BULL. AUSTRAL. MATH. SOC. 46M18, 46H05, 46H20
VOL. 65 (2002) [191–197]

AMENABILITY AND TOPOLOGICAL CENTRES OF THE SECOND DUALS OF BANACH ALGEBRAS

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Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{A}^{**}$ be the second dual algebra of $\mathfrak{A}$ endowed with the first or the second Arens product. We investigate relations between amenability of $\mathfrak{A}^{**}$ and Arens regularity of $\mathfrak{A}$ and the rôle of topological centres in amenability of $\mathfrak{A}^{**}$. We also find conditions under which weak amenability of $\mathfrak{A}^{**}$ implies weak amenability of $\mathfrak{A}$.

INTRODUCTION AND PRELIMINARIES

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}^{**}$ be the second dual space of $\mathfrak{A}$ endowed with the first or the second Arens product. In [**2**] the first author, Loy and Willis studied some implications of amenability and weak amenability of $\mathfrak{A}^{**}$; special emphasis was put on the case when $\mathfrak{A}$ was a Banach algebra related to a locally compact group. These studies have lead to the work done in [**1, 4, 7, 8, 9**]. In this paper we consider the following two questions

1. Is $\mathfrak{A}$ Arens regular when $\mathfrak{A}^{**}$ is amenable?
2. Is $\mathfrak{A}$ weakly amenable when $\mathfrak{A}^{**}$ is weakly amenable?

For the origin of these questions see [**2, 3**]. We show that under certain additional assumptions on $\mathfrak{A}$ or $\mathfrak{A}^{**}$ the answer to either one of these questions is positive. We also explore the rôle of the topological centres in amenability of $\mathfrak{A}^{**}$.

Throughout this paper, the first (second) Arens product is denoted by $\Box$ (respectively $\Diamond$). These products can be defined by

$$F \Box G = \text{weak}^* \lim_i \lim_j \widehat{f}_i \widehat{g}_j \quad \text{and } F \Diamond G = \text{weak}^* \lim_j \lim_i \widehat{f}_i \widehat{g}_j ,$$

where (f_i) and (g_j) are nets of elements of $\mathfrak{A}$ such that $\widehat{f}_i \to F$ and $\widehat{g}_i \to G$, in the weak* topology. The first topological centre of $\mathfrak{A}^{**}$ is

$$\begin{aligned} Z_1 &= \{y \in \mathfrak{A}^{**} : x \mapsto y \Box x \text{ is weak* continuous } \} \\ &= \{y \in \mathfrak{A}^{**} : y \Box x = y \Diamond x, \text{ for all } x \in \mathfrak{A}^{**}\} , \end{aligned}$$

Received 26th April, 2001.
The first author's research was supported by NSERC grant 36640-98.

and the second topological centre is defined by

$$Z_2 = \{y \in \mathfrak{A}^{**} : x \mapsto x \lozenge y \text{ is weak* continuous }\}.$$

We note that $\mathfrak{A}$ is Arens regular if and only if $Z_1 = \mathfrak{A}^{**}$, or $Z_2 = \mathfrak{A}^{**}$. See [2] and [7] for properties of Arens products and topological centres.

A Banach algebra $\mathfrak{A}$ is amenable if every continuous derivation $D : \mathfrak{A} \to X^*$ is inner, for every Banach $\mathfrak{A}$-bimodule X. If all the continuous derivations from $\mathfrak{A}$ into $\mathfrak{A}^*$ (the special case of $X = \mathfrak{A}$) are inner, then $\mathfrak{A}$ is weakly amenable. There are alternative formulations of the notion of amenability, of which we need the following two, first introduced in [6]. The Banach algebra $\mathfrak{A}$ is amenable if and only if either, and hence both, of the following hold,

(i) $\mathfrak{A}$ has an *approximate diagonal*, that is a bounded net $(m_i) \subset (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that for each $x \in \mathfrak{A}$, $x \cdot m_i - m_i \cdot x \to 0$, $\pi(m_i)x \to x$;

(ii) $\mathfrak{A}$ has a *virtual diagonal*, that is an element $M \in (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that for each $x \in \mathfrak{A}$, $x \cdot M = M \cdot x$, and $(\pi^{**}M) \cdot x = \widehat{x}$; here $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is specified by $\pi(a \otimes b) = ab$ $(a, b \in \mathfrak{A})$.

1. Amenability of certain subalgebras of $\mathfrak{A}^{**}$

In [2, Theorem 1.8] it was shown that if $\mathfrak{A}^{**}$ is amenable (with either one of the Arens products), then $\mathfrak{A}$ is amenable. In the following proposition we strengthen the above cited result. We have assumed that $\mathfrak{A}^{**}$ has first Arens product. The image of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ under the canonical mapping is denoted by $\widehat{\mathfrak{A}}$.

Proposition 1.1. *Let B be a closed subalgebra of $\mathfrak{A}^{**}$ such that $\widehat{\mathfrak{A}} \subseteq B$. If B is amenable, then $\mathfrak{A}$ is amenable.*

Proof: By [2, Lemma 1.7] there is a continuous linear mapping $\psi : \mathfrak{A}^{**}\widehat{\otimes}\mathfrak{A}^{**} \to (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that for $a, b, x \in \mathfrak{A}$ and $m \in \mathfrak{A}^{**}\widehat{\otimes}\mathfrak{A}^{**}$ the following hold.

(i) $\psi(\widehat{a} \otimes \widehat{b}) = (a \otimes b)^\wedge$;

(ii) $\psi(m) \cdot x = \psi(m \cdot x)$;

(iii) $x \cdot \psi(m) = \psi(x \cdot m)$;

(iv) $(\pi_{\mathfrak{A}})^{**}(\psi(m)) = \pi_{\mathfrak{A}^{**}}(m)$.

From the definition of projective tensor norm we see that when both $B\widehat{\otimes}B$ and $\mathfrak{A}^{**}\widehat{\otimes}\mathfrak{A}^{**}$ are equipped with the projective tensor norm, then the mapping $J : B\widehat{\otimes}B \to \mathfrak{A}^{**}\widehat{\otimes}\mathfrak{A}^{**}$ specified by $J(b_1 \otimes b_2) = b_1 \otimes b_2$, $(b_1, b_2 \in B)$ is norm decreasing. Let (m_i) be an approximate diagonal for B, and set $\Phi = \psi \circ J$. Then for all $x \in \mathfrak{A}$, $\Phi(m_i) \cdot x - x \cdot \Phi(m_i) \to 0$ and $\pi_{\mathfrak{A}}^{**}(\Phi(m_i)) \cdot x \to x$. If M is a weak*-cluster point of $(\Phi(m_i))$ in $(\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$, then, for each $x \in \mathfrak{A}$, $M \cdot x = x \cdot M$ and $\pi_{\mathfrak{A}}^{**}(M) \cdot x = x$, and so M is a virtual diagonal for $\mathfrak{A}$. ∎

Corollary 1.2. *Suppose that Z_1 (or Z_2) is amenable. Then $\mathfrak{A}$ is amenable.*

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For a Banach algebra $\mathfrak{A}$, let $\mathfrak{A}^{op}$ be the Banach algebra with underlying Banach space same as $\mathfrak{A}$ and with product $\circ$ given by $a \circ b = ba$.

PROPOSITION 1.3. *Let $\mathfrak{A}$ be a Banach algebra. Then*

(i) *$\mathfrak{A}$ is amenable if and only if $\mathfrak{A}^{op}$ is amenable.*

(ii) *Let $\mathfrak{A}$ be commutative. Then $(\mathfrak{A}^{**}, \Box)$ is amenable if and only if $(\mathfrak{A}^{**}, \Diamond)$ is amenable.*

PROOF:

(i) This is trivial.

(ii) Take $F, G \in \mathfrak{A}^{**}$, and let $(f_i), (g_j)$ be nets in $\mathfrak{A}$ such that $w^* - \lim_i \widehat{f_i} = F$ and $w^* - \lim_j \widehat{g}_j = G$. Then

$$F \Box G = w^* - \lim_i w^* - \lim_j \widehat{f_i}\widehat{g}_j = w^* - \lim_i w^* - \lim_j \widehat{g}_j \circ \widehat{f_i} = G \Diamond F,$$

and so $(\mathfrak{A}^{**}, \Box) = (\mathfrak{A}^{**}, \Diamond)^{op}$. By (i), $(\mathfrak{A}^{**}, \Box)$ is amenable if and only if $(\mathfrak{A}^{**}, \Diamond)$ is amenable.

∎

PROPOSITION 1.4. *Let $\mathfrak{A}$ be a Banach algebra with a continuous anti-isomorphism $\lambda : \mathfrak{A} \to \mathfrak{A}$. Then, $(\mathfrak{A}^{**}, \Box)$ is amenable if and only if $(\mathfrak{A}^{**}, \Diamond)$ is amenable. A similar conclusion holds if λ is a continuous involution.*

PROOF: Let $\lambda : \mathfrak{A} \to \mathfrak{A}$ be a continuous anti-isomorphism. Set $(\mathfrak{A}^{**}, \Box) = A$ and $(\mathfrak{A}^{**}, \Diamond) = B$. Take F, G in A and let (f_i) and (g_j) be nets in $\mathfrak{A}$ such that weak$^*-\lim_i \widehat{f_i} = F$, weak$^* - \lim_j \widehat{g}_j = G$.

Let $\lambda^{**} : A \to B$, be the second adjoint of λ. Then

$$\begin{aligned}\lambda^{**}(F \Box G) &= w^* - \lim_i w^* - \lim_j \lambda^{**}(\widehat{f_i}\widehat{g}_j) \\ &= w^* - \lim_i w^* - \lim_j (\lambda(f_i)\lambda(g_j))^\wedge \\ &= w^* - \lim_i w^* - \lim_j \lambda^{**}(\widehat{g}_j)\lambda^{**}(\widehat{f_i}) \\ &= \lambda^{**}(G) \Diamond \lambda^{**}(F).\end{aligned}$$

Hence λ^{**} is an isomorphism from A onto B^{op} and so by Proposition 1.3 (i) B is amenable.

The proof in the case when λ is an involution follows similar lines. ∎

Recall that $\mathfrak{A}^*$ is said to *factor on the left* if $\mathfrak{A}^* \cdot \mathfrak{A} = \mathfrak{A}^*$, [**10**]. When $\mathfrak{A}$ has a bounded approximate identity and $\mathfrak{A}^{**}$ has an identity, $\mathfrak{A}^*$ factors on the left [**10**].

THEOREM 1.5. *Suppose that $(\mathfrak{A}^{**}, \Box)$ is amenable and $\widehat{\mathfrak{A}} \Box \mathfrak{A}^{**} \subset Z_1$. Then $\mathfrak{A}$ is Arens regular.*

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PROOF: Since $(\mathfrak{A}^{**}, \Box)$ is amenable it has an identity [**2**]. Also, amenability of $\mathfrak{A}^{**}$ necessitates amenability of $\mathfrak{A}$, ([**2**]), and so $\mathfrak{A}$ has a bounded approximate identity. Hence $\mathfrak{A}^{\star}$ factors on the left; $\mathfrak{A}^{\star} \cdot \mathfrak{A} = \mathfrak{A}^{\star}$. Let $f \in \mathfrak{A}^{\star}$. Then $f = g \cdot a$, for some $g \in \mathfrak{A}^{\star}$ and $a \in \mathfrak{A}$. Let $m, n \in \mathfrak{A}^{**}$, and $f \in \mathfrak{A}^{\star}$. Then, since $\widehat{a} \Box m \in Z_1$ and $\widehat{a} \Box m = \widehat{a} \Diamond m$, we have

$$\begin{aligned}\langle m \Box n, f\rangle &= \langle m \Box n, g \cdot a\rangle \\ &= \big\langle \widehat{a} \Box (m \Box n), g\big\rangle = \big\langle (\widehat{a} \Box m) \Box n, g\big\rangle \\ &= \big\langle (\widehat{a} \Box m) \Diamond n, g\big\rangle = \big\langle (\widehat{a} \Diamond m) \Diamond n, g\big\rangle \\ &= \big\langle \widehat{a} \Diamond (m \Diamond n), g\big\rangle = \langle m \Diamond n, g \cdot a\rangle \\ &= \langle m \Diamond n, f\rangle,\end{aligned}$$

and so $m \Box n = m \Diamond n$, showing that $\mathfrak{A}$ is Arens regular. ∎

2. Weak amenability of $\mathfrak{A}^{**}$

Let $\mathfrak{A}^2 = \operatorname{span}\{ab : a, b \in \mathfrak{A}\}$.

It is known that if the Banach algebra $\mathfrak{A}$ is weakly amenable, then $\mathfrak{A}^2$ is dense in $\mathfrak{A}$. The following is a positive result in the direction of answering the question of whether weak amenability of $\mathfrak{A}^{**}$ implies weak amenability of $\mathfrak{A}$.

PROPOSITION 2.1. *Suppose that $\mathfrak{A}^{**}$ is weakly amenable. Then $\mathfrak{A}^2$ is dense in $\mathfrak{A}$.*

PROOF: Let $a \in \mathfrak{A}$. Since $\mathfrak{A}^{**}$ is weakly amenable $\mathfrak{A}^{**}$ is equal to the closure of $(\mathfrak{A}^{**})^2$. Hence there exists a sequence $(s_n) \subset (\mathfrak{A}^{**})^2$ such that $s_n = \sum_{k=1}^{K(n)} M_{n,k} \Box N_{n,k}$ and norm$-\lim_n s_n = \widehat{a}$.

On the other hand, for each n and k, there exist nets $\{a_{n,k,i} : i \in I\}$ and $\{b_{n,k,j} : j \in J\}$ such that weak$^\star - \lim_i \widehat{a}_{n,k,i} = M_{n,k}$ and weak$^\star - \lim_j \widehat{b}_{n,k,j} = N_{n,k}$. Hence

$$M_{n,k} \Box N_{n,k} = w^\star - \lim_i w^\star - \lim_j \widehat{a}_{n,k,i} \Box \widehat{b}_{n,k,j}$$

and so

$$\widehat{a} = \text{norm} - \lim_n w^\star - \lim_i w^\star - \lim_j \widehat{a}_{n,k,i} \Box \widehat{b}_{n,k,j}.$$

This shows that $\widehat{a}$ belongs to the weak* closure of the set $\widehat{\mathfrak{A}} \Box \widehat{\mathfrak{A}}$; this means that a belongs to the weak closure of $\mathfrak{A}\mathfrak{A}$. Hence a is in the weak closure of $\operatorname{span}(\mathfrak{A}\mathfrak{A})$. Since $\operatorname{span}(\mathfrak{A}\mathfrak{A})$ is convex, it follows that a belongs to the norm-closure of $\operatorname{span}(\mathfrak{A}\mathfrak{A})$. ∎

Recall that a Banach algebra $\mathfrak{A}$ is a *dual Banach algebra* if $\mathfrak{A} = X^\star$ for some Banach space X and $\widehat{X}$ is a submodule of the dual $\mathfrak{A}$-bimodule $\mathfrak{A}^\star$.

THEOREM 2.2. *Suppose that $\mathfrak{A}$ is a dual Banach algebra. If $\mathfrak{A}^{**}$ is weakly amenable then $\mathfrak{A}$ is weakly amenable.*

PROOF: Let $\mathfrak{A} = B^\star$, for some Banach space B, such that $\widehat{B}$ is a submodule of the dual module $\mathfrak{A}^\star$. Let $i : B \to \mathfrak{A}^\star$ be the canonical mapping and let $i^\star$ be the adjoint of i. First we show that $i^\star$ is a homomorphism from $(\mathfrak{A}^{\star\star}, \Box)$ onto $\mathfrak{A}$. If $a \in \mathfrak{A}$, then for $b \in B$, we have

$$\langle i^\star(\widehat{a}), b\rangle = \langle \widehat{a}, i(b)\rangle = \langle a, b\rangle.$$

Hence $i^\star(\widehat{a}) = a$. Now for $F, G \in \mathfrak{A}^{\star\star}$, take two nets (f_α), (g_β) of $\mathfrak{A}$ such that $F = \text{weak}^\star - \lim_\alpha \widehat{f}_\alpha$, $G = \text{weak}^\star - \lim_\beta \widehat{g}_\beta$. Then

$$\begin{aligned} i^\star(F \Box G) &= i^\star(w^\star - \lim_\alpha w^\star - \lim_\beta \widehat{f}_\alpha \widehat{g}_\beta) = w^\star - \lim_\alpha w^\star - \lim_\beta i^\star\big((f_\alpha\, g_\beta)\big) \\ &= w^\star - \lim_\alpha w^\star - \lim_\beta (f_\alpha\, g_\beta) = w^\star - \lim_\alpha f_\alpha\, w^\star - \lim_\beta g_\beta \\ &= w^\star - \lim_\alpha \big(i^\star(f_\alpha)\, w^\star - \lim_\beta i^\star(g_\beta)\big) = w^\star - \lim_\alpha (f_\alpha) i^\star(G)) \\ &= i^\star(F)\, i^\star(G). \end{aligned}$$

Hence $i^\star$ is an algebra homomorphism from $\mathfrak{A}^{\star\star}$ onto $\mathfrak{A}$.

Now let $D : \mathfrak{A} \to \mathfrak{A}^\star$ be a derivation. Then $\overline{D} = i^{\star\star} \circ D \circ i^\star : \mathfrak{A}^{\star\star} \to \mathfrak{A}^{\star\star\star}$ is a derivation. In fact, let $m, n, p \in \mathfrak{A}^{\star\star}$. Then

$$\begin{aligned} \langle \overline{D}(m \Box n), p\rangle &= \langle (i^{\star\star} \circ D \circ i^\star)(m \Box n), p\rangle \\ &= \Big\langle D\big(i^\star(m)\, i^\star(n)\big), i^\star(p)\Big\rangle \\ &= \Big\langle D\big(i^\star(m)\big)\, i^\star(n) + i^\star(m)\, D\big(i^\star(n)\big), i^\star(p)\Big\rangle \\ &= \Big\langle D\big(i^\star(m)\big), i^\star(n)\, i^\star(p)\Big\rangle + \Big\langle D\big(i^\star(n)\big), i^\star(p)\, i^\star(m)\Big\rangle \\ &= \Big\langle D\big(i^\star(m)\big), i^\star(n \Box p)\Big\rangle + \Big\langle D\big(i^\star(n)\big), i^\star(p \Box m)\Big\rangle \\ &= \Big\langle i^{\star\star}\big(D\big(i^\star(m)\big)\big) n \Box p\Big\rangle + \Big\langle i^{\star\star}\big(D\big(i^\star(n)\big)\big), p \Box m\Big\rangle \\ &= \Big\langle (i^{\star\star} \circ D \circ i^\star)(m) \cdot n + m \cdot i^{\star\star}\, D\big(i^\star(n)\big), p\Big\rangle \\ &= \langle \overline{D}(m) \cdot n + m \cdot \overline{D}(n), p\rangle. \end{aligned}$$

Hence $\overline{D}$ is a derivation, and so from the assumption of weak amenability of $\mathfrak{A}^{\star\star}$, there exists $F \in \mathfrak{A}^{\star\star\star}$ such that

$$\overline{D}(m) = m \cdot F - F \cdot m \quad (m \in \mathfrak{A}^{\star\star}).$$

Now $\mathfrak{A}^{\star\star}$ is naturally an $\mathfrak{A}$-bimodule and the canonical mapping $j : \mathfrak{A} \to \mathfrak{A}^{\star\star}$ is an $\mathfrak{A}$-bimodule morphism, and hence so is $j^\star : \mathfrak{A}^{\star\star\star} \to \mathfrak{A}^\star$. Set $f = j^\star(F)$. Then if $a, b \in \mathfrak{A}$, we

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have

$$\begin{aligned}\langle D(a),b\rangle &= \left\langle D(i^\star(\widehat{a})),i^\star(\widehat{b})\right\rangle\\ &= \left\langle i^{\star\star}D(i^\star(\widehat{a})),\widehat{b}\right\rangle\\ &= \langle \overline{D}(\widehat{a}),j(b)\rangle\\ &= \langle \widehat{a}\cdot F - F\cdot \widehat{a}, j(b)\rangle\\ &= \langle j^\star(\widehat{a}\cdot F - F\cdot\widehat{a}),b\rangle\\ &= \langle a\cdot j^\star(F) - j^\star(F)\cdot a, b\rangle.\end{aligned}$$

Hence $D(a) = a\cdot f - f\cdot a$, and $\mathfrak{A}$ is weakly amenable. $\square$

In the proof of the next theorem we adopt the following notation. Let $\mathfrak{A}^{\star\star}$ have the first Arens product $\Box$. Then for $F\in\mathfrak{A}^{\star\star\star} = (\mathfrak{A}^{\star\star})^\star$ and $m\in\mathfrak{A}^{\star\star}, m \boxdot F$ is the element of $\mathfrak{A}^{\star\star\star}$ specified by $\langle m\boxdot F, n\rangle = \langle F, n\Box m\rangle$ $(m\in\mathfrak{A}^{\star\star})$. $F\boxdot m$, $m \diamond F$ and $F\diamond m$, follow similar convention and should be clear from the context.

THEOREM 2.3. *Let $\mathfrak{A}$ be a Banach algebra admitting a continuous anti-homomorphism φ such that $\varphi^2 = 1_{\mathfrak{A}}$. Then $(\mathfrak{A}^{\star\star},\Box)$ is weakly amenable if and only if $(\mathfrak{A}^{\star\star},\Diamond)$ is weakly amenable.*

PROOF: Let $\varphi^{\star\star}:\mathfrak{A}^{\star\star}\to\mathfrak{A}^{\star\star}$ be the second adjoint of φ. For clarity, we introduce the following notation. $A = (\mathfrak{A}^{\star\star},\Box)$, $A^{op} = (\mathfrak{A}^{\star\star},\overline{\Box})$, $B = (\mathfrak{A}^{\star\star},\Diamond)$,$B^{op} = (\mathfrak{A}^{\star\star},\overline{\Diamond})$, so that if $F,G\in\mathfrak{A}^{\star\star}$, then $F\,\overline{\Box}\,G = G\Box F$ and $F\,\overline{\Diamond}\,G = G\Diamond F$. Using weak* limits we have $(\varphi^{\star\star})^2 = 1_{\mathfrak{A}^{\star\star}}$. Let $F,G\in\mathfrak{A}^{\star\star}$. Then $\varphi^{\star\star}(F\Box G) = \varphi^{\star\star}(F)\,\overline{\Diamond}\,\varphi^{\star\star}(G)$. In fact let (f_i) and (g_j) be nets in $\mathfrak{A}$, such that $w^\star - \lim_i \widehat{f_i} = F$, $w^\star - \lim_j \widehat{g}_j = G$. Then

$$\begin{aligned}\varphi^{\star\star}(F\Box G) &= w^\star - \lim_i w^\star - \lim_j \varphi^{\star\star}(\widehat{f_i}\,\widehat{g}_j) = w^\star - \lim_i w^\star - \lim_j \left[\varphi(f_i\,g_j)\right]^\wedge\\ &= w^\star - \lim_i w^\star - \lim_j \varphi(g_j)^\wedge\,\varphi(f_i)^\wedge = \varphi^{\star\star}(G)\Diamond\varphi^{\star\star}(F)\\ &= \varphi^{\star\star}(F)\,\overline{\Diamond}\,\varphi^{\star\star}(G).\end{aligned}$$

Similarly,

$$\varphi^{\star\star}(F\Diamond G) = \varphi^{\star\star}(F)\,\overline{\Box}\,\varphi^{\star\star}(G)\quad (F,G\in\mathfrak{A}^{\star\star}).$$

Now suppose that $A = (\mathfrak{A}^{\star\star},\Box)$ is weakly amenable. Then A^{op} is weakly amenable. Let D be a derivation from B into $B^\star$. Then $\overline{D} = \varphi^{\star\star\star}\circ D\circ\varphi^{\star\star}$ is a derivation from A^{op} into $(A^{op})^\star$. In fact, for $m,n\in A^{op}$, we have

$$\begin{aligned}(\varphi^{\star\star\star}\circ D\circ\varphi^{\star\star})(m\,\overline{\Box}\,n) &= \varphi^{\star\star\star}\left[D\left(\varphi^{\star\star}(m)\Diamond\varphi^{\star\star}(n)\right)\right]\\ &= \varphi^{\star\star\star}\left[D\left(\varphi^{\star\star}(m)\right)\diamond\varphi^{\star\star}(n) + \varphi^{\star\star}(m)\diamond D\left(\varphi^{\star\star}(n)\right)\right]\\ &= \varphi^{\star\star\star}\left[D(\varphi^{\star\star}(m)\right]\,\overline{\boxdot}\,n + m\,\overline{\boxdot}\,\varphi^{\star\star\star}D\left(\varphi^{\star\star}(n)\right).\end{aligned}$$

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Hence there exists an element $\psi \in \mathfrak{A}^{***}$ such that for every $F \in \mathfrak{A}^{**}$

$$(\varphi^{***} \circ D \circ \varphi^{**})(F) = F \,\overline{\boxdot}\, \psi - \psi \,\overline{\boxdot}\, F.$$

Applying φ^{***} to the two sides of the above equation and using $(\varphi^{***})^2 = 1_{\mathfrak{A}^{***}}$, we obtain

$$D\left(\varphi^{**}(F)\right) = \varphi^{**}(F) \diamondsuit \varphi^{***}(\psi) - \varphi^{***}(\psi) \diamondsuit \varphi^{**}(F).$$

Since φ^{**} is surjective, D is inner. □

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