta S \backslash S) \subseteq \{a, b\}$ so that $\text{cl}_\theta S \backslash S$ is θ-closed. By (2.5) and (2.3), S is an open filter adherence.

REMARK (2.7). The subset $C \cup \{a\}$ of hW is an H-set; for if $N \in \eta_a$, cl N contains all but finitely many points of C. However, if $(C \cup \{a\}) = \text{cl}_\theta S$ for some $S \subseteq hW$, then $C \subseteq S$; for if $y \in C$, there exists $N_y \in \eta_y$ such that $\{y\} = [(\text{cl } N_y) \cap (C \cup \{a\})] = [\text{cl } N_y \cap \text{cl}_\theta S] \supseteq [\text{cl } N_y \cap S] \neq \emptyset$. Hence $b \in \text{cl}_\theta C \subseteq \text{cl}_\theta S = (C \cup \{a\})$. This contradiction establishes that $C \cup \{a\}$ is not of the form $\text{cl}_\theta S$ for any $S \subseteq hW$. However, by (2.6), $C \cup \{a\}$ is an open filter adherence. Thus (2.2) extends (4.15) of [3].

EXAMPLE (2.8). Fix $n \in \mathbf{N}$. Let X_n denote a copy of hW and let A_n (resp. B_n, C_n) denote the copy of A (resp. B, C) in X_n. The point $a \in hW$ (resp. $b \in hW$) will be denoted a^n (resp. b^n). Let D_n and E_n be distinct copies of

$$\{0\} \cup \left\{\frac{1}{m} | m \in \mathbf{N}\right\}$$

with the topology inherited from the real line, and let Y_n denote the quotient space obtained from the disjoint union $X_n \uplus D_n \uplus E_n$ by identifying $0 \in D_n$ with a^n and $0 \in E_n$ with b^n. Let $Z \equiv [(\uplus_{n \in \mathbf{N}} Y_n) \cup \{\alpha, \beta, \delta, \sigma\}]$ where a basic neighborhood of α (resp. β) is of the form

$$N_k(\alpha) = \{\alpha\} \cup \left(\bigcup_{n \geqq k} A_n\right) \left(\text{resp. } N_k(\beta) = \{\beta\} \cup \left(\bigcup_{n \geqq k} B_n\right)\right),$$

and a basic neighborhood of δ (resp. σ) is of the form

$$N_k(\delta) = \{\delta\} \cup \left(\bigcup_{n \geqq k} D_n \backslash \{a^n\}\right) \left(\text{resp. } N_k(\sigma) = \{\sigma\} \cup \left(\bigcup_{n \geqq k} E_n \backslash \{b^n\}\right)\right).$$

The space Z is H-closed.

REMARK (2.9). It is easy to verify that $S \equiv \{\beta, \delta\} \cup [\cup_{n \in \mathbf{N}}(\{a^n\} \cup C_n)]$ is an H-set of Z and $\text{cl}_\theta S = S \cup \{\alpha\} \cup \{b^n | n \in \mathbf{N}\}$. For $N_\alpha \in \eta_\alpha$, N_α contains some A_k, and for $N_{b^k} \in \eta_b k$, $\text{int}(\text{cl } N_\alpha \cup \text{cl } N_{b^k}) \cap S \neq \emptyset$. Therefore, by (2.3), S is not an open filter adherence.

PROPOSITION (2.10). *Let S be an H-set of X and let $\mathscr{F}$ be an open filter base such that $S \subsetneq \text{ad}\,\mathscr{F}$. There exists an open filter base $\mathscr{G}$ such that $S \subseteq \text{ad}\,\mathscr{G}$ and $\text{ad}\,\mathscr{F} \not\subseteq \text{ad}\,\mathscr{G} \subseteq \text{cl}_\theta\, S$.*

PROOF. Let $x \in (\text{ad}\,\mathscr{F})\backslash S$. By (2.5) there exists $N_x \in \eta_x$ such that $\text{int}(\text{cl}\,N_x) \cap S = \emptyset$. Let $\mathscr{G} \equiv \{N\backslash \text{cl}\,N_x | N \in \eta_S\}$ and let $G \equiv \cap_{i=1}^n (N_i \backslash \text{cl}\,N_x)$ be an arbitrary finite intersection of elements of $\mathscr{G}$. For $s \in S$, $s \in [(\cap_{i=1}^n N_i)\backslash \text{int}(\text{cl}\,N_x)] \subseteq \text{cl}\,G$. It follows that $\mathscr{G}$ is an open filter base and that $S \subseteq \text{ad}\,\mathscr{G} \subseteq \text{cl}_\theta\, S$. Since N_x fails to meet each $(N\backslash \text{cl}\,N_x) \in \mathscr{G}$, $x \notin \text{ad}\,\mathscr{G}$ so that $\text{ad}\,\mathscr{F} \not\subseteq \text{ad}\,\mathscr{G}$.

COROLLARY (2.11). *Let S be an H-set of X which is not an open filter adherence. There does not exist an open filter base $\mathscr{F}$ such that $S \subseteq \text{ad}\,\mathscr{F} \subseteq \text{ad}\,\mathscr{G}$ for all open filter bases $\mathscr{G}$ such that $S \subseteq \text{ad}\,\mathscr{G}$.*

For an H-set S of a space X, $\mathscr{R}_S$ will denote the family of all $x \in (\text{cl}_\theta\, S\backslash S)$ for which there exists $N_x \in \eta_x$ such that $[\text{cl}_\theta(\text{cl}\,N_x)] \cap [(\text{cl}_\theta\, S)\backslash(S \cup \{x\})] = \emptyset$.

PROPOSITION (2.12). *Let S be an H-set of X, and let $\mathscr{F}$ be an open filter base on X such that $S \subsetneq \text{ad}\,\mathscr{F}$. If $\mathscr{R}_S \cap (\text{ad}\,\mathscr{F}) \neq \emptyset$, then there exists an open filter base $\mathscr{G}$ such that $S \subseteq \text{ad}\,\mathscr{G} \subsetneq \text{ad}\,\mathscr{F}$.*

PROOF. Let $x \in (\mathscr{R}_S \cap \text{ad}\,\mathscr{F})$, and let $N_x \in \eta_x$ such that $[\text{cl}_\theta(\text{cl}\,N_x)] \cap [\text{cl}_\theta\, S\backslash(S \cup \{x\})] = \emptyset$. By (2.5), we may assume that $\text{int}(\text{cl}\,N_x) \cap S = \emptyset$. If $(\text{cl}_\theta\, S)\backslash(S \cup \{x\}) = \emptyset$ then the proposition follows from (2.10). So assume that $(\text{cl}_\theta\, S)\backslash(S \cup \{x\}) \neq \emptyset$. For $y \in [(\text{cl}_\theta\, S)\backslash(S \cup \{x\})]\backslash \text{ad}\,\mathscr{F}$, let $N_y \in \eta_y$ such that $\text{cl}\,N_y \cap \text{cl}\,N_x = \emptyset$, and $\text{cl}\,N_y \cap F_y = \emptyset$ for some $F_y \in \mathscr{F}$. Let $\mathscr{G} \equiv \{N\backslash[\text{cl}\,N_x \cup \text{cl}\,N_y] | N \in \eta_S, y \in [(\text{cl}_\theta\, S)\backslash(S \cup \{x\})]\backslash \text{ad}\,\mathscr{F}\}$, and let $G \equiv \cap_{i=1}^n [N_i \backslash(\text{cl}\,N_x \cup \text{cl}\,N_{y_i})]$ be an arbitrary finite intersection of elements of $\mathscr{G}$. For $s \in S$, $s \in \text{cl}(\cap_{i=1}^n F_{y_i})$, so that $s \notin \text{int}(\cup_{i=1}^n \text{cl}\,N_{y_i})$. Moreover, since $(\cup_{i=1}^n \text{cl}\,N_{y_i}) \cap (\text{cl}\,N_x) = \emptyset$ and $s \notin \text{int}(\text{cl}\,N_x)$, $s \notin \text{int}[(\text{cl}\,N_x) \cup (\cup_{i=1}^n \text{cl}\,N_{y_i})] = \text{int}[\cup_{i=1}^n(\text{cl}\,N_x \cup \text{cl}\,N_{y_i})]$. Hence $s \in (\cap_{i=1}^n N_i)\backslash \text{int}[\cup_{i=1}^n(\text{cl}\,N_x \cup \text{cl}\,N_{y_i})] \subseteq \text{cl}\,G$. It follows that $\mathscr{G}$ is an open filter base and that $S \subseteq \text{ad}\,\mathscr{G}$. Since for $y \in [(\text{cl}_\theta\, S)\backslash(S \cup \{x\})]\backslash \text{ad}\,\mathscr{F}$, N_x and N_y fail to meet each $[N\backslash(\text{cl}\,N_x \cup \text{cl}\,N_y)] \in \mathscr{G}$, $y \notin \text{ad}\,\mathscr{G}$ and $x \notin \text{ad}\,\mathscr{G}$. Thus $\text{ad}\,\mathscr{G} \subseteq \text{ad}\,\eta_S\backslash[([\text{cl}_\theta\, S\backslash(S \cup \{x\})]\backslash \text{ad}\,\mathscr{F}) \cup \{x\}] = (\text{cl}_\theta\, S)\backslash[(\text{cl}_\theta\, S\backslash \text{ad}\,\mathscr{F}) \cup \{x\}] \subseteq \text{ad}\,\mathscr{F}\backslash\{x\} \subsetneq \text{ad}\,\mathscr{F}$. This completes the proof.

COROLLARY (2.13). *If S is an H-set of X, and $\mathscr{F}$ has a minimal adherence with respect to S, then $\mathscr{R}_S \cap \text{ad}\,\mathscr{F} = \emptyset$.*

REMARK (2.14). Let Z and S be as defined in (2.8) and (2.9). It is easy to verify that $\mathscr{R}_S = \{b^n | n \in N\}$.

Let S be an H-set of a space X. $\mathscr{E}_S$ will denote the family of all $x \in (\text{cl}_\theta\, S)\backslash S$

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such that for $N_x \in \eta_x$ there exists a finite subset $E \subseteq (\mathscr{R}_S \backslash \{x\})$ which satisfies:

(i) for $N_E \in \eta_E$, $\operatorname{int}(\mathrm{cl}\, N_x \cup \mathrm{cl}\, N_E) \cap S \neq \emptyset$.

REMARK (2.15). $\mathscr{R}_S \cap \mathscr{E}_S = \emptyset$. To see this, let $x \in \mathscr{R}_S$ and let $N_x \in \eta_x$ such that $[\mathrm{cl}_\theta(\mathrm{cl}\, N_x)] \cap [(\mathrm{cl}_\theta\, S) \backslash (S \cup \{x\})] = \emptyset$. By (2.5) we may assume that $\operatorname{int}(\mathrm{cl}\, N_x) \cap S = \emptyset$. If E is a finite subset of $(\mathscr{R}_S \backslash \{x\})$, then $E \subseteq [(\mathrm{cl}_\theta\, S) \backslash (S \cup \{x\})]$. Hence there exists, for $y \in E$, a neighborhood $M_y \in \eta_y$ such that $\mathrm{cl}\, M_y \cap \mathrm{cl}\, N_x = \emptyset$. Moreover, by (2.5), there exists $M_E \in \eta_E$ such that $\operatorname{int}(\mathrm{cl}\, M_E) \cap S = \emptyset$. Consider $N_E \equiv [M_E \cap (\cup_{y \in E} M_y)] \in \eta_E$. Since $(\mathrm{cl}\, N_x \cap \mathrm{cl}\, N_E) \subseteq [\mathrm{cl}\, N_x \cap (\cup_{y \in E} \mathrm{cl}\, M_y)] = \emptyset$, and $[\operatorname{int}(\mathrm{cl}\, N_E) \cap S] \subseteq [\operatorname{int}(\mathrm{cl}\, M_E) \cap S] = \emptyset$, $\operatorname{int}(\mathrm{cl}\, N_x \cup \mathrm{cl}\, N_E) \cap S = \emptyset$. It follows that $x \notin \mathscr{E}_S$ and, therefore $\mathscr{R}_S \cap \mathscr{E}_S = \emptyset$.

REMARK (2.16). For Z and S as defined in (2.8) and (2.9), $\mathscr{E}_S = \{\alpha\}$. So, $[(\mathrm{cl}_\theta\, S) \backslash S] = \mathscr{E}_S \cup \mathscr{R}_S$.

The next lemma is easily proven.

LEMMA (2.17). *If E is a finite subset of an H-closed space X, and $\mathscr{F}$ is an open filter base on X which meets η_E, then $\mathscr{F}$ has a cluster point in E.*

PROPOSITION (2.18). *Let S be an H-set of an H-closed space such that $(\mathrm{cl}_\theta\, S) \backslash S = (\mathscr{E}_S \cup \mathscr{R}_S)$. If $\mathscr{F}$ is an open filter base such that* ad $\mathscr{F} = (S \cup \mathscr{E}_S)$, *then $\mathscr{F}$ has a minimal adherence with respect to S.*

PROOF. Let $\mathscr{F}$ be an open filter base such that ad $\mathscr{F} = (S \cup \mathscr{E}_S)$, and suppose that $\mathscr{G}$ is an open filter base such that $S \subseteq$ ad $\mathscr{G} \subsetneq$ ad $\mathscr{F}$. Let $x \in$ (ad $\mathscr{F} \backslash$ad $\mathscr{G}$). Since $x \notin$ ad $\mathscr{G}$, there exists $N \in \eta_x$ and $G_x \in \mathscr{G}$ such that $\mathrm{cl}\, N \cap G_x = \emptyset$. Now $x \in \mathscr{E}_S \subseteq (\mathrm{cl}_\theta\, S) \backslash S \subseteq \mathrm{cl}_\theta\, S$ so that $\mathrm{cl}\, N \cap S \neq \emptyset$, and there exists a finite subset $E \subseteq (\mathscr{R}_S \backslash \{x\})$ which satisfies (i). Now $S \subseteq$ ad $\mathscr{G}$ so that for $N_E \in \eta_E$, $\mathscr{G}$ meets $\operatorname{int}(\mathrm{cl}\, N \cup \mathrm{cl}\, N_E)$ and, therefore, meets $\mathrm{cl}\, N \cup \mathrm{cl}\, N_E$. Since $G_x \cap \mathrm{cl}\, N = \emptyset$, $\mathscr{G}$ meets $\mathrm{cl}\, N_E$ and, therefore, meets N_E. Hence, $\mathscr{G}$ meets η_E, and by (2.17), $\mathscr{G}$ has a cluster point $y \in E \subseteq \mathscr{R}_S$. But then, $y \notin$ ad $\mathscr{F}$ so that ad $\mathscr{G} \not\subseteq$ ad $\mathscr{F}$, a contradiction. It follows that there does not exist an open filter base $\mathscr{G}$ such that $S \subseteq$ ad $\mathscr{G} \subsetneq$ ad $\mathscr{F}$. Hence, $\mathscr{F}$ has a minimal adherence with respect to S.

REMARK (2.19). Note that by the proof of (2.18), if S is an H-set of an H-closed space, and $\mathscr{G}$ is an open filter base such that $S \subseteq$ ad $\mathscr{G}$ and $(\mathscr{E}_S \backslash$ad $\mathscr{G}) \neq \emptyset$, then $\mathscr{G}$ has a cluster point in $\mathscr{R}_S$.

REMARK (2.20). Let Z and S be as defined in (2.8) and (2.9). By (2.16), $[(\mathrm{cl}_\theta\, S) \backslash S] = (\mathscr{E}_S \cup \mathscr{R}_S)$. By (2.18), if $\mathscr{F}$ is an open filter base on Z such that ad $\mathscr{F} = S \cup \mathscr{E}_S = S \cup \{\alpha\}$, then $\mathscr{F}$ has a minimal adherence with respect to S.

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To prove the existence of such an $\mathscr{F}$, (2.3) will be used. By (1.1)(*c*) of [2], $S \cup \{a\}$ is an H-set. Moreover, $(\mathrm{cl}_\theta\, S)\backslash(S \cup \{\alpha\}) = \{b^n | n \in \mathbf{N}\} = \mathscr{R}_S$. For $n \in \mathbf{N}$, let $N_{b^n} \in \eta_{b^n}$ such that $N_{b^n} \subseteq (\{b^n\} \cup B_n \cup E_n)$. If E is any finite subset of $\mathscr{R}_S$, then $N_E \equiv \cup\{N_{b^n} | b^n \in E\}$ is an element of η_E such that $\mathrm{int}(\mathrm{cl}\, N_E) \cap S = \emptyset$. By (2.3), $(S \cup \{\alpha\}) = \mathrm{ad}\, \mathscr{F}$ for some open filter base $\mathscr{F}$ on Z; and, by the argument above, $\mathscr{F}$ has a minimal adherence with respect to S.

Note also that, if $\mathscr{G}$ is any open filter base on Z such that $S \subseteq \mathrm{ad}\, \mathscr{G}$ and $\mathrm{ad}\, \mathscr{G} \neq (S \cup \{\alpha\})$, then $\mathscr{G}$ does not have a minimal adherence with respect to S. For, either $\alpha \in \mathrm{ad}\, \mathscr{G}$ or $\alpha \notin \mathrm{ad}\, \mathscr{G}$. If $\alpha \in \mathrm{ad}\, \mathscr{G}$, then $S \subseteq \mathrm{ad}\, \mathscr{F} \subsetneq \mathrm{ad}\, \mathscr{G}$. And, if $\alpha \notin \mathrm{ad}\, \mathscr{G}$, then $\mathrm{ad}\, \mathscr{G} \cap \mathscr{R}_S \neq \emptyset$ (see (2.19)), so that by (2.12), $\mathscr{G}$ does not have a minimal adherence with respect to S.

3. **A related result.** In [8], G. Viglino defined a space X to be *seminormal* if every closed subset of X possesses a neighborhood base consisting of regular open sets. Viglino also introduced C-compact spaces [7]. A space X is *C-compact* if every closed subset of X is an H-set. Every H-closed and seminormal space is C-compact [8]; and, easily, every C-compact space is H-closed. It is well-known that neither of these implications can be reversed.

PROPOSITION (3.1). *Let C be a closed subset of an H-closed space X; and let $\mathscr{F}$ be an open filter base on X which meets C, but has no cluster point in C. Then no regular open neighborhood of* ad $\mathscr{F}$ *is contained in* $X\backslash C$.

PROOF. Let $N \in \eta_{\mathrm{ad}\mathscr{F}}$, and assume that $\mathrm{int}(\mathrm{cl}\, N) \cap C = \emptyset$. Let $V = X\backslash \mathrm{cl}\, N$. Since $C \subseteq \mathrm{cl}\, V$ and $\mathscr{F}$ meets C, it follows that $\mathscr{F}$ meets $\mathrm{cl}\, V$ and hence meets V. As X is H-closed, $\mathrm{ad}\, \mathscr{F} \cap \mathrm{cl}\, V \neq \emptyset$. But $\mathrm{cl}\, V = X\backslash\mathrm{int}(\mathrm{cl}\, N) \subseteq X\backslash N$. So $\mathrm{ad}\, \mathscr{F} \cap (X\backslash N) \neq \emptyset$, a contradiction as $\mathrm{ad}\, \mathscr{F} \subseteq N$. It follows that $\mathrm{int}(\mathrm{cl}\, N) \cap C \neq \emptyset$ for every $N \in \eta_{\mathrm{ad}\mathscr{F}}$.

A space X will be called *H-seminormal* if every H-set of X possesses a neighborhood base consisting of regular open sets.

PROPOSITION (3.2). If X is H-closed and H-seminormal, then X is C-compact.

PROOF. Let X be H-closed, and suppose that X is not C-compact. By (2.1), there exists a closed subset $C \subseteq X$ and an open filter base $\mathscr{F}$ which meets C, but has no cluster point in C. By (2.2), ad $\mathscr{F}$ is an H-set; and, by (3.1), X is not H-seminormal. This proves the assertion.

COROLLARY (3.3). *An H-closed space X is seminormal if and only if X is H-seminormal.*

PROOF. Necessity is trivial since H-sets are closed. On the other hand, if X is H-seminormal, then by (3.2), X is C-compact. Hence, every closed subset of X is an H-set, and therefore, possesses a neighborhood base consisting of regular open sets.

References

1. M. Berri, J. Porter, and R. Stephenson, *A survey of minimal topological spaces*, Proc. Kanpur Topology Conference, 1968, Academic Press, 1970, pp. 93-114.

2. R. Dickman, Jr. and J. Porter, *Between minimal Hausdorff and compact Hausdorff spaces*, Auburn Topology Conference, Spring Meeting, 1984.

3. J. Joseph, *Multifunctions and cluster sets*, Proc. Amer. Math. Soc. **74** (1979), pp. 329-337.

4. J. Porter and J. Thomas, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. **138** (1969), pp. 159-170.

5. N. Veličko, *H-closed topological spaces*, Mat. Sb. **70** (1966), pp. 98-112; Amer. Math. Soc. Transl. **78** (1968), pp. 103-118.

6. J. Vermeer, *Closed subspaces of H-closed spaces*, Pacific J. Math. **1** (1985), pp. 229-247.

7. G. Viglino, *C-compact spaces*, Duke J. Math. **36** (1969), pp. 761-764.

8. G. Viglino, *Seminormal and C-compact spaces*, Duke J. Math. **38** (1971), pp. 57-61.

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