LACUNARITY OF DEDEKIND η-PRODUCTS

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Dedicated to the memory of Kim Hughes

1. Introduction. The Dedekind η-function is defined by

$$\eta(\tau) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n),$$

where τ lies in the upper half plane $\mathcal{H} = \{\tau \mid \operatorname{Im}(\tau) > 0\}$, and $x = e^{2\pi i \tau}$. It is a modular form of weight $\frac{1}{2}$ with a multiplier system. We define an η-product to be a function $f(\tau)$ of the form

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta}, \tag{1}$$

where $r_\delta \in \mathbb{Z}$. This is a modular form of weight $k = \frac{1}{2} \sum_{\delta \mid N} r_\delta$ with a multiplier system. The Fourier coefficients of η-products are related to many well-known number-theoretic functions, including partition functions and quadratic form representation numbers. They also arise from representations of the "monster" group [**3**] and the Mathieu group M_{24} [**13**]. The multiplicative structure of these Fourier coefficients has been extensively studied. Recent papers include [**1**], [**4**], [**5**] and [**6**]. Here we study the connections between the density of the non-zero Fourier coefficients of $f(\tau)$ and the representability of $f(\tau)$ as a linear combination of Hecke character forms (defined in Section 4 below). We first make the following definition.

Definition. A power series is called *lacunary* if the arithmetic density of its non-zero coefficients is zero. More precisely, the series $x^\nu \sum_{n=0}^{\infty} c(n)x^n$ is lacunary if

$$\lim_{t\to\infty} \frac{\operatorname{card}\{n \mid n \le t \text{ and } c(n) \ne 0\}}{t} = 0.$$

Serre [**17**] has determined all the even integers r for which $\eta(\tau)^r$ is lacunary. The result is as follows.

Theorem 1. (Serre). *Suppose $r > 0$ is even. Then $\eta(\tau)^r$ is lacunary if and only if* $r = 2, 4, 6, 8, 10, 14$ *or* 26.

We will extend Theorem 1 to the η-products $\eta(\tau)^r\eta(2\tau)^s$ $(r, s \in \mathbb{Z})$, a reasonable next case in view of the fact that powers of the classical theta-function

$$\theta(-x) = \theta_3(2\tau + 1) = \sum_{-\infty}^{\infty} (-x)^{n^2} = \prod_{n=1}^{\infty} (1 - x^n)^2(1 - x^{2n})^{-1} = \eta(\tau)^2\eta(2\tau)^{-1}$$

and many partition functions are of this type. Our main result is the following.

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THEOREM 2. *Suppose that $r+s$ is even and $rs \neq 0$. Then $\eta(\tau)^r\eta(2\tau)^s$ is lacunary if and only if (r,s) is one of the following 45 pairs:*

$k=1$: $(1,1)$ $(3,-1)$ $(-1,3)$ $(4,-2)$ $(-2,4)$

$k=2$: $(2,2)$ $(3,1)$ $(1,3)$ $(5,-1)$ $(-1,5)$ $(6,-2)$ $(-2,6)$ $(7,-3)$ $(-3,7)$

$k=3$: $(3,3)$ $(4,2)$ $(2,4)$ $(5,1)$ $(1,5)$ $(7,-1)$ $(-1,7)$ $(8,-2)$ $(-2,8)$ $(9,-3)$ $(-3,9)$ $(10,-4)$ $(-4,10)$ $(11,-5)$ $(-5,11)$

$k=5$: $(5,5)$ $(7,3)$ $(3,7)$ $(14,-4)$ $(-4,14)$ $(15,-5)$ $(-5,15)$ $(16,-6)$ $(-6,16)$ $(17,-7)$ $(-7,17)$ $(18,-8)$ $(-8,18)$ $(19,-9)$ $(-9,19)$

$k=9$: $(9,9)$.

If (r,s) is in this list, so is (s,r). This fact emerges upon applying the canonical involution $\tau \to -1/(N\tau)$ to the Riemann surface $X_0(N)$ of $\Gamma_0(N)$.

In a later paper we will obtain the analogue of Theorem 3 for the forms $\eta(\tau)^r\eta(q\tau)^s$ with q an odd prime ≤ 23, and also for the forms $\eta(\tau)^r\eta(2\tau)^s\eta(4\tau)^t$. In principle the same methods can be used to determine all lacunary η-products (1) for any given N.

2. Reduction of the problem. We begin by recalling some results from [**6**]. Suppose that the weight $k=\frac{1}{2}\sum_{\delta|N} r_\delta$ is an integer. Put

$$\prod_{\delta|N} \delta^{r_\delta} = \Delta, \tag{2}$$

$$\frac{1}{24}\sum_{\delta|N} \delta r_\delta = \frac{c}{e}, \tag{3}$$

$$\frac{1}{24}\sum_{\delta|N} \frac{N}{\delta} r_\delta = \frac{c_0}{e_0}, \tag{4}$$

where the fractions c/e and c_0/e_0 are in lowest terms. Put $M = Nee_0$ and let ε be the Dirichlet character (mod M) defined by $\varepsilon(p) = \left(\frac{(-1)^k\Delta}{p}\right)$ for primes p not dividing M. It is known ([**6**, p. 174]) that if $f(\tau)$ is the η-product (1), then $F(\tau) = f(e\tau)$ is in the vector space $\mathcal{M}(\Gamma_0(M), k, \varepsilon)$ of modular forms on $\Gamma_0(M)$ with weight k and Nebentypus ε, holomorphic in $\mathcal{H}$ and meromorphic at the cusps of $X_0(M)$. These cusps can be represented by rational numbers $\kappa = \frac{\lambda}{\mu}$, where $\mu > 0$, $\mu \mid M$ and $(\lambda,\mu)=1$. The order of $F(\tau)$ at the cusp κ is

$$\operatorname{ord}_\kappa(f) = \frac{M}{24\left(\frac{M}{\mu},\mu\right)} \sum_{\delta|M} \frac{(\delta,\mu)^2}{\delta\mu} r_\delta. \tag{5}$$

Therefore $F(\tau)$ belongs to the subspace $\mathcal{S}(\Gamma_0(M), k, \varepsilon)$ of cusp forms in $\mathcal{M}(\Gamma_0(M), k, \varepsilon)$ if and only if the sums in (5) are all positive.

In this paper we are concerned with the case $N=2$, $r_1=r$ and $r_2=s$. We then have $k=\frac{1}{2}(r+s)$, so our assumption that $r+s$ is even amounts to requiring that if $f_{r,s}(\tau)=\eta(\tau)^r\eta(2\tau)^s$, the corresponding form $F_{r,s}(\tau)=f_{r,s}(e\tau)$ on $\Gamma_0(M)$ has integral weight. Since e and e_0 are divisors of 24, $M=2ee_0$ is of the form $2^\alpha 3^\beta$. Moreover $\Delta=2^s$ and $\varepsilon(p)=\left(\frac{(-1)^k 2^s}{p}\right)$ for $p\nmid M$. Using (5), we find that $F_{r,s}(\tau)\in\mathcal{S}(\Gamma_0(M),\,k,\,\varepsilon)$ if and only if

$$2r+s>0,\qquad r+2s>0. \tag{6}$$

The proof of Theorem 2 now breaks down into three parts. In Section 3 we show that if $F_{r,s}(\tau)$ is lacunary but not a cusp form, then $(r,s)=(4,-2)$ or $(-2,4)$. In Section 4 we show that if $F_{r,s}(\tau)$ is a lacunary cusp form, then (r,s) must be one of the remaining 43 pairs in the statement of Theorem 2. Finally, in Section 5 we show that $F_{r,s}$ is indeed lacunary for all these pairs (r,s).

3. Lacunary non-cusp forms. We now consider the case where one of the inequalities (6) fails to hold. We continue to assume that $r+s$ is even and $rs\neq 0$. For convenience, put $(r,s)=\eta(\tau)^r\eta(2\tau)^s$. It should be clear from context whether the symbol (r,s) is being used to denote a lattice point or the corresponding η-product. Clearly the lacunarity of a series $f(\tau)$ is preserved if τ is replaced by $\tau+\frac{1}{2}$, or equivalently if x is replaced by $-x$. Let $(r,s)^*=\eta(\tau+\frac{1}{2})^r\eta(2\tau)^s$ denote the image of (r,s) under this replacement. We will make use of the classical identities

$$G(x)=x^{-1/8}(-1,2)=\prod_{m=1}^{\infty}(1-x^m)^{-1}(1-x^{2m})^2=\sum_{n=0}^{\infty}x^{(n^2+n)/2},$$

$$\theta(x)=(2,-1)^*=\prod_{m=1}^{\infty}(1+x^{2m-1})^2(1-x^{2m})=\sum_{n=-\infty}^{\infty}x^{n^2}.$$

We also require the functions

$$P(x)=x^{1/24}(-1,0),$$

$$Q(x)=x^{-1/24}(-1,1)=\prod_{m=1}^{\infty}(1+x^m),$$

$$Q_0(x)=x^{1/24}(1,-1)^*=Q(-x)^{-1}=\prod_{m=1}^{\infty}(1+x^{2m-1}).$$

The Fourier expansions of these functions are

$$P(x)=\sum_{n=0}^{\infty}p(n)x^n,\qquad Q(x)=\sum_{n=0}^{\infty}q(n)x^n,\qquad Q_0(x)=\sum_{n=0}^{\infty}q_0(n)x^n,$$

where $p(n)$ is the partition function, $q(n)$ is the number of partitions of n into distinct parts and $q_0(n)$ is the number of partitions of n into distinct odd parts. Clearly $p(n)$, $q(n)$ and $q_0(n)$ tend to infinity with n. Therefore every non-constant function

$$(0,-a)(-b,b)(c,-c)^*=x^{(2a-b-c)/24}P(x^2)^aQ(x)^bQ_0(x)^c$$

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where a, b, $c \geq 0$, is non lacunary. Moreover $(-d, 2d)$ and $(2d, -d)^*$ are lacunary for $d = 2$ [**11**], but not for $d > 2$, since every positive integer n is the sum of three triangular numbers, and is also the sum of three squares unless $n = 4^{\alpha}(8\beta + 7)$.

To show that (r, s) is nonlacunary when (6) does not hold, we suppose first that $r + s \leq 0$. If $r < 0$, the equation

$$(r, s) = (r, -r)(0, r + s)$$

shows that (r, s) is nonlacunary, while if $r > 0$, the equation

$$(r, s)^* = (r, -r)^*(0, r + s)$$

implies the nonlacunarity of $(r, s)^*$, hence that of (r, s).

We may therefore suppose henceforth that $r + s > 0$. If $2r + s \leq 0$, we write

$$(r, s) = (2r + s, -2r - s)(-r - s, 2r + 2s).$$

By the above remarks, this is lacunary if and only if $2r + s = 0$ and $r + s = 2$, i.e. $(r, s) = (-2, 4)$. If $r + 2s \leq 0$, we write

$$(r, s)^* = (-r - 2s, r + 2s)^*(2r + 2s, -r - s)^*.$$

This is lacunary if and only if $r + 2s = 0$ and $r + s = 2$, i.e. $(r, s) = (4, -2)$.

4. Lacunary cusp forms. In this section we consider the case where the inequalities (6) hold, i.e. $F_{r,s}(\tau) \in \mathscr{S}(\Gamma_0, k, \varepsilon)$. It is known [**17**] that all forms in $\mathscr{S}(\Gamma_0, 1, \varepsilon)$ are lacunary, so we assume henceforth that $k > 1$. To obtain a useful criterion for lacunarity when $k > 1$, we introduce the class of *Hecke character forms*, defined as follows. Let K be a number field, O_K its ring of integers and $\mathfrak{f}$ an ideal of O_K. A Hecke character (=Grössencharacter) (mod $\mathfrak{f}$) of exponent $k - 1$ is a homomorphism of the group $I(\mathfrak{f})$ of fractional ideals prime to $\mathfrak{f}$ into $\mathbb{C}$ such that $c(\mathfrak{a}) = \alpha^{k-1}$ for principal ideals $\mathfrak{a} = (\alpha)$ with α totally positive and $\alpha \equiv 1$ (mod$^{\times}$ $\mathfrak{f}$). As with Dirichlet characters, two Hecke characters $c_1(\mathfrak{a})$ (mod $\mathfrak{m}_1$) and $c_2(\mathfrak{a})$ (mod $\mathfrak{m}_2$) can be regarded as equal if they agree on $I(\mathfrak{m}_1\mathfrak{m}_2)$. From this point of view, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are just two different "definition moduli" for the same Hecke character $c_1(\mathfrak{a}) = c_2(\mathfrak{a})$ (mod $\mathfrak{m}_1\mathfrak{m}_2$). Every Hecke character $c(\mathfrak{a})$ has a (multiplicatively) smallest definition modulus $\mathfrak{f} = \mathfrak{f}(c)$, called its conductor.

Now suppose that K is a quadratic imaginary field of discriminant d, $\mathfrak{m}$ an ideal of O_K, $c(\mathfrak{a})$ a Hecke character (mod $\mathfrak{m}$) and δ a positive integer. Put

$$\phi_{K,c,\delta}(\tau) = \phi_{K,c}(\delta\tau) = \sum_{(\mathfrak{a},\mathfrak{m})=1} c(\mathfrak{a})x^{\delta N(\mathfrak{a})},$$

where the sum is over all integral ideals $\mathfrak{a}$ prime to $\mathfrak{m}$, and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$. Hecke and Shimura have shown that if M is any multiple of $\delta\,|d|\,N(\mathfrak{m})$, then $\phi_{K,c,\delta}(\tau)$ is in $\mathscr{S}(\Gamma_0(M), k, \varepsilon_c)$, where

$$\varepsilon_c(p) = \left(\frac{d}{p}\right)\frac{c((p))}{p^{k-1}}$$

for all primes $p \nmid M$.

For a given $k \geq 2$, M and Dirichlet character ε (mod M), the forms $\phi_{K,c,\delta}(\tau)$ with

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$\delta \,|d|\, N(\mathfrak{m}) \mid M$ and $\varepsilon_c = \varepsilon$ span a subspace $\mathcal{S}_{cm}(\Gamma_0(M), k, \varepsilon)$ of $\mathcal{S}(\Gamma_0(M), k, \varepsilon)$. The elements of $\mathcal{S}_{cm}(\Gamma_0(M), k, \varepsilon)$ are called CM-*forms*. For convenience we recall the following theorem of Serre [**16**].

THEOREM 3. *Suppose* $F(\tau) = \sum_{n=1}^{\infty} c(n)x^n \in \mathcal{S}(\Gamma_0(M), k, \varepsilon)$, *with* $k \geq 2$, *and put*

$$M_f(t) := \operatorname{card}\{n \mid 0 \leq n \leq t \text{ and } c(n) \neq 0\}.$$

(i) *If* $F(\tau) \notin \mathcal{S}_{cm}(\Gamma_0(M), k, \varepsilon)$, *then* $M_f(t) \asymp t$ *for* $t \to \infty$.

(ii) *If* $F(\tau) \in \mathcal{S}_{cm}(\Gamma_0(M), k, \varepsilon)$ *and* $F(\tau) \neq 0$, *then* $M_f(t) \asymp t/(\log t)^{1/2}$ *for* $t \to \infty$, *where* $\phi(t) \asymp \psi(t)$ *means that* $\phi(t) = O(\psi(t))$ *and* $\psi(t) = O(\phi(t))$.

Thus $F(\tau)$ is lacunary if and only if it is a CM-form. We will also make use of the following theorem of Ribet [**14**, p. 35].

THEOREM 4. *If p is inert in the imaginary quadratic field K, then* $\phi_{K,c,\delta}(\tau) \mid T_p = 0$.

Hence the CM-form $F(\tau) = \sum_{\nu} \alpha_\nu \phi_{K_\nu, c_\nu, \delta_\nu}$ is annihilated by T_p if p is inert in all the fields K_ν.

We now require some further notation. Define the Fourier coefficients $a_{r,s}(n)$ by:

$$\eta(\tau)^r \eta(2\tau)^s = x^{(r+2s)/24} \prod_{n=1}^{\infty} (1 - x^n)^r (1 - x^{2n})^s$$

$$= x^{(r+2s)/24} \sum_{n=0}^{\infty} a_{r,s}(n)x^n.$$

Let

$$\frac{r+2s}{24} = \frac{c_{r,s}}{e_{r,s}} \tag{7}$$

in lowest terms. Recall the notation $f_{r,s}(\tau) = \eta(\tau)^r \eta(2\tau)^s$ and $F_{r,s}(\tau) = f(e_{r,s}\tau)$. Then

$$F_{r,s}(\tau) = \sum_{n=0}^{\infty} a_{r,s}(n)x^{c+en} = \sum_{n=0}^{\infty} b_{r,s}(n)x^n, \tag{8}$$

say. We write a, b, c, e, f and F instead of $a_{r,s}$, $b_{r,s}$, $c_{r,s}$, $e_{r,s}$, $f_{r,s}$ and $F_{r,s}$ if the subscripts are clear from context.

If $F(\tau)$ is lacunary, then by Theorem 3 it is a CM-form:

$$F(\tau) = \sum_{\nu} \alpha_\nu \phi_{K_\nu, c_\nu, \delta_\nu}.$$

As remarked in Section 2, $M = 2ee_0$ is of the form $2^\alpha 3^\beta$. Since the discriminant d_ν of K_ν divides M, the only possibilities for d_ν are -3, -4, -8 or -24, giving $K_\nu = \mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-6})$ respectively. Every prime $p \equiv 23 \pmod{24}$ is inert in all four of these fields. This proves the following result.

LEMMA 1. *If $F_{r,s}(\tau)$ is a cusp form of integral weight $k \geq 2$ and is lacunary, then* $F_{r,s}(\tau) \mid T_p = 0$ *for all* $p \equiv 23 \pmod{24}$.

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LEMMA 2. *Suppose* $f(\tau) \mid T_{23} = 0$.

(i) *If* $r + 2s \geq 3$, *then* $a_{r,s}(m) = a_{r,s}(m + 23) = 0$, *where* $1 \leq m \leq 20$ *and* $m \equiv -(r + 2s) \pmod{23}$.

(ii) *If* $r + 2s = 2$, *then* $a_{r,s}(21) = 0$.

(iii) *If* $r + 2s = 1$, *then* $a_{r,s}(45) = 0$.

Proof. (i) Let $G(\tau) = F(24\tau)$. Then $G(\tau)$ is on $\Gamma_0(24M)$ and is lacunary if and only if $f(\tau)$ is lacunary. We apply the Hecke operator $T_{23} = U_{23} + V_{23}$ to $G(\tau)$:

$$G(\tau) \mid T_{23} = \sum_{r+2s+24n \equiv 0 \pmod{23}} a_{r,s}(n) x^{(r+2s+24n)/23} + \varepsilon(23) 23^{k-1} \sum_{n=0}^{\infty} a_{r,s}(n) x^{23(r+2s+24n)}.$$

The lowest term in $G(\tau) \mid U_{23}$ is $a_{r,s}(m) x^{(r+2s+24m)/23}$, where m is the least non-negative integer such that $r + 2s + 24m \equiv 0 \pmod{23}$. We have $m \equiv -(r + 2s) \pmod{23}$ and $0 \leq m \leq 22$. Since $r + 2s \geq 3$, we have $m \leq 20$, and

$$\frac{r + 2s + 24(m + 23)}{23} \leq \frac{r + 2s + 1032}{23} < 23(r + 2s).$$

Thus the first two terms in $G(\tau) \mid U_{23}$ appear before the first term in $G(\tau) \mid V_{23}$, proving (i).

(ii) If $r + 2s = 2$, then $e = 12$ and $m \equiv -2 \pmod{23}$, giving $m = 21$. Therefore

$$G(\tau) \mid T_{23} = a_{r,s}(21) x^{22} + (a_{r,s}(44) + \varepsilon(23) 23^{k-1}) x^{46} + \dots,$$

which proves (ii).

(iii) If $r + 2s = 1$, then $e = 24$ and $m \equiv -1 \pmod{23}$, giving $m = 22$. Therefore

$$G(\tau) \mid T_{23} = (a_{r,s}(22) + \varepsilon(23) 23^{k-1}) x^{23} + a_{r,s}(45) x^{47} + \dots,$$

which proves (iii). □

Define

$$a^*_{r,s}(m) = \begin{cases} a_{r,s}(m) & \text{if } m \text{ is even} \\ a_{r,s}(m)/r & \text{if } m \text{ is odd.} \end{cases}$$

Using Maple, it can be shown that the coefficients $a^*_{r,s}(m)$, $0 \leq m \leq 45$ are irreducible polynomials in r and s. Hence the algebraic curves $\mathscr{C}_m$, $0 \leq m \leq 45$ defined by

$$\mathscr{C}_m := \{(r, s) \in \mathbb{C}^2 \mid a^*_{r,s}(m) = 0\}$$

are also irreducible. The first few polynomials $a_{r,s}(m)$ are as follows.

$$a_{r,s}(0) = 1.$$

$$a_{r,s}(1) = -r.$$

$$2!\, a_{r,s}(2) = r^2 - 3r - 2s.$$

$$3!\, a_{r,s}(3) = 9r^2 - 8r - r^3 + 6rs.$$

$$4!\, a_{r,s}(4) = 36rs - 12r^2 s - 36s + 12s^2 - 18r^3 + 59r^2 - 42r + r^4.$$

$$5!\, a_{r,s}(5) = 340rs - 60rs^2 + 30r^4 - 215r^3 + 450r^2 - 144r - r^5 - 180r^2 s + 20r^3 s.$$

The remaining polynomials $a_{r,s}(m)$, $6 \leq m \leq 45$, are quite cumbersome to write down and we omit them.

We can now combine Lemmas 1 and 2 to obtain the following result.

LEMMA 3. *Suppose $\eta(\tau)^r\eta(2\tau)^s$ is lacunary.*

(i) *If $r+2s\geq 3$, then (r,s) is in the intersection of the curves $\mathscr{C}_m$ and $\mathscr{C}_{m+23}$ for some m with $0\leq m\leq 20$.*

(ii) *If $r+2s=2$, then (r,s) is on the curve $a^*_{2-2s,s}(21)=0$.*

(iii) *If $r+2s=1$, then (r,s) is on the curve $a^*_{1-2s,s}(45)=0$.* □

Since the curves $\mathscr{C}_m$, $2\leq m\leq 45$, are irreducible and distinct, Bezout's theorem [**7**] can be applied to show that there are only finitely many points satisfying Lemma 3. In fact we can explicitly find these points using resultants. This reduces the possible pairs (r,s) to the list given in Theorem 2. To prove that for these pairs the forms $\eta(\tau)^r\eta(2\tau)^s$ are indeed lacunary, we exhibit them as linear combinations of Hecke character forms in the next section.

5. Hecke character forms. In this section we show that $F_{r,s}(\tau)$ is indeed lacunary for the 45 pairs (r,s) listed in Theorem 2. For notational convenience, put

$$[r,s]=F_{r,s}(\tau)=\eta(e\tau)^r\eta(2e\tau)^s.$$

For example, $[1,1]=\eta(8\tau)\eta(16\tau)$. We extend this notation by putting

$$[r,s,t]=\eta(e\tau)^r\eta(2e\tau)^s\eta(4e\tau)^t,$$

where $e=24/\gcd(r+2s+4t,24)$.

Let K be an imaginary quadratic field with ring of integers O_K, and $\mathfrak{m}$ an ideal of O_K. Let $R(\mathfrak{m})$ be the group of reduced residue classes (mod $\mathfrak{m}$). For simplicity of notation we let α denote the residue class $\alpha+\mathfrak{m}$ when working in $R(\mathfrak{m})$. Let $G(\mathfrak{m})$ be the multiplicative group of all $\alpha\in K^*$ prime to $\mathfrak{m}$ and $I(\mathfrak{m})$ the group of fractional ideals prime to $\mathfrak{m}$. A general way to construct a Hecke character $c(\mathfrak{a})$ (mod $\mathfrak{m}$) with exponent $k-1$ is to start with an ordinary character $\chi(\alpha)$ of $R(\mathfrak{f})$, lift it to a character $s(\alpha)$ of $G(\mathfrak{f})$ and then define $c(\mathfrak{a})=s(\alpha)\alpha^{k-1}$ for principal ideals $\mathfrak{a}=(\alpha)$. For this definition to be independent of the particular generator α of $\mathfrak{a}$, it is necessary and sufficient that $s(\varepsilon)\varepsilon^{k-1}=1$ for the units ε of O_K. The extension of $c(\mathfrak{a})$ to non-principal ideals is carried out using the structure of the ideal class group of K. In the present situation, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ have class number 1. However $\mathbb{Q}(\sqrt{-6})$ has class number 2, so once a Hecke character $c(\mathfrak{a})$ has been defined for its principal ideals $\mathfrak{a}$, there are two extensions to the non-principal ideals.

If $\chi(\alpha)$ is a primitive character of $R(\mathfrak{f})$, the associated Hecke characters $c(\mathfrak{a})$ of exponent $k-1$ have conductor $\mathfrak{f}$. Most of the examples in this section are of this type.

[1, 1]: Take $K=\mathbb{Q}(i)$ and $\mathfrak{f}=((1+i)^5)$. The group $R(\mathfrak{f})$ is the direct product $\langle i\rangle\times\langle 1+2i\rangle\times\langle 1+4i\rangle$, where the generators i, $1+2i$, $1+4i$ have orders 4, 2, 2 respectively. Define two characters $\chi_\pm$ of $R(\mathfrak{f})$ by putting

$$\chi_\pm(i)=1,\qquad \chi_\pm(1+2i)=\pm 1,\qquad \chi_\pm(1+4i)=1. \tag{13}$$

Starting with these characters of $R(\mathfrak{f})$, construct Hecke characters $c_\pm(\mathfrak{a})$ of exponent 0 as explained above. It turns out that

$$\phi_{K,c}(\tau)=[1,1].$$

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(Verification of this and all similar equations below is carried out by comparing enough coefficients to exceed the dimension of the relevant vector space of forms.)

[2, 2]: Take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = ((1+i)^3)$. Then $R(\mathfrak{f}) = \langle i \rangle$, a cyclic group of order 4. Define the character χ of $R(\mathfrak{a})$ by putting $\chi(i) = -i$. Let $c(\mathfrak{a})$ be the corresponding Hecke character of exponent 1. Then

$$\phi_{K,c}(\tau) = [2, 2].$$

[3, 3]: Take $K = \mathbb{Q}(\sqrt{-2})$ and $\mathfrak{f} = (4)$. The group of reduced residues mod 4 is the direct product $\langle -1 \rangle \times \langle 1 + i\sqrt{2} \rangle$; the generators have orders 2 and 4 respectively. Define two characters $\chi_{\pm}(\alpha)$ of $R(\mathfrak{f})$ by putting

$$\chi_{\pm}(-1) = -1, \qquad \chi_{\pm}(1 + i\sqrt{2}) = \pm i. \tag{14}$$

Let $c_{\pm}(\mathfrak{a})$ be the corresponding Hecke characters of exponent 2. Then

$$\phi_{K,c_{\pm}}(\tau) = [9, -3] + 32[1, -3, 8] \mp 4\sqrt{2}[3, 3].$$

[5, 5]: Take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = ((1+i)^5)$. Let $\chi_{\pm}(\alpha)$ be the characters (13), and $c_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of exponent 4; thus $c_{\pm}(\mathfrak{a}) = s_{\pm}(\alpha)\alpha^4$, where $\mathfrak{a} = (\alpha)$. Then

$$\phi_{K,c_{\pm}}(\tau) = [17, -7] - 64[-7, 17] \mp 48i[5, 5].$$

[9, 9]: This is a linear combination of four Hecke forms, arising in pairs from the fields $K = \mathbb{Q}(i)$ and $L = \mathbb{Q}(\sqrt{-2})$. Put

$$A(\tau) = [9, 9],$$

$$B(\tau) = [17, -9, 8] + 2^8[17, -15, 16] + 2^{12}[-15, 33],$$

$$C(\tau) = [21, -3] - 2^6[-3, 21],$$

$$D(\tau) = [27, -9] + 3 \cdot 2^5[19, -9, 8] + 3 \cdot 2^{10}[11, -9, 16] + 2^{15}[3, -9, 24].$$

It turns out that $F_{\pm}(\tau) = -6544A(\tau) + B(\tau) \pm 672C(\tau)$ and $G_{\pm}(\tau) = 18544A(\tau) + B(\tau) \pm 112\sqrt{2}D(\tau)$ are Hecke forms. Hence $A(\tau) = (50176)^{-1}\{G_+(\tau) + G_-(\tau) - F_+(\tau) - F_-(\tau)\}$ is lacunary.

To express $F_{\pm}(\tau)$ as Hecke forms, take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = ((1+i)^5)$. Let $\chi_{\pm}(\alpha)$ be the characters (13), and $c_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of exponent 8. Then

$$\phi_{K,c_{\pm}}(\tau) = F_{\pm}(\tau).$$

To express $G_{\pm}(\tau)$ as Hecke forms, take $L = \mathbb{Q}(\sqrt{-2})$ and $\mathfrak{f} = (4)$. Let $\chi'_{\pm}(\alpha)$ be the characters of $R(\mathfrak{f})$ defined by (14), and $c'_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of

$$\phi_{L,c'_{\pm}}(\tau) = G_{\pm}(\tau).$$

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[3, 1], [1, 3], [7, −3], and [−3, 7]: Take $K=\mathbb{Q}(\sqrt{-6})$ and $\mathfrak{f}=(4)\mathfrak{p}$, where $\mathfrak{p}^2=(3)$. (K has class number 2 and $\mathfrak{p}$ is non-principal.) The group $R(\mathfrak{f})$ is the direct product $\langle 5\rangle\times\langle 7\rangle\times\langle 1+i\sqrt{6}\rangle$; the generators $5, 7, 1+i\sqrt{6}$ have orders 2, 2, 4 respectively. Define two characters $\chi_{\pm}(\alpha)$ of $R(\mathfrak{f})$ by:

$$\chi_{\pm}(5)=-1,\qquad \chi_{\pm}(7)=1,\qquad \chi_{\pm}(1+i\sqrt{6})=\pm i. \tag{15}$$

As explained above, $\chi_{+}(\alpha)$ gives rise to two Hecke characters $c_{+,\pm}(\mathfrak{a})$ of exponent 1 and conductor $\mathfrak{f}$. Similarly, $\chi_{-}(\alpha)$ gives rise to two Hecke characters $c_{-,\pm}(\mathfrak{a})$. It turns out that

$$\phi_{K,c_{+,\pm}}(\tau)=[7,-3]-2\sqrt{6}[1,3]\pm 2i\sqrt{3}[3,1]\mp 4i\sqrt{2}[-3,7],$$
$$\phi_{K,c_{-,\pm}}(\tau)=[7,-3]+2\sqrt{6}[1,3]\pm 2i\sqrt{3}[3,1]\pm 4i\sqrt{2}[-3,7].$$

[5, 1] and [1, 5]: As in the previous case, take $K=\mathbb{Q}(\sqrt{-6})$ and $\mathfrak{f}=(4)\mathfrak{p}$, where $\mathfrak{p}^2=(3)$. Define characters $\chi_{\pm}(\alpha)$ of $R(\mathfrak{f})$ by putting

$$\chi_{\pm}(5)=\chi_{\pm}(7)=-1,\qquad \chi_{\pm}(1+i\sqrt{6})=\pm 1.$$

These give rise to Hecke characters $c_{+,\pm}(\mathfrak{a})$ and $c_{-,\pm}(\mathfrak{a})$ with conductor $\mathfrak{f}$ and exponent 2. Then

$$\phi_{K,c_{+,\pm}}(\tau)=[11,-5]+32[3,-5,8]\pm 2i\{[7,-1]+32[-1,-1,8]\}$$
$$+4i\sqrt{6}[5,1]\pm 8\sqrt{6}[1,5],$$
$$\phi_{K,c_{-,\pm}}(\tau)=[11,-5]+32[3,-5,8]\pm 2i\{[7,-1]+32[-1,-1,8]\}$$
$$-4i\sqrt{6}[5,1]\mp 8\sqrt{6}[1,5].$$

[3, −1] and [−1, 3]: Take $K=\mathbb{Q}(i)$ and $\mathfrak{f}=(3(1+i)^5)$. If $(\alpha,\mathfrak{f})=1$, then

$$\alpha\equiv i^a(1+2i)^b(1+4i)^c \pmod{(1+i)^5}$$
$$\alpha\equiv(1-i)^d \pmod{3},$$

where a is mod 4, b and c are mod 2 and d is mod 8. Define characters $\chi_{+,\pm}(\alpha)$ and $\chi_{-,\pm}(\alpha)$ of $R(\mathfrak{f})$ by putting

$$\chi_{+,\pm}(\alpha)=(-1)^{a+c}(\pm i)^d,$$
$$\chi_{-,\pm}(\alpha)=(-1)^{a+b+c}(\pm i)^d.$$

These give rise to Hecke characters $c_{+,\pm}(\mathfrak{a})$ and $c_{-,\pm}(\mathfrak{a})$ with conductor $\mathfrak{f}$ and exponent 0. We have

$$\phi_{K,c_{+,\pm}}(\tau)=\phi_{K,c_{-,\mp}}(\tau)=[3,-1]\pm 2i[-1,3].$$

[7, 3] and [3, 7]: Take $K=\mathbb{Q}(i)$ and $\mathfrak{f}=(3(1+i)^5)$. Let $\chi_{+,\pm}(\alpha)$ and $\chi_{-,\pm}(\alpha)$ be the characters defined in the previous case, and let $c_{+,\pm}(\mathfrak{a})$ and $c_{-,\pm}(\mathfrak{a})$ be the corresponding Hecke characters of exponent 3. Then

$$\phi_{K,c_{+,\pm}}(\tau)=[19,-9]+448[-5,15]\pm 2i\{7[15,-5]+64[-9,19]\}$$
$$-240i[7,3]\pm 480[3,7],$$
$$\phi_{K,c_{-,\pm}}(\tau)=[19,-9]+448[-5,15]\mp 2i\{7[15,-5]+64[-9,19]\}$$
$$+240i[7,3]\mp 480[3,7].$$

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[7, −1], [−1, 7], [11, −5] and [−5, 11]: Take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = (3(1+i)^5)$. Let $\chi_{+,\pm}(\alpha)$ and $\chi_{-,\pm}(\alpha)$ be the characters of $R(\mathfrak{f})$ used in the previous two cases, and let $c_{+,\pm}(\mathfrak{a})$ and $c_{-,\pm}(\mathfrak{a})$ be the corresponding Hecke characters of exponent 2. Then

$$\phi_{K,c_{-,\pm}}(\tau) = [11, -5] \pm 6i[7, -1] + 24i[-1, 7] \pm [-5, 11],$$
$$\phi_{K,c_{+,\pm}}(\tau) = [11, -5] \mp 6i[7, -1] - 24i[-1, 7] \mp [-5, 11].$$

[5, −1] and [−1, 5]: Take $K = \mathbb{Q}(\sqrt{-2})$ and $\mathfrak{f} = (4)$. Then $R(\mathfrak{f}) = \langle -1 \rangle \times \langle 1 + i\sqrt{2} \rangle$, where the generators have orders 2 and 4 respectively. Define characters $\chi_\pm(\alpha)$ of $R(\mathfrak{f})$ by putting $\chi_\pm(-1) = -1$, $\chi_\pm(1 + i\sqrt{2}) = \pm 1$, and let $c_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 1. Then

$$\phi_{K,c_\pm}(\tau) = [5, -1] \pm 2i[-1, 5].$$

[4, 2] and [−4, 10]: Take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = (3)$. Then $R(\mathfrak{f}) = \langle 1 - i \rangle$ is cyclic of order 8. Define two characters $\chi_\pm(\alpha)$ of $R(\mathfrak{f})$ by putting $\chi_\pm(1 - i) = \pm i$, and let $c_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 2. Then

$$\phi_{K,c_\pm}(\tau) = [4, 2] \pm 2[-4, 10].$$

[2, 4] and [10, −4]: Take $K = \mathbb{Q}(i)$, and $\mathfrak{m} = (3(1+i))$. Let $c_\pm(\mathfrak{a})$ be the Hecke characters of the previous case restricted to ideals prime to $(3(1+i))$. Then

$$\phi_{K,c_\pm}(\tau) = [10, -4] \pm 8[2, 4]$$

[8, −2] and [−2, 8]: Take $K = \mathbb{Q}(\sqrt{-3})$ and $\mathfrak{f} = (4\sqrt{-3})$. If $(\alpha, \mathfrak{f}) = 1$, then

$$\alpha \equiv \zeta^a (1 - 2\zeta)^b \pmod 4,$$
$$\alpha \equiv (-1)^c \pmod{\sqrt{-3}}.$$

Here $\zeta = (1 + i\sqrt{3})/2$ is a primitive 6th root of unity, a is mod 6, and b, c are mod 2. Define characters $\chi_\pm(\alpha)$ on $R(\mathfrak{f})$ by

$$\chi_+(\alpha) = \zeta^a (-1)^c \alpha^2,$$
$$\chi_-(\alpha) = \zeta^a (-1)^{b+c},$$

and let $c_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 2. Then

$$\phi_{c_\pm}(\tau) = [10, -4] + 32[2, -4, 8] \pm 8i\sqrt{3}[-2, 8].$$

This shows that $[-2, 8]$ and $[10, -4] + 32[2, -4, 8]$ are lacunary. Since

$$[8, -2] = [10, -4] + 32[2, -4, 8] - 8[-2, 8],$$

the same holds for $[8, -2]$.

[9, −3] and [−3, 9]: Take $K = \mathbb{Q}(i)$ and $\mathfrak{f} = ((1+i)^5)$. Define characters $\chi_\pm(\alpha)$ on the group $R(\mathfrak{f}) = \langle i \rangle \times \langle 1 + 2i \rangle \times \langle 1 + 4i \rangle$ by putting

$$\chi_+(\alpha) = (-1)^{a+c},$$
$$\chi_-(\alpha) = (-1)^{a+b+c}.$$

Let $c_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 2. Then

$$\phi_{c_\pm}(\tau) = [9, -3] \pm [-3, 9].$$

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[6, −2] and [−2, 6]: Here $K=\mathbb{Q}(i)$ and $\mathfrak{f}=(3(1+i)^3)$. For $\mathfrak{a}=(\alpha)$ with $(\alpha,6)=1$, define

$$c_{\pm}(\mathfrak{a})=i^{a\pm b}\alpha,$$

where

$$\alpha\equiv i^a\ (\mathrm{mod}(1+i)^3),$$

$$\alpha\equiv(1-i)^b(\mathrm{mod}\ 3).$$

Then $c_{\pm}(\mathfrak{a})$ depends only on $\mathfrak{a}$, not on the particular generator α. We have

$$\sum_{(\mathfrak{a},\mathfrak{f})=1} c_{\pm}(\mathfrak{a})x^{N(\mathfrak{a})}=\eta(12\tau)^6\eta(24\tau)^{-2}\pm 4\eta(12\tau)^{-2}\eta(24\tau)^6.$$

[14, −4]: Take $K=\mathbb{Q}(i)$ and $\mathfrak{m}=(1+i)$. Let $\chi_0(\alpha)$ be the principal character of $R(\mathfrak{m})$ and $c(\mathfrak{a})$ the corresponding Hecke character of exponent 4. Then

$$\phi_{K,c}(\tau)=[14,-4].$$

[−4, 14]: Take $K=\mathbb{Q}(\mathrm{i})$ and $\mathfrak{f}=(1)$. Let $\chi_0(\alpha)$ be the principal character of $R(\mathfrak{f})$ and $c(\mathfrak{a})$ the corresponding Hecke character of exponent 4. Then

$$\phi_{K,c}(2\tau)+[14,-4]=[-4,14]$$

[17, −7] and [−7, 17]: These are linear combinations of four Hecke forms, arising in pairs from the fields $K=\mathbb{Q}(\sqrt{-2})$ and $L=\mathbb{Q}(i)$. First take $K=\mathbb{Q}(\sqrt{-2})$ and $\mathfrak{f}=(4)$. Let $\chi_{\pm}(\alpha)$ be the characters (14) of $R(\mathfrak{f})$, and $c_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of exponent 4. Then

$$\phi_{K,c_{\pm}}(\tau)=[17,-7]+64[-7,17]\pm 8\{[11,-1]+32[3,-1,8]\}.$$

Next take $L=\mathbb{Q}(i)$ and $\mathfrak{f}=((1+i)^5)$. Let $\chi'_{\pm}(\alpha)$ be the characters (13) of $R(\mathfrak{f})$, and $c'_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of exponent 4. Then

$$\phi_{L,c'_{\pm}}(\tau)=[17,-7]-64[-7,17]\mp 48i[5,5].$$

(This was already noted in the case [5, 5].) Thus both $[17,-7]+64[-7,17]$ and $[17,-7]-64[-7,17]$ are lacunary, which implies that [17, −7] and [−7, 17] are lacunary.

[18, −8] and [−6, 16]: These are linear combinations of four Hecke forms arising in pairs from the fields $K=\mathbb{Q}(i)$ and $L=\mathbb{Q}(\sqrt{-3})$. First take $K=\mathbb{Q}(i)$ and $\mathfrak{f}=(3(1+i)^3)$. Let $\chi_{18\pm}$ be the characters of $R(\mathfrak{f})$ defined by

$$\chi_{18\pm}(\alpha)=(-1)^a(\pm i)^b \quad \text{if}$$

$$\alpha\equiv i^a\ (\mathrm{mod}(1+i)^3)$$

$$\alpha\equiv(1-i)^b\ (\mathrm{mod}\ 3),$$

and $c_{\pm}(\mathfrak{a})$ the corresponding Hecke characters of exponent 4. Then

$$\phi_{K,c_{\pm}}(\tau)=[18,-8]+256[-6,16]\pm 48[10,0].$$

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(This case is in Serre's paper.) Next, let $L = \mathbb{Q}(\sqrt{-3})$, and $\mathfrak{f} = (4\sqrt{-3})$. Let $\chi'_{19\pm}$ be the characters of $R(\mathfrak{f})$ defined by

$$\chi'_+(\alpha) = \zeta^{-a}(-1)^c, \qquad \chi'_-(\alpha) = \zeta^{-a}(-1)^{b+c},$$

where

$$\alpha \equiv \zeta^a(1+2\zeta)^b \pmod 4$$
$$\alpha \equiv (-1)^c \pmod 3.$$

Let $c'_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 4. Then

$$\phi_{L,c'_\pm}(\tau) = [18,-8] - 128[-6,16] \pm 48\sqrt{3}\{[6,4] + 32[-2,4,8]\}.$$

Thus both $[18,-8] + 256[-6,16]$ and $[18,-8] - 128[-6,16]$ are lacunary, which implies that $[18,-8]$ and $[-6,16]$ are also lacunary.

[−8, 18] and [16, 6]: These are lacunary by the previous case, since

$$[-8,18] = [6,4] + 32[-2,4,8] + 8[-6,16],$$
$$[16,-6] = [18,-8] + 128[-6,16] - 16[-8,18].$$

[19, −9], [−9, 19], [15, −5] and [−5, 15]: These also require four forms. First take $K = \mathbb{Q}(\sqrt{-6})$, $\mathfrak{f} = (4)\mathfrak{p}$, where $\mathfrak{p}^2 = (3)$. Let $\chi_\pm(\alpha)$ be the characters (15) of $R(\mathfrak{f})$, and for principal ideals $\mathfrak{a}$, let $c_\pm(\mathfrak{a})$ be the corresponding Hecke characters of exponent 4. Since K has class number 2, each of these characters has 2 extensions to the set of all integral ideals prime to $\mathfrak{f}$. This gives four Hecke characters $c_{\pm,\pm}$ of $\mathfrak{a}$. The Hecke forms $\phi_{K,c_{\pm,\pm}}(\tau)$ comprise the four consistent sign combinations of

$$[19,-9] - 1472[-5,15] \pm 46i\{23[15,-5] + 64[-9,19]\}$$
$$\pm 40i\sqrt{6}\{[13,-3] + 32[5,-3,8]\} \pm 80\sqrt{6}\{[9,1] + 32[1,1,8]\}.$$

Hence $[19,-9] - 1472[-5,15]$ and $23[15,-5] + 64[-9,19]$ are lacunary. By the case $[7,3]$, $[19,-9] + 448[-5,15]$ and $7[15,-5] + 64[-9,19]$ are lacunary. Hence $[19,-9]$, $[-9,19]$, $[15,-5]$ and $[-5,15]$ are all lacunary.

[4, −2] and [−2, 4]: Although we have already proved the lacunarity of these forms in Section 3, we include them here for completeness. While they are of weight 1 and therefore do not fall under the general theory of Hecke and Shimura, they can nevertheless be expressed in terms of sums resembling Hecke character forms.

We have

$$[4,-2] = \theta(-x)^2 = \sum_{n=0}^{\infty} (-1)^n r_2(n) x^n,$$

where $r_2(n)$ is the number of representations of n as the sum of 2 squares. Removing the constant term and dividing by -4 we get

$$\frac{1}{4}(1 - \eta(\tau)^4\eta(2\tau)^{-2}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{r_2(n)}{4} x^n = 2\phi_{K,c'}(\tau) - \phi_{K,c}(\tau),$$

where $K = \mathbb{Q}(i)$, $c(\mathfrak{a})$ is the trivial character mod(1) and $c'(\mathfrak{a})$ is the trivial character mod

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$(1+i)$. Next, we have $[-2,4]=G(x^4)^2=\sum_{n=0}^{\infty} t_2(n)x^{2n+1}$, where $t_2(n)$ is the number of representations of n as the sum of two triangular numbers. Since $n=\frac{1}{2}(a^2+a)+\frac{1}{2}(b^2+b)$ if and only if $8n+2=(2a+1)^2+(2b+1)^2$, we easily find that $t_2(n)=\frac{1}{4}r_2(8n+2)$. This implies that

$$[-2,4]=\phi_{K,c}(\tau).$$

6. A combinatorial application. The lacunarity of the "diagonal cases" (s,s) for $s=1,2,3,5$ and 9 gives the following:

COROLLARY. *For $q\geq 0$, let $T(q)=\binom{q+1}{2}$ be the qth triangular number. Let $\mathcal{P}$ be the set of all partitions of the form $\pi:n=n_\square+n_\Delta$, where $n_\square=p_1^2+p_2^2+\cdots+p_s^2$, $n_\Delta=T(q_1)+T(q_2)+\cdots+T(q_s)$, with p_i, $q_j\in\mathbb{Z}$ and $q_j\geq 0$. Let $\alpha_s(n)$ be the number of such partitions with $n_\square$ even and $\beta_s(n)$ the number of them with $n_\square$ odd. Then*

$$\alpha_s(n)=\beta_s(n)\quad \textit{for almost all } n$$

if and only if $s=1,2,3,5$ or 9.

Proof. As noted in Section 3, we have

$$\sum_{p=-\infty}^{\infty}(-x)^{p^2}=(2,-1)$$

$$\sum_{q=0}^{\infty}x^{T(q)}=x^{-1/8}(-1,2).$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty}(\alpha(n)-\beta(n))x^n &= \sum_{\pi\in\mathcal{P}}(-1)^{n_\square}x^n\\ &= \sum_{p_1,\ldots,p_s}(-1)^{p_1^2+\ldots+p_s^2}x^{p_1^2+\ldots+p_s^2}\sum_{q_1,\ldots,q_s\geq 0}x^{T(q_1)+\ldots+T(q_s)}\\ &= \left(\sum_{p=-\infty}^{\infty}(-x)^{p^2}\right)^s\left(\sum_{q=0}^{\infty}x^{T(q)}\right)^s\\ &= x^{-s/8}(2s,-s)(-s,2s)\\ &= x^{-s/8}(s,s).\end{aligned}$$

By Theorem 2, this is lacunary if and only if $s=1,2,3,5$ or 9. □

The diagonal form $(2,2)$ is also of particular interest because it is the inverse Mellin transform of the Hasse-Weill L-function of the curve $y^2=x^3-x$. It is the image under the Shimura map of the forms of weight 3/2 which arise in Tunnel's work on the congruent number problem (see **[10]** and **[18]**).

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