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GROUPS COVERED BY FINITELY MANY NILPOTENT SUBGROUPS

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Let G be a finitely generated soluble group. Lennox and Wiegold have proved that G has a finite covering by nilpotent subgroups if and only if any infinite set of elements of G contains a pair $\{x, y\}$ such that $\langle x, y\rangle$ is nilpotent. The main theorem of this paper is an improvement of the previous result: we show that G has a finite covering by nilpotent subgroups if and only if any infinite set of elements of G contains a pair $\{x, y\}$ such that $[x,_n y] = 1$ for some integer $n = n(x, y) \geqslant 0$.

1. INTRODUCTION AND RESULTS

Let x and y be elements of a group G and let n be a non-negative integer. As usual, $[x,_n y]$ is defined inductively by $[x,_0 y] = x$ and $[x,_{n+1} y] = [[x,_n y], y]$, where $[x, y] = x^{-1}y^{-1}xy$. We say that G is *covered* by a family of subgroups $(H_i)_{i\in I}$ if $G = \bigcup_{i\in I} H_i$. The family $(H_i)_{i\in I}$ is called a *covering* of G. The following characterisation for finitely generated soluble groups covered by finitely many nilpotent subgroups was obtained by Lennox and Wiegold [4]:

THEOREM A. *Let G be a finitely generated soluble group. Then the following properties are equivalent:*

(i) *G is finite-by-nilpotent (that is, G has a finite covering by nilpotent subgroups, by Lemma 5 below).*
(ii) *Any infinite set of elements of G contains a pair $\{x, y\}$ which generate a nilpotent subgroup.*

The main purpose of this note is to improve the previous result. We shall prove:

THEOREM 1. *Let G be a finitely generated soluble group. Then the following properties are equivalent:*

(i) *G has a finite covering by nilpotent subgroups.*
(ii) *Any infinite set of elements of G contains a pair $\{x, y\}$ such that $[x,_n y] = 1$ for some integer $n = n(x, y) \geqslant 0$.*

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Note that this theorem is not true for an arbitrary group: the standard wreath product of a group of prime order p and an infinite elementary abelian p-group satisfies (ii) (this group is locally nilpotent) but does not satisfy (i) by Lemma 5 below (the centre is trivial).

The origin of the previous results is a problem of P. Erdös [**6**]. Associate with a group G a graph $\Gamma(G)$ in this way: the vertices of $\Gamma(G)$ are the elements of G, and two vertices x, y are connected by an edge if and only if $[x, y] \neq 1$.

Suppose that $\Gamma(G)$ contains no infinite complete subgraph (that is, any infinite set of elements of G contains a pair $\{x, y\}$ such that $[x, y] = 1$); is there then a finite bound on the cardinality of complete subgraphs of $\Gamma(G)$?

Neumann [**6**] solved the problem in the affirmative by proving that if $\Gamma(G)$ contains no infinite complete subgraph, then G has a finite covering by abelian subgroups. Therefore, if G is covered by n abelian subgroups, the order of a complete subgraph of $\Gamma(G)$ is at most n. Now consider the graph $\Gamma^*(G)$, where the vertices are the elements of G, and two vertices x, y are connected by an edge if and only if $[x,_n y] \neq 1$ and $[y,_n x] \neq 1$ for every integer $n \geqslant 0$. By observing that $\Gamma^*(G)$ contains no infinite complete subgraph if and only if G satisfies the property (ii) of Theorem 1, we obtain at once the following consequence of the Theorem 1:

COROLLARY. *Let G be a finitely generated soluble group. Suppose that the graph $\Gamma^*(G)$ defined above contains no infinite complete subgraph. Then, there exists a finite bound on the cardinality of complete subgraphs of $\Gamma^*(G)$.*

Now, consider an infinite group G. As was observed in [**5**], if for every pair $\{X, Y\}$ of infinite subsets of G there exists $x \in X$, $y \in Y$ such that $[x, y] = 1$, then G is abelian. For finitely generated soluble groups, this result was extended in this way:

THEOREM B. [**9**] *Let $k > 0$ be an integer. Let G be an infinite finitely generated soluble group such that, whenever X, Y are infinite subsets of G, there exist $x \in X$, $y \in Y$ such that $[x,_k y] = 1$. Then G is a k-Engel group (that is, $[x,_k y] = 1$ for all x, y in G)*

By a result of Gruenberg [**2**], it is well-known that every finitely generated soluble Engel group is nilpotent. Therefore, under the assumptions of Theorem B, the group G is nilpotent. As a consequence of Theorem 1, we shall prove a result of a similar nature:

THEOREM 2. *Let G be an infinite finitely generated soluble group such that, whenever X, Y are infinite subsets of G, there exist $x \in X$, $y \in Y$ and an integer $n \geqslant 0$ such that $[x,_n y] = 1$. Then G is nilpotent.*

2. Some preliminary lemmas

Let u be an element of a group G. An element x of G is called a *right Engel element with respect to* u if there exists an integer $n \geqslant 0$ such that $[x,_n u] = 1$. Let $R_u(G)$ denote the set of all such elements. An element of $R(G) := \bigcap_{u\in G} R_u(G)$ is called a *right Engel element.* If the derived subgroup G' is nilpotent (in particular if G is metabelian), then $R_u(G)$ is a subgroup of G [7].

Lemma 1. *Let $u, u_1, \dots, u_k$ be arbitrary elements of a metabelian group G. Then*

(i) $R_{u^{-1}}(G) = R_u(G)$.

(ii) $\bigcap_{t\in G} t^{-1}\{R_{u_1}(G) \cap \dots \cap R_{u_k}(G)\}t \subseteq \bigcap_{t\in G} t^{-1} R_{u_1\dots u_k}(G)t$.

(iii) *If $G = \langle w_1, \dots, w_q\rangle$ is finitely generated, we have*

$$R(G) = \bigcap_{t\in G} t^{-1}\{R_{w_1}(G) \cap \dots \cap R_{w_q}(G)\}t.$$

Proof: (i) It suffices to show the relation

$$[x,_n u^{-1}] = u^n [x,_n u]^{(-1)^n} u^{-n}$$

for arbitrary $u, x \in G$ and $n \geqslant 0$. Observe that our relation is true for $n \in \{0, 1\}$ and suppose that $[x,_{n-1} u^{-1}] = u^{n-1}[x,_{n-1} u]^{(-1)^{n-1}} u^{-n+1}$ for an integer $n > 1$. Then

$$\begin{aligned}[x,_n u^{-1}] &= [[x,_{n-1} u^{-1}], u^{-1}] = [u^{n-1}[x,_{n-1} u]^{(-1)^{n-1}} u^{-n+1}, u^{-1}] \\ &= u^{n-1}[[x,_{n-1} u]^{(-1)^{n-1}}, u^{-1}]u^{-n+1}.\end{aligned}$$

Since $[x,_{n-1} u]$ commutes with its conjugates, we can write

$$[x,_n u^{-1}] = u^{n-1}[[x,_{n-1} u], u^{-1}]^{(-1)^{n-1}} u^{-n+1}.$$

But $[[x,_{n-1} u], u^{-1}] = u[[x,_{n-1} u], u]^{-1} u^{-1}$, hence we obtain

$$[x,_n u^{-1}] = u^{n-1}\{u[[x,_{n-1} u], u]^{-1} u^{-1}\}^{(-1)^{n-1}} u^{-n+1} = u^n [x,_n u]^{(-1)^n} u^{-n}.$$

(ii) We show the assertion in the case $k = 2$: the assertion in the general case will follow at once by an easy induction on k. For convenience denote u_1 by u and u_2 by v. Let x be an element of $\bigcap_{t\in G} t^{-1}\{R_u(G)\cap R_v(G)\}t$. Since $\bigcap_{t\in G} t^{-1}\{R_u(G)\cap R_v(G)\}t$ is a normal subgroup of G, it suffices to prove that x belongs to $R_{uv}(G)$. First note that $[x, uv]$ is an element of $\bigcap_{t\in G} t^{-1}\{R_u(G) \cap R_v(G)\}t$. Thus there exists an integer $n > 0$

such that $[x, uv,_n u] = [x, uv,_n v] = 1$. From the relations $[y, uv] = [y, u][y, v][y, u, v]$ and $[y, u, v] = [y, v, u](y \in G')$, we deduce that $[x,_{2n} uv]$ is a product of commutators of the form $[x, uv,_r u',_s v']$, where $r + s \geqslant 2n - 1$, $r \geqslant s$ and $\{u', v'\} = \{u, v\}$. But the previous inequalities imply $r \geqslant n$, hence $[x,_{2n} uv] = 1$ and so $x \in R_{uv}(G)$ as required.

(iii) Clearly, we have the inclusion $R(G) \subseteq \bigcap_{t \in G} t^{-1}\{R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)\}t$. Conversely, to prove the inclusion $\bigcap_{t \in G} t^{-1}\{R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)\}t \subseteq R(G)$, it must be shown that $\bigcap_{t \in G} t^{-1}\{R_{w1}(G) \cap \ldots \cap R_{w_q}(G)\}t \subseteq R_u(G)$ for an arbitrary element $u \in G$. Write u in the form of a product of elements in $\{w_1, \ldots, w_q\} \cup \{w_1^{-1}, \ldots, w_q^{-1}\}$ and apply (i) (ii): it follows that

$$\bigcap_{t \in G} t^{-1}\{R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)\}t \subseteq \bigcap_{t \in G} t^{-1} R_u(G) t.$$

Hence $\bigcap_{t \in G} t^{-1}\{R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)\}t \subseteq R_u(G)$, so (iii) is proved. □

LEMMA 2. *Let G be a metabelian group satisfying the property (ii) of Theorem 1. Then*

- (i) *$R_u(G)$ has finite index in G for every $u \in G$.*
- (ii) *If G is finitely generated, $R(G)$ has finite index in G.*

PROOF: (i) Suppose there exists $u \in G$ such that $|G: R_u(G)|$ is infinite and choose a right transversal T of $R_u(G)$ in G. If $x^{-1}ux = y^{-1}uy$ $(x, y \in T)$, then $[xy^{-1}, u] = 1$, hence $x = y$ since $xy^{-1} \in R_u(G)$. Therefore, the set of conjugates of u by elements of T is infinite. Hence there exist $x, y \in T$ $(x \neq y)$ and $n > 0$ such that $[x^{-1}ux,_n y^{-1}uy] = 1$. We have

$$\begin{aligned} 1 &= [yx^{-1}uxy^{-1},_n u] = [u[u, xy^{-1}],_n u] = [[u, xy^{-1}],_n u] \\ &= [[xy^{-1}, u]^{-1},_n u] = [[xy^{-1}, u],_n u]^{-1} = [xy^{-1},_{n+1} u]^{-1} \end{aligned}$$

and so $xy^{-1} \in R_u(G)$, a contradiction.

(ii) Suppose that $G = \langle w_1, \ldots, w_q \rangle$. By (i), every subgroup $R_{w_1}(G), \ldots, R_{w_q}(G)$ has finite index in G, hence also $R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)$ and $\bigcap_{t \in G} t^{-1}\{R_{w_1}(G) \cap \ldots \cap R_{w_q}(G)\}t$. Using Lemma 1 (iii), we obtain the required result. □

The following result is due to Lennox [4]:

LEMMA 3. *Let G be a finitely generated soluble group and A an abelian normal subgroup such that G/A is polycyclic and $\langle a, x \rangle$ is polycyclic whenever $a \in A$, $x \in G$. Then G is polycyclic.*

Lemma 4. *Let G be a finitely generated soluble group satisfying the property (ii) of Theorem 1. Then G is polycyclic.*

Proof: Denote by d the derived length of G. First we show the lemma in the case $d \leqslant 2$. If $d \leqslant 1$, the result is obvious. Suppose now that $d = 2$. By Lemma 2, $|G\colon R(G)|$ is finite; hence $R(G)$ is finitely generated. Moreover $R(G)$ is a soluble Engel group and hence $R(G)$ is nilpotent [**2**]. Therefore we can say that G is polycyclic-by-polycyclic so G is polycyclic. Finally, use induction on d in the general case. If $d > 0$, put $A = G^{(d-1)}$. It follows from the inductive hypothesis that G/A is polycyclic. Clearly, the derived length of $\langle a, x\rangle$ is at most 2 whenever $a \in A$, $x \in G$, hence $\langle a, x\rangle$ is polycyclic. Lemma 3 permits us to conclude that G is polycyclic. □

Finally, we shall need the following characterisation of groups covered by finitely many nilpotent subgroups (see [**10**] for the equivalence of (i) and (ii) and [**3**] for the equivalence of (ii) and (iii)):

Lemma 5. *For an arbitrary group G, the following properties are equivalent:*

(i) *G has a finite covering by nilpotent subgroups.*

(ii) *For some integer $c \geqslant 0$, the term $\zeta_c(G)$ of the upper central series of G has finite index in G.*

(iii) *G is finite-by-nilpotent.*

3. Proofs of the theorems

Proof of Theorem 1: We have only to show that (ii) implies (i) since the converse is clearly true. Use induction on the derived length d of G, the case $d = 0$ being trivial. For $d > 0$, it follows from the inductive hypothesis and Lemma 5 that there exists an integer $c \geqslant 0$ such that $\left|G/G^{(d-1)}\colon \zeta_c\left(G/G^{(d-1)}\right)\right| < \infty$. But in a finitely generated soluble group, the hypercentre coincides with the set of right Engel elements [**1**]; hence $\left|G/G^{(d-1)}\colon R\left(G/G^{(d-1)}\right)\right|$ is finite. Let e denote the exponent of the quotient group $\left(G/G^{(d-1)}\right)/R\left(G/G^{(d-1)}\right)$. Therefore, for all $x, y \in G$, there exists an integer $m \geqslant 0$ such that $[x^e,_m y] \in G^{(d-1)}$. The subgroup $H = \langle [x^e,_m y], y\rangle$ is clearly metabelian. Hence $R(H)$ has finite index in H by Lemma 2. Denote by f the exponent of $H/R(G)$. Thus there exists an integer $n \geqslant 0$ such that $[[x^e,_m y]^f,_n y] = 1$. Since $[x^e,_m y]$ commutes with its conjugates, we obtain

$$[[x^e,_m y]^f,_n y] = [[x^e,_m y],_n y]^f = 1.$$

In other words, $[x^e,_{m+n} y]$ belongs to the torsion group $\tau\left(G^{(d-1)}\right)$ of $G^{(d-1)}$. This means that the quotient group $\{G/\tau\left(G^{(d-1)}\right)\}/R\left(G/\tau\left(G^{(d-1)}\right)\right)$ has exponent dividing e and so is finite. But $R\left(G/\tau\left(G^{(d-1)}\right)\right)$ coincides with the hypercentre of $G/\tau\left(G^{(d-1)}\right)$ by the result quoted above. Moreover, $G/\tau\left(G^{(d-1)}\right)$ satisfies the maximal condition on subgroups by Lemma 4. Therefore we have $R\big(G/\tau(G^{(d-1)})\big) = \zeta_{c'}\big(G/\tau(G^{(d-1)})\big)$ for some integer $c' \geqslant 0$ and $\big|G/\tau(G^{(d-1)}) : \zeta_{c'}\big(G/\tau(G^{(d-1)})\big)\big|$ is finite. We deduce from Lemma 5 that $G/\tau(G^{(d-1)})$ is finite-by-nilpotent. But G satisfies the maximal condition (Lemma 4) hence $\tau(G^{(d-1)})$ is finite and so G is finite-by-nilpotent. Finally, Lemma 5 shows that G has a finite covering by nilpotent subgroups, as required. $\square$

PROOF OF THEOREM 2: It suffices to show that $\zeta^*(G) = G$, where $\zeta^*(G)$ is the hypercentre of G. Clearly, G satisfies the property (ii) of Theorem 1, hence G has a finite covering by nilpotent subgroups. It follows from Lemma 5 that $\zeta^*(G)$ has finite index in G. In particular, $\zeta^*(G)$ is infinite. Let x, y be elements of G. Subsets $x\zeta^*(G)$ and $y\zeta^*(G)$ are infinite, hence there exist $u, v \in \zeta^*(G)$, $n \geqslant 0$, such that $[xu,_n yv] = 1$. This implies $[x,_n y] \in \zeta^*(G)$, so $G/\zeta^*(G)$ is an Engel group. But it is well-known that finite Engel groups are nilpotent (for example [**8**, 7.21]), so $G/\zeta^*(G)$ is nilpotent. Since the centre of $G/\zeta^*(G)$ is trivial, we obtain $\zeta^*(G) = G$. $\square$

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