Canad. Math. Bull. Vol. 63 (1), 2020 pp. 31–45
http://dx.doi.org/10.4153/S0008439519000468

Linear Independence of Logarithms of Cyclotomic Numbers and a Conjecture of Livingston

Tapas Chatterjee and Sonika Dhillon

Abstract. In 1965, A. Livingston conjectured the $\overline{\mathbb{Q}}$-linear independence of logarithms of values of the sine function at rational arguments. In 2016, S. Pathak disproved the conjecture. In this article, we give a new proof of Livingston's conjecture using some fundamental trigonometric identities. Moreover, we show that a stronger version of her theorem is true. In fact, we modify this conjecture by introducing a co-primality condition, and in that case we provide the necessary and sufficient conditions for the conjecture to be true. Finally, we identify a maximal linearly independent subset of the numbers considered in Livingston's conjecture.

1 Introduction

In a written communication with A. Livingston in 1965, Erdős made the following conjecture (see [9]).

Conjecture A (Erdős) *If q is a positive integer and f is a number-theoretic function with period q for which $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots, q-1$ and $f(q) = 0$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

In 1965, Livingston tried to settle the above conjecture. He predicted that Erdős' conjecture is true if one can prove the following conjecture.

Conjecture B (Livingston) *Let $q \geq 3$ be a positive integer. The numbers*

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}\Big\} \cup \{\pi\},$$

when q is odd, and

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}\Big\} \cup \{\pi, \log 2\},$$

when q is even, are linearly independent over the field of algebraic numbers.

Received by the editors December 17, 2018; revised July 31, 2019.
Published online on Cambridge Core December 12, 2019.
AMS subject classification: **11J81**, 11J72, 11J86, 11J91, 11R27.
Keywords: Baker's theory, linear forms in logarithm, primitive root, units in cyclotomic field.

In 2016, S. Pathak gave necessary conditions on the positive integer q under which Conjecture B is true (for a proof, see [12]). She also observed that Livingston's conjecture is not sufficient to prove Erdős conjecture. In fact, to prove Erdős' conjecture, we still need to prove that if f is an Erdős function, then at least one of

$$\sum_{a=1}^{q-1} f(a)\cot\Big(\frac{\pi a}{q}\Big), \quad \sum_{a=1}^{q-1} f(a)\cos\Big(\frac{2ab\pi}{q}\Big),$$

and

$$T_q = \begin{cases} \frac{\log 2}{q}\big(\sum_{k=1}^{q-1}(-1)^k f(k)\big) & \text{if } q \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

is not zero (see [6,7]).

In the direction of Livingston's conjecture, she proved the following theorems.

Theorem 1.1 *Conjecture B does not hold for $q \geq 6$ and q not prime. In fact for a composite positive integer $q \geq 6$, the numbers*

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}\Big\}$$

are $\mathbb{Q}$-linearly dependent.

Theorem 1.2 *Let p be an odd prime. The numbers*

$$\Big\{\log\Big(2\sin\frac{a\pi}{p}\Big) : 1 \leq a < \frac{p-1}{2}\Big\} \cup \{\pi\}$$

are $\mathbb{Q}$-linearly independent. Thus, Conjecture B is true when the modulus p is prime.

In that article [12], the author proved that Conjecture B is true when q is prime using the Dedekind determinant and provided an explicit counterexample when q is composite.

In this article, we give a new proof that involves the identities of the sine function at rational arguments.

Observe that when q is a multiple of 4, then for $a/q = 1/4$, we have a rational multiple of $\log 2$ in the set

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}\Big\}.$$

Also, when q is a multiple of 6, for $a/q = 1/6$ we have $\log(2\sin(\pi/6)) = 0$. To avoid these ambiguities in Conjecture B, we can rewrite Livingston's conjecture in a more suitable manner and ask a similar question.

Question 1 Let $q \geq 2$ be an integer. Are the numbers

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}, a/q \neq \frac{1}{\pi}\Big(\sin^{-1}\frac{1}{2^\alpha}\Big),\ \alpha \in \mathbb{Q}\Big\} \cup \{\pi,\ \log 2\}$$

linearly independent over the field of algebraic numbers?

In Section 3, we begin with a necessary and sufficient condition such that Question 1 has an affirmative answer. In particular, our theorem is as follows.

Theorem 1.3 *Let $q \geq 2$ be an integer. Then the numbers*

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2}, a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^{\alpha}}\right), \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\},$$

are linearly independent over the field of algebraic numbers if and only if q is a prime or $q \in \{4, 6\}$.

Now instead of taking all the residue classes mod q in Question 1, one can think of asking a similar question for the co-prime residue classes mod q.

Question 2 Let $q \geq 2$ be an integer. Then are the numbers

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2},\ (a,q) = 1,\ \ a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^{\alpha}}\right),\ \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\}$$

linearly independent over the field of algebraic numbers?

In our next theorem, we give a necessary and sufficient condition such that Question 2 has an affirmative answer.

Theorem 1.4 *Let $q \geq 2$ be an integer. Then the numbers*

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \leq a < \frac{q}{2},\ (a,q) = 1,\ \ a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^{\alpha}}\right), \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\}$$

are linearly independent over the field of algebraic numbers if and only if q is a prime power or $q = 6$.

Now note that when q is not a prime power, the sine function satisfies the following identity at rational arguments (for a proof see Section 3):

$$2^{\phi(q)} \prod_{\substack{k=1,\\ (k,q)=1}}^{q-1} \sin\left(\frac{k\pi}{q}\right) = 1. \tag{1.1}$$

Thus, the numbers $\{\log \sin \frac{k\pi}{q}\}$ where $(k,q) = 1$ satisfy a non-trivial relation when q is not a prime power. Now we can again modify Question 2 and can ask the similar question by excluding one of the terms among the numbers $\{\log \sin \frac{k\pi}{q}\}$ where $(k,q) = 1$.

Question 3 Let $q \geq 2$ be an integer. Then are the numbers

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 < a < \frac{q}{2},\ (a,q) = 1,\ \ a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^{\alpha}}\right),\ \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\},$$

linearly independent over the field of algebraic numbers?

Before doing this, note that Pei and Feng [13] gave a necessary and sufficient condition on $q \not\equiv 2 \pmod 4$ such that the cyclotomic numbers

$$\left\{\frac{1-\zeta_q^h}{1-\zeta_q}\ \middle|\ (h,q) = 1, 2 \leq h < q/2\right\} \tag{1.2}$$

are multiplicatively independent. Note that the multiplicative independence of these cyclotomic numbers is closely related to the linear independence of the numbers considered in Livingston's conjecture, $\log \sin(a\pi/q)$ where $1 < a \le q/2$ and $(a, q) = 1$. Since $|1 - \zeta_q^a| = 2 \sin a\pi/q$, therefore, the cyclotomic numbers numbers are multiplicatively independent if and only if $\log \sin(a\pi/q)$, where $1 \le a \le q/2$ with $(a, q) = 1$, are linearly independent over $\mathbb{Q}$ (for a proof, see Section 3).

The proposition by Pei and Feng [13] regarding the necessary and sufficient condition for multiplicatively independent cyclotomic units is also of great importance in proving our theorems (for a proof, see [13]). We say that n is a semi-primitive root modulo q if the order of $n \pmod q$ is $\frac{\phi(q)}{2}$.

Proposition 1.5 (Pei and Feng) *For a composite number $q \not\equiv 2 \pmod 4$, the system*

$$\left\{ \frac{1-\zeta_q^h}{1-\zeta_q} \;\middle|\; (h, q) = 1, 2 \le h < q/2 \right\}$$

of cyclotomic units of field $\mathbb{Q}(\zeta_q)$ is independent if and only if one of the following conditions are satisfied (here $\alpha_0 \ge 3; \alpha_1, \alpha_2, \alpha_3 \ge 1; p_1, p_2, p_3$ are odd primes):

(i) $q = 4p_1^{\alpha_1}$; *and*
  (a) 2 *is a primitive root* mod $p_1^{\alpha_1}$; *or*
  (b) 2 *is a semi-primitive root* mod $p_1^{\alpha_1}$ *and* $p_1 \equiv 3 \pmod 4$.
(ii) $q = 2^{\alpha_0} p_1^{\alpha_1}$; *the order of* $p_1 \pmod{2^{\alpha_0}}$ *is* $2^{\alpha_0 - 2}$, $2^{\alpha_0-3} p_1 \not\equiv -1 \pmod{2^{\alpha_0}}$, *and*
  (a) 2 *is a primitive root mod* $p_1^{\alpha_1}$; *or*
  (b) 2 *is a semi-primitive root mod* $p_1^{\alpha_1}$ *and* $p_1 \equiv 3 \pmod 4$.
(iii) $q = p_1^{\alpha_1} p_2^{\alpha_2}$; *and*
  (a) *when* $p_1 \equiv p_2 \equiv 3 \pmod 4$: p_1 *is a semi-primitive root* mod $p_2^{\alpha_2}$ *and* p_2 *is a semi-primitive root* mod $p_1^{\alpha_1}$, *or vice versa.*
  (b) *otherwise:* p_1 *and* p_2 *are primitive roots* mod $p_2^{\alpha_2}$ *and* mod $p_1^{\alpha_1}$ *respectively.*
(iv) $q = 4p_1^{\alpha_1} p_2^{\alpha_2}$; $(p_1 - 1, p_2 - 1) = 2$ *and*
  (a) *when* $p_1 \equiv p_2 \equiv 3 \pmod 4$: 2 *is a primitive root for one* p *and a semi-primitive root for another* p; p_1 *is a primitive root* mod $2p_2^{\alpha_2}$ *and* p_2 *is a semi-primitive root* mod $2p_1^{\alpha_1}$ *or vice versa.*
  (b) *when* $p_1 \equiv 1, p_2 \equiv 3 \pmod 4$: 2 *is a primitive root* mod $p_2^{\alpha_2}$; p_1 *and* p_2 *are primitive roots* mod $p_2^{\alpha_2}$ *and* mod $p_1^{\alpha_1}$, *respectively.*
(v) $q = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$; $p_1 \equiv p_2 \equiv p_3 \equiv 3 \pmod 4$: $(p^i - 1)/2 (1 \le i \le 3)$ *are co-prime to each other; and*
  (a) p_1, p_2, p_3 *are primitive roots* mod $p_2^{\alpha_2}$, mod $p_3^{\alpha_3}$, mod $p_1^{\alpha_1}$, *respectively and semi-primitive roots* mod $p_3^{\alpha_3}$, mod $p_1^{\alpha_1}$, mod $p_2^{\alpha_2}$, *respectively.*

Note that the $\frac{\phi(q)}{2} - 1$ many numbers in equation (1.2) do not form a set of multiplicatively independent units in the cyclotomic fields for any $q \in \mathbb{N}$. In 1966, Ramachandra [14] exhibited a set of real independent units, popularly known as Ramachandra units, in the cyclotomic fields defined as: suppose $q = \prod_{i=1}^k p_i^{a_i}$ and let s be such that $1 < s < q/2$ with $(s, q) = 1$. Define

$$v_s = \prod_{e_i} \left(\frac{1 - a^{s p_1^{a_1 e_1} \cdots p_k^{a_k e_k}}}{1 - a^{p_1^{a_1 e_1} \cdots p_k^{a_k e_k}}} \right),$$

where the product is extended over all $e_i = 0$ or 1, $i = 1, 2, \ldots, k$ except $e_1 = e_2 = \cdots = e_k = 1$. Then these numbers form a set of multiplicatively independent units.

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

Using the Proposition 1.5 and some trigonometric identities, we give the necessary and sufficient condition on $q \equiv 2 \pmod 4$ for which the system (1.2) is multiplicatively independent. Our next theorem will be an important ingredient in resolving Question 3.

Proposition 1.6 *For any composite number $q \equiv 2 \pmod 4$, the system*

$$\Big\{\frac{1-\zeta_q^h}{1-\zeta_q}\Big|\ (h,q)=1,\ 1<h<q/2\Big\}$$

is multiplicatively independent if and only if q satisfies one of the following conditions:

(i) *$q = 2p^n$, where p is an odd prime;*
(ii) *$q = 2m$, where m satisfies conditions* (iii) *and* (v) *in Proposition* 1.5.

Now we are in a position to give the necessary and sufficient conditions on q such that Question 3 has an affirmative answer.

Theorem 1.7 *Let $q \geq 2$ be an integer. Then the numbers*

$$\Big\{\log\Big(2\sin\frac{a\pi}{q}\Big) : 1<a<\frac{q}{2},\ (a,q)=1, a/q \neq \frac{1}{\pi}\Big(\sin^{-1}\frac{1}{2^{\alpha}}\Big), \alpha\in\mathbb{Q}\Big\} \cup \{\pi,\ \log 2\}$$

are linearly independent over the field of algebraic numbers if and only if q satisfies one of the following conditions:

(i) *q is a prime power;*
(ii) *$q = 2p^n$, where p is an odd prime and $n \in \mathbb{N}$;*
(iii) *q satisfies the conditions in Proposition* 1.5*;*
(iv) *$q = 2m$ where m satisfies conditions* (iii) *and* (v) *in Proposition* 1.5.

Now in our next theorem, we will construct a maximal linearly independent subset of the set M defined as follows:

$$M = \Big\{\pi,\ \log 2,\ \log\Big(2\sin\frac{a\pi}{q}\Big) : 1 \leq a < \frac{q}{2}\Big\}.$$

We prove the following theorem assuming q satisfies the conditions given in Proposition 1.5.

Theorem 1.8 *Let $q = p_1^{a_1}p_2^{a_2}$ be a positive integer that satisfies Proposition* 1.5(iii)*. Then out of the set M, the subset*

$$M_1 = \Big\{\log\Big(2\sin\frac{a\pi}{q}\Big), \log\Big(2\sin\frac{\pi}{p_1^{a_1}}\Big), \log\Big(2\sin\frac{\pi}{p_2^{a_2}}\Big) : 1<a<\frac{q}{2},\ (a,q)=1\Big\} \cup \{\pi,\ \log 2\} \tag{1.3}$$

is a maximal linearly independent subset over the field of algebraic numbers.

2 Notation and Preliminaries

This section is comprised of some of the known results in transcendental number theory.

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

An important ingredient is Baker's theorem about the linear forms in logarithms of algebraic numbers (see [1]).

Proposition 2.1 *If $\alpha_1, \dots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the field of rational numbers, then $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the field of algebraic numbers.*

The next proposition is an application of Baker's theorem by Murty and Saradha that will play a key role in proving our theorems (see [11]).

Proposition 2.2 *Let $\alpha_1, \dots, \alpha_n$ be positive algebraic numbers. If $c_0, c_1, \dots, c_n$ are algebraic numbers with $c_0 \neq 0$, then*

$$c_0 \pi + \sum_{j=1}^{n} c_j \log \alpha_j$$

is a transcendental number and hence non-zero.

The next proposition, due to Chatterjee and Gun [5], about the multiplicatively independence of cyclotomic units, will be of great importance in proving our theorems.

Proposition 2.3 *For any finite set J of primes in $\mathbb{N}$ with $p_i \in J$ and $q_i = p_i^{m_i}$, where $m_i \in \mathbb{N}$, and let ζ_{q_i} be a primitive q_i-th root of unity. Then the numbers $1 - \zeta_{q_i}$, $(1 - \zeta_{q_i}^{a_{j_i}})/(1 - \zeta_{q_i})$, where*

$$1 < a_{j_i} < q_i/2, \quad (a_{j_i}, q_i) = 1, \quad \textit{and} \quad 1 < j_i < q_i/2, \quad \forall p_i \in J,$$

are multiplicatively independent.

(For a proof see [5]).

The next lemma plays a pivotal role in establishing Proposition 1.6.

Lemma 2.4 *For any positive composite number $q \equiv 2 \pmod 4$ with $q = 2m$ for some odd integer m, the system*

$$S = \left\{ \log\left(2 \sin \frac{a\pi}{q}\right) \,\middle|\, (a, q) = 1, 1 < a < q/2 \right\}$$

is linearly independent over $\mathbb{Q}$ if and only if the system

$$T = \left\{ \log\left(2 \sin \frac{h\pi}{m}\right) \,\middle|\, (h, m) = 1, 1 < h < m/2 \right\}$$

is linearly independent over $\mathbb{Q}$.

Proof Suppose m is a composite number and the system S is linearly independent over $\mathbb{Q}$. Now for any $2 \le a \le q/2$ with $(a, q) = 1$,

$$\log\left(2 \sin \frac{a\pi}{q}\right) = -\log\left(2 \sin \frac{(m-a)\pi}{q}\right) + \log\left(2 \sin \frac{(2m-2a)\pi}{q}\right).$$

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

This implies that every element of S can be written as a $\mathbb{Q}$-linear combination of elements of the T and $\log(2\sin\frac{\pi}{m})$. Since m is composite, by using equation (1.1) we therefore have

$$\log\left(2\sin\frac{\pi}{m}\right) = -\sum_{\substack{a=2\\(a,m)=1}}^{m-1} \log\left(2\sin\frac{a\pi}{m}\right). \tag{2.1}$$

Thus, every element of the set S can be written as a linear combination of elements of the set T. Hence, the set T is linearly independent. Now assume that the set T is linearly independent. Suppose there exist rationals $c_1, c_2, \ldots, c_r$ where $r = \phi(q)/2 - 1$ such that

$$\sum_{i=1}^{r} c_i \log\left(2\sin\frac{b_i\pi}{q}\right) = 0,$$

where $2 \le b_i \le q/2$ with $b_1 \le b_2 \le \cdots \le b_r$ and $(b_i, q) = 1$. Rewriting this equation, we get

$$\sum_{i=1}^{r} c_i \log\left(2\sin\frac{(m-b_i)\pi}{q}\right) = \sum_{i=1}^{r} c_i \log\left(2\sin\frac{2(m-b_i)\pi}{q}\right). \tag{2.2}$$

Since m is composite, by using equation (1.1) we therefore have

$$\log\left(2\sin\frac{\pi}{m}\right) = -\sum_{\substack{a=2\\(a,m)=1}}^{(m-1)/2} \log\left(2\sin\frac{a\pi}{m}\right). \tag{2.3}$$

Also observe that $b_r = m - 2$, and rewriting equation (2.2), we get

$$\sum_{i=1}^{r-1} c_i \log\left(2\sin\frac{(m-b_i)\pi}{q}\right) + c_r \log\left(2\sin\frac{2\pi}{q}\right) = \\ \sum_{i=1}^{r} c_i \log\left(2\sin\frac{2(m-b_i)\pi}{q}\right).$$

Now, substituting the value of $\log(2\sin\pi/m) = \log(2\sin 2\pi/q)$ from equation (2.3) in the above equation, we get

$$\sum_{i=1}^{r-1}(-c_r + c_i) \log\left(2\sin\frac{(m-b_i)\pi}{q}\right) - c_r \log\left(2\sin\frac{(m-1)\pi}{q}\right) = \\ \sum_{i=1}^{r} c_i \log\left(2\sin\frac{2(m-b_i)\pi}{q}\right).$$

Note that both sides of equation (2.2) represent all the terms of the set T. Using the fact that the set T is linearly independent, we must have $c_r = -c_{t_1}$ for some t_1. Considering the coefficient of $\log(2\sin\frac{(q-b_{t_1})\pi}{m})$, we must have $-c_r + c_{t_1} = c_{t_2}$, that is, $c_{t_2} = -2c_r$. Again considering the coefficient of $\log(2\sin\frac{(q-b_{t_2})\pi}{m})$, we get $c_{t_3} = -3c_r$. Repeating this process and since T is a finite linearly independent set, we get $c_r = -rc_r$, that is, $c_r = 0$, and hence $c_i = 0$ for all i. Thus, the set S is linearly independent.

For the case when $q = 2m$ and m is an odd prime power, by [15, theorem 8.3] the set $T \cup \{\log(2\sin\frac{\pi}{m})\}$ is linearly independent. The rest follows from the similar path as above. This completes the proof. ∎

Now we will prove another important lemma that will play a crucial role in proving Theorem 1.8.

Lemma 2.5 *Let q be a positive integer that satisfy Proposition 1.5*(iii)*. Then the set*

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 < a < \frac{q}{2}, \quad (a,q)=1\right\},\ \pi,$$
$$\log 2,\ \log\left(2\sin\frac{\pi}{p_1^{a_1}}\right),\ \log\left(2\sin\frac{\pi}{p_2^{a_2}}\right),$$

is linearly independent over the field of algebraic numbers.

Proof Suppose there exist integers d_1, d_2, d_3, and c_a, where $1 < a < q/2$ with $(a,q) = 1$ such that

$$(2.4) \qquad (2^{d_1})\left(2\sin\left(\frac{\pi}{p_1^{a_1}}\right)\right)^{d_2}\left(2\sin\left(\frac{\pi}{p_2^{a_2}}\right)\right)^{d_3}\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}\left(2\sin\frac{a\pi}{q}\right)^{c_a} = 1.$$

Taking the norm on both sides, we get

$$(2^{d_1r_1+d_2r_2+d_3r_3})(p_1^{d_2r_2})(p_2^{d_3r_3}) = 1,$$

where $r_i \in \mathbb{N}$. This implies that $d_1 = d_2 = d_3 = 0$. Thus, equation (2.4) reduces to

$$\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}\left(2\sin\frac{a\pi}{q}\right)^{c_a} = 1.$$

Since the numbers $2\sin(\pi a/q)$, where $2 \le a \le q/2$ with $(a,q) = 1$, are multiplicatively independent by using Theorem 1.7, we therefore get $c_a = 0$ for all a. Thus, the numbers

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 < a < \frac{q}{2},\ (a,q)=1\right\},\ \pi,$$
$$\log 2,\ \log\left(2\sin\frac{\pi}{p_1^{a_1}}\right), \log\left(2\sin\frac{\pi}{p_2^{a_2}}\right)$$

are linearly independent over $\mathbb{Q}$, and hence over $\overline{\mathbb{Q}}$, by using Baker's theorem and Proposition 2.2. ∎

Since we are dealing with the cyclotomic fields, one important ingredient is the cyclotomic polynomial $\Phi_q(x)$ at $x = 1$, where $\Phi_q(x)$ is defined as

$$\Phi_q(x) = \prod_{\substack{k=1,\\(k,q)=1}}^{q-1}\left(x - e^{2\pi ik/q}\right).$$

Note that at $x = 1$, the cyclotomic polynomial $\Phi_q(x)$ satisfies the following relations:

$$\Phi_q(1) = \begin{cases} p & \text{if } q = p^n, \text{ where } p \text{ is a prime},\\ 1 & \text{otherwise.}\end{cases}$$

For a proof, see [2, Lemma 7.3].

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

3 Proofs of the Main Theorems

Before proving the theorems in Section 1, we will make an important observation. Suppose q is a positive integer. Then

$$1 + x + x^2 + \cdots + x^{q-1} = \prod_{k=1}^{q-1} (x - \zeta_q^k),$$

where $\zeta_q = e^{2\pi i/q}$. For $x = 1$, we get

$$q = \prod_{k=1}^{q-1} (1 - \zeta_q^k).$$

Since, $|1 - \zeta_q^k| = 2\sin(\frac{k\pi}{q})$, we have

$$q = 2^{q-1} \prod_{k=1}^{q-1} \sin\left(\frac{k\pi}{q}\right).$$

If q is not a prime power, then $\Phi_q(1) = 1$ where $\Phi_q(x)$ is the q-th cyclotomic polynomial. Thus, we have

$$1 = \prod_{\substack{k=1,\\ (k,q)=1}}^{q-1} |1 - \zeta_q^k|, \quad \text{where } \zeta_q = e^{2\pi i/q}. \tag{3.1}$$

Since $|1 - \zeta_q^k| = 2\sin(\frac{k\pi}{q})$, we get

$$2^{\phi(q)} \prod_{\substack{k=1,\\ (k,q)=1}}^{q-1} \sin\left(\frac{k\pi}{q}\right) = 1. \tag{3.2}$$

3.1 Proof of Theorem 1.3

Proof When $q = 6$, the set

$$\left\{ \log\left(2\sin\frac{a\pi}{q}\right) : 1 \le a < \frac{q}{2},\ a/q \ne \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^\alpha}\right),\ \alpha \in \mathbb{Q} \right\} \tag{3.3}$$

contains only one element, namely, $a = 2$. Since $\log 2$ and $\log 3$ are linearly independent, by Proposition 2.2 we will get the desired result. For $q = 4$, the set in equation (3.3) contains no element. Thus, $\log 2$ and π are linearly independent by using Proposition 2.2.

When q is not a prime power and $q \ne 6$, consider the identity

$$2^{\phi(q)} \prod_{\substack{k=1,\\ (k,q)=1}}^{q-1} \sin\left(\frac{k\pi}{q}\right) = 1,$$

where $\phi(n)$ is the Euler–Phi function. Thus, taking log on both sides gives us a nontrivial relation among the numbers in equation (3.3).

Suppose q is a prime power and q itself is not a prime. Then for any divisor $b > 1$ of q and $x = 1, 2, \ldots, q-1$, we have (see [8])

$$\log|1-\zeta_q^{(q/b)x}| = \sum_{\substack{u=1 \\ u\equiv x \pmod b}}^{q-1} \log|1-\zeta_q^u|, \tag{3.4}$$

where $\zeta_q = e^{2\pi i/q}$. Also, for any $1 \le k \le q-1$,

$$\sin\Big(\frac{k\pi}{q}\Big) = \frac{e^{-ik\pi/q}(\zeta_q^k - 1)}{2i}.$$

Thus, for any divisor b of q, equation (3.4) becomes

$$\log\Big(2\sin\frac{bx\pi}{q}\Big) = \sum_{\substack{u=1 \\ u\equiv x \bmod q/b}}^{q-1} \log\Big(2\sin\frac{u\pi}{q}\Big). \tag{3.5}$$

From here we can conclude that $\log(2\sin\frac{k\pi}{q})$ where $(k,q) > 1$ and $1 \le k \le \frac{q-1}{2}$ can be written as a $\mathbb{Q}$-linear combination of $\log(2\sin\frac{r\pi}{q})$ where $(r,q) = 1$ and $1 \le r \le \frac{q-1}{2}$.

Now assume q is a prime number; then the cyclotomic units

$$\frac{1-\zeta_q^a}{1-\zeta_q} \text{ where } 1 < a < q/2,$$

are multiplicatively independent (see [15, Theorem 8.3]). Thus, the numbers

$$1-\zeta_q, \frac{1-\zeta_q^a}{1-\zeta_q} \text{ where } 1 < a < q/2,$$

are also multiplicatively independent (see [10, Lemma 14]). From this, one can easily deduce that the numbers

$$1-\zeta_q^a \text{ where } 1 \le a < q/2$$

are multiplicatively independent. For if there exist integers c_a such that

$$\prod_{a=1}^{q/2}(1-\zeta_q^a)^{c_a} = (1-\zeta_q)^{c_1 - \sum_{i\neq 1} c_i}\prod_{a=2}^{q/2}\Big(\frac{1-\zeta_q^a}{1-\zeta_q}\Big)^{c_a} = 1.$$

Thus, by Proposition 2.3, we have $c_a = 0$ for all a. Since $2\sin(\frac{a\pi}{q}) = |1-\zeta_q^a|$ and $1 \le a < q/2$, taking log on both sides therefore implies that the numbers

$$\log\Big(2\sin\frac{a\pi}{q}\Big), \text{ where } 1 \le a < q/2,$$

are linearly independent over $\mathbb{Q}$. Thus, by Baker's theorem Proposition 2.1, the numbers are linearly independent over $\overline{\mathbb{Q}}$. This completes the proof. ∎

3.2 Proof of Theorem 1.4

Proof The case when $q = 6$ follows from Theorem 1.3.

Now suppose that q is not a prime power; then, using equation (3.2), we get

$$2^{\phi(q)} \prod_{\substack{k=1,\\(k,q)=1}}^{q-1} \sin\left(\frac{k\pi}{q}\right) = 1.$$

Taking log on both sides gives a non-trivial relation among the numbers in Theorem 1.4.

For the case when q is a prime power, then by using Proposition 1.5, the numbers

$$\frac{1-\zeta_q^a}{1-\zeta_q} \text{ where } 1 < a < q/2,\ (a,q) = 1$$

are multiplicatively independent. Assuming $J = \{q\}$ in Proposition 2.3 implies that the numbers

$$1-\zeta_q, \frac{1-\zeta_q^a}{1-\zeta_q} \text{ where } 1 < a < q/2, (a,q) = 1$$

are multiplicatively independent. Now, using a method similar to the one we used in Theorem 1.3, the numbers

$$\log\left(2\sin\frac{a\pi}{q}\right), \text{ where } 1 \le a < q/2, (a,q) = 1$$

are linearly independent over $\mathbb{Q}$, and hence over $\overline{\mathbb{Q}}$, by Baker's theorem. This completes the proof. ∎

Remark 3.1 Note that, using Theorems 1.3 and 1.4, we can conclude that when q is a prime power, then out of the set

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : 1 \le a < \frac{q}{2}, a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^\alpha}\right), \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\},$$

the subset

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right) : (a,q) = 1, a/q \neq \frac{1}{\pi}\left(\sin^{-1}\frac{1}{2^\alpha}\right), \alpha \in \mathbb{Q}\right\} \cup \{\pi, \log 2\},$$

is a maximal linearly independent subset. Also, a variant of this result has been proved in [3] and has been extended in [4].

3.3 Proof of Proposition 1.6

Proof Suppose $q = 2m$ where $m > 1$ is an odd positive integer. First, we will show that to prove the numbers

$$\left\{\frac{1-\zeta_q^h}{1-\zeta_q} \mid (h,q) = 1,\ 1 < h < q/2\right\}$$

are multiplicatively independent, it is sufficient to show that the numbers

$$\left\{1-\zeta_q^h \mid (h,q) = 1,\ 1 < h < q/2\right\} \tag{3.6}$$

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

are multiplicatively independent. Suppose the numbers in equation (3.6) are multiplicatively independent, and, if possible, let there be integers c_h such that

$$\prod_{\substack{h=2,\\(h,q)=1}}^{q-1/2} \left(\frac{1-\zeta_q^h}{1-\zeta_q}\right)^{c_h} = 1.$$

Since q is a composite number which is not a prime power, by using equation (3.1) and substituting the value of $1-\zeta_q$ in the above equation, we get

$$\prod_{\substack{h=2,\\(h,q)=1}}^{q-1/2} \left(1-\zeta_q^h\right)^{\left(\sum_i c_i\right)+c_h} = 1.$$

Note that we are not considering the negative terms in the above formula, as it is not going to affect the multiplicative independence. Since the numbers in equation (3.6) are multiplicatively independent, we get

$$\Bigg(\sum_{\substack{i=2,\\(i,q)=1}}^{q/2} c_i\Bigg) + c_h = 0 \text{ for all } 1 < h < q/2, (h,q) = 1.$$

Solving the above system of linear homogeneous equations in the variables c_h, it is not difficult to see that $c_h = 0$ for all h. Thus, to prove Proposition 1.6, we need to show that the numbers

$$\left\{\log\left(2\sin\frac{h\pi}{q}\right) \;\middle|\; (h,q) = 1, 1 < h < q/2\right\}$$

are linearly independent over $\mathbb{Q}$, and hence by using Lemma 2.4, the above numbers are linearly independent if and only if the numbers

$$\left\{\log\left(2\sin\frac{h\pi}{m}\right) \;\middle|\; (h,m) = 1, 1 < h < m/2\right\} \tag{3.7}$$

are linearly independent over $\mathbb{Q}$. Since m is an odd integer, the numbers in equation (3.7) are linearly independent over $\mathbb{Q}$ if and only if m is a prime power or m satisfies conditions (iii) and (v) of Proposition 1.5. This completes the proof. ■

3.4 Proof of Theorem 1.7

Proof Suppose q does not satisfy any of the given conditions in Theorem 1.7; then the numbers

$$\left\{\frac{1-\zeta_q^a}{1-\zeta_q} \;\middle|\; (a,q) = 1,\ 1 < a < q/2\right\}$$

are multiplicatively dependent by using Proposition 1.5 and 1.6. Let c_a be integers, not all zero, such that

$$\prod_{\substack{a=2,\\(a,q)=1}}^{q/2} \left(\frac{1-\zeta_q^a}{1-\zeta_q}\right)^{c_a} = (1-\zeta_q)^{-\sum_i c_i} \prod_{\substack{a=2,\\(a,q)=1}}^{q/2} (1-\zeta_q^a)^{c_a} = 1.$$

Substituting the value of $1-\zeta_q$ in this equation, we get

$$\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}(1-\zeta_q^a)^{c_a+\left(\sum_i c_i\right)}=1.$$

Now, as we did in the proof of Proposition 1.6, the system of homogeneous equations $c_a+\sum_i c_i=0$ for all $1<a<q/2$ with $(a,q)=1$ has a non-trivial solution if and only if there exists an h such that $c_h\neq 0$. Since all the c_a are not zero, the numbers

$$\log\left(2\sin\frac{a\pi}{q}\right)\quad 1<a\leq q/2,\quad (a,q)=1$$

are linearly dependent where $|1-\zeta_q^a|=2\sin(a\pi/q)$.

When q is a prime power, then by using Theorem 1.4, the result holds. Suppose q is not a prime power and q satisfies one of the conditions in Theorem 1.7. If possible, let there be integers b_a such that

$$\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}(1-\zeta_q^a)^{b_a}=(1-\zeta_q)^b\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}\left(\frac{1-\zeta_q^a}{1-\zeta_q}\right)^{b_a}=1, \tag{3.8}$$

where

$$b=\sum_{\substack{i=2\\(i,q)=1}}^{q/2}b_i.$$

Now by using equation (3.1), we get

$$\left(1-\zeta_q\right)^{-\phi(q)/2}=\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}\left(\frac{1-\zeta_q^a}{1-\zeta_q}\right).$$

Substituting the value of $1-\zeta_q$ and rewriting equation (3.8), we get

$$\prod_{\substack{a=2,\\(a,q)=1}}^{q/2}\left(\frac{1-\zeta_q^a}{1-\zeta_q}\right)^{\left(-b_a\phi(q)/2\right)+b}=1.$$

Since q satisfies one of the conditions (ii), (iii), or (iv), the above numbers are multiplicatively independent by using Propositions 1.5 and 1.6, and thus we get $-b_a\phi(q)/2+b=0$ for all $1<a<q/2$ with $(a,q)=1$. Solving this linear homogeneous system of equations as we did in Proposition 1.6, we get $b_a=0$ for all a. Thus, the numbers

$$\left\{\log\left(2\sin\frac{a\pi}{q}\right):1<a<\frac{q}{2},\ (a,q)=1,\right\}$$

are linearly independent. Now by using the idea of norm as we did in Lemma 2.5 and by using Proposition 2.2, the numbers

$$\left\{\log 2,\log\left(2\sin\frac{a\pi}{q}\right):1<a<\frac{q}{2},\ (a,q)=1,\right\}\cup\{\pi\}$$

are linearly independent over $\mathbb{Q}$, and hence over $\overline{\mathbb{Q}}$, by Baker's Theorem. This completes the proof. ∎

https://doi.org/10.4153/S0008439519000468 Published online by Cambridge University Press

3.5 Proof of Theorem 1.8

Proof First observe that in order to prove Theorem 1.8, by using Lemma 2.5, it is sufficient to show that every element of the form $\log(2\sin(d\pi/q))$ where $(d,q)>1$ can be written as an algebraic linear combination of the elements of the set M_1. Let $(d,q)=b>1$ and m be the positive integer such that $d=mb$.

If $b=p_1^{t_1}p_2^{t_2}$ where $t_1\neq a_1$ and $t_2\neq a_2$, then by using equation (3.5), we get

$$\log\Big(2\sin\frac{bm\pi}{q}\Big)=\sum_{\substack{u=1\\ u\equiv m \bmod q/b}}^{q-1}\log\Big(2\sin\frac{u\pi}{q}\Big). \tag{3.9}$$

Since $t_1\neq a_1$ and $t_2\neq a_2$, $(m,q)=1$, and hence $(u,q)=1$. Now if u does not assume the value 1, then all the numbers on the right hand side in equation (3.9) belong to the set M_1. If it does, then, by using equation (2.1), we get the desired result.

Now when $b=p_1^{a_1}$ or $p_2^{a_2}$, without loss of generality assume that $b=p_2^{a_2}$. Then by using (3.9), we get $u=m+p_1^{a_1}k$ where $0\le k\le b-1$. Suppose there exists a u such that $(u,q)<p_2^{a_2}$; then $(u,q)=p_1^{t_1}p_2^{t_2}$ where $t_1\neq a_1$ and $t_2\neq a_2$. Thus, by using similar ideas as above, we can write the number $\log\sin u\pi/q$ on the right-hand side of equation (3.9) as a linear combination of the set M_1. Suppose for some u we have $(u,q)\ge p_2^{a_2}$; then u has to be unique, since $0\le k\le b-1$; that is, there can be at most one u on the right-hand side of equation (3.9) such that $(u,q)\ge p_2^{a_2}$. In fact, the choice of $k=(p_2^{a_2}t-m)/p_1^{a_1}$, where $t\equiv p_2^{-a_2} \bmod p_1^{a_1}$ gives us the unique value.

Observe that when $b=p_2^{a_2}$, if we vary m, then for each m, we can get at most one u satisfying $(u,q)\ge p_2^{a_2}$. Then we will show that out of these numbers we can choose one of them as the representative and write the rest of the numbers as linear combinations of the representative and the elements in the set M_1.

For this, first choose $m=1$, that is, $d=p_2^{a_2}$. Then by equation (3.9), we have

$$\log\Big(2\sin\frac{d\pi}{q}\Big)=\sum_{\substack{u=1\\ u\equiv 1 \bmod p_1^{a_1}}}^{q-1}\log\Big(2\sin\frac{u\pi}{q}\Big). \tag{3.10}$$

Let $u_1=1+p_1^{a_1}k$ for some $0\le k\le b-1$ be the unique element on the right-hand side of equation (3.10) such that $(u_1,q)\ge p_2^{a_2}$. Assume that $u_1=p_2^{a_2}t_1$ for some t_1. Then we have $p_2^{a_2}t_1\equiv 1 \pmod{p_1^{a_1}}$. Now, substituting the value $d=u_1$ in equation (3.10) and repeating the same process for $d=u_1$, we get a unique u_2 of the form $u_2=t_1+p_1^{a_1}k$ for some $0\le k\le b-1$ such that $p_2^{a_2}$ divides u_2 and assume $u_2=p_2^{a_2}t_2$. Then

$$p_2^{a_2}t_2\equiv t_1 \pmod{p_1^{a_1}}.$$

Thus, we get $t_2\equiv t_1^2 \bmod p_1^{a_1}$. Without loss of generality, let us assume $t_2=t_1^2$. Again repeat this process for $\log(2\sin\frac{u_2\pi}{q})$, that is, for $d=u_2$ in equation (3.10), and continuing in this manner, we will get a least positive integer n such that $u_n=p_2^{a_2}t_1^n$ and

$$\sin\Big(\frac{\pi}{p_1^{a_1}}\Big)=\sin\Big(\frac{u_n\pi}{q}\Big)$$

and $t_1^n\equiv\pm1 \bmod p_1^{a_1}$. Since q satisfies (iii)(a) or (iii)(b) in Proposition 1.5 and $p_2^{a_2}t_1\equiv 1 \pmod{p_1^{a_1}}$, we have $n\ge\phi(p_1^{a_1})/2$. Thus, after n iterations we can extract all the u such that $(u,q) \geq p_2^{a_2}$, and without loss of generality, we can take $m = 1$ to be the representative that is the element $\log 2\sin p_2^{a_2}\pi/q = \log 2\sin \pi/p_1^{a_1}$. Thus, every element of the form $\log(2\sin\frac{d\pi}{q})$, where $(d,q) = p_2^{a_2}$, can be written as an algebraic linear combination of the elements in equation (1.3). This gives us the desired form. The case when $b = p_1^{a_1}$ can be done in a similar way. Thus, every element on the right-hand side of equation (3.9) can be written as a linear combination of the elements in the set M_1, and hence the number on the left-hand side can be written in the desired form.

Now we are left with the case where $b = p_1^{a_1}p_2^r$ and $0 < r < a_2$. Then by (3.9) we have $u = m + p_2^{a_2-r}k$, where $0 \leq k \leq b-1$, and $(m,p_2) = 1$ and hence $(u,p_2) = 1$. If possible, let $u = p_1^a t$ where $(t,p_1) = 1$. If $a < a_1$, then $(u,q) = p_1^{t_1}p_2^{t_2}$ where $t_1 \neq a_1$ and $t_2 \neq a_2$. Thus, by our first case, we can write $\log 2\sin u\pi/q$ in the desired form. If $a > a_1$, then $(u,q) = p_1^{a_1}$, and thus, again by our earlier case, we get the desired form. Similarly, we can prove the case when $b = p_1^r p_2^{a_2}$ where $0 < r < a_1$. This completes the proof. ■

References

[1] A. Baker, *Transcendental number theory*. Cambridge University Press, London–New York, 1975.

[2] H. Bass, *Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups*. Topology 4(1965), 391–410. https://doi.org/10.1016/0040-9383(66)90036-X

[3] T. Chatterjee and S. Dhillon, *Linear independence of harmonic numbers over the field of algebraic numbers*. Ramanujan J. (2019). https://doi.org/10.1007/s11139-018-0124-6

[4] T. Chatterjee and S. Dhillon, *Linear independence of harmonic numbers over the field of algebraic numbers II*. Ramanujan J. https://doi.org/10.1007/s11139-019-00183-8

[5] T. Chatterjee and S. Gun, *The digamma function, Euler–Lehmer constants and their p-adic counterparts*. Acta Arith. 162(2014), 197–208. https://doi.org/10.4064/aa162-2-4

[6] T. Chatterjee and M. R. Murty, *Non-vanishing of Dirichlet series with periodic coefficients*. J. Number Theory 145(2014), 1–21. https://doi.org/10.1016/j.jnt.2014.05.010

[7] T. Chatterjee, M. R. Murty, and S. Pathak, *A vanishing criterion for Dirichlet series with periodic coefficients*. In: *Number theory related to modular curves—Momose memorial volume*, Contemp. Math., 701, American Mathematical Society, Providence, RI, 2018, pp. 69–80. https://doi.org/10.1090/conm/701/14141

[8] V. Ennola, *On relations between cyclotomic units*. J. Number Theory 4(1972), 236–247. https://doi.org/10.1016/0022-314X(72)90050-9

[9] A. Livingston, *The series $\sum_{n=1}^{\infty} f(n)/n$ for periodic f*. Canad. Math. Bull. 8(1965), 413–432. https://doi.org/10.4153/CMB-1965-029-2

[10] M. R. Murty and N. Saradha, *Transcendental values of the p-adic digamma function*. Acta Arith. 133(2008), 349–362. https://doi.org/10.4064/aa133-4-4

[11] M. R. Murty and N. Saradha, *Euler–Lehmer constants and a conjecture of Erdős*. J. Number Theory 130(2010), 2671–2682. https://doi.org/10.1016/j.jnt.2010.07.004

[12] S. Pathak, *On a conjecture of Livingston*. Canad. Math. Bull. 60(2017), 184–195. https://doi.org/10.4153/CMB-2016-065-1

[13] D. Y. Pei and K. Feng, *A note on the independence of the cyclotomic units*. Acta Math. Sinica 23(1980), 773–778.

[14] K. Ramachandra, *On the units of cyclotomic fields*. Acta Arith. 12(1966/67), 165–173. https://doi.org/10.4064/aa-12-2-165-173

[15] L. C. Washington, *Introduction to cyclotomic fields*, Second ed., Graduate Texts in Mathematics, 83, Springer, New York, 1997. https://doi.org/10.1007/978-1-4612-1934-7

Department of Mathematics, Indian Institute of Technology Ropar, Punjab, India
e-mail: tapasc@iitrpr.ac.in sonika@iitrpr.ac.in